UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y – CALCULUS II
FALL-WINTER 2007-08
ASSIGNMENT #1. DUE ON OCTOBER 11.
PROBLEMS AND SOLUTIONS

1. A brief review of single variable calculus.

a) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{x^2+8\sin x}{1-\cos(4x)}\right)$, if the limit exists.

Solution:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{x^2+8\sin x}{1-\cos(4x)}\right)=\lim_{x\to 0}\left(\frac{1-\cos(4x)-x^3-8x\sin x}{x-x\cos(4x)}\right)$$

Now, $\lim\limits_{x\to 0}(1-\cos(4x)-x^3-8x\sin x)=0=\lim\limits_{x\to 0}(x-x\cos(4x))$ and the given limit can be computed

applying l'Hospital's Rule a few times.

$$\lim_{x\to 0}\left(\frac{1-\cos(4x)-x^3-8x\sin x}{x-x\cos(4x)}\right)=\lim_{x\to 0}\left(\frac{4\sin(4x)-3x^2-8\sin x-8x\cos x}{1-\cos(4x)+4x\sin(4x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{16\cos(4x)-6x-16\cos x+8x\sin x}{8\sin(4x)+16x\cos(4x)}\right)=\lim_{x\to 0}\left(\frac{-64\sin(4x)-6+24\sin x+8x\cos x}{48\cos(4x)-16x\sin(4x)}\right)=-\frac{1}{8}.$$

b) Find the absolute maximum and the absolute minimum, if any, of the function $f(x)=e^{6|x-1|-x^2}$.

Solution:

$$f(x)=\begin{cases} e^{6(1-x)-x^2} & \text{if } x<1 \\ e^{6(x-1)-x^2} & \text{if } x\ge 1\end{cases}, \text{ therefore, } \quad f'(x)=\begin{cases}(-6-2x)e^{6(1-x)-x^2} & \text{if } x<1\\ \text{undefined} & \text{if } x=1\\ (6-2x)e^{6(x-1)-x^2} & \text{if } x>1\end{cases}$$

The function $f(x)$ is continuous everywhere, with $f'(-3)=0=f'(3)$, $f'(x)>0$ when $x<-3$ and when $1<x<3$, and $f'(x)<0$ when $-3<x<1$ and when $x>3$.

So, $f(x)$ has local maxima at $x=-3$ and at $x=3$ and a local minimum at $x=1$.

The absolute maximum of $f(x)$ is $f(-3)=e^{15}$.

There is no absolute minimum of $f(x)$ because $\lim\limits_{x\to\pm\infty} f(x)=0$ and $f(x)>0$ for all x.

2. More review on single variable calculus.

a) Evaluate the improper integral $\int_{-1}^{\infty} (\frac{x^4}{1+x^6})^2 \, dx$.

Solution:

We use integration by parts ($\int u\,dv = u\,v - \int v\,du$) with $u = x^3$ and $dv = \frac{x^5}{(1+x^6)^2} dx$.

Then $du = 3x^2 dx$, $v = -\frac{1}{6(1+x^6)}$ and $\int (\frac{x^4}{1+x^6})^2 \, dx = -\frac{x^3}{6(1+x^6)} + \int \frac{x^2}{2(1+x^6)} \, dx$.

But $\int \frac{x^2}{2(1+x^6)} \, dx = \frac{1}{6} \arctan(x^3) + C$.

Therefore , $\int_{-1}^{\infty} (\frac{x^4}{1+x^6})^2 \, dx = \left[\frac{1}{6} (-\frac{x^3}{1+x^6} + \arctan(x^3)) \right]_{-1}^{\infty} = \frac{3\pi - 2}{24}$.

b) Let W be the region consisting of all the points (x, y) such that $1 \le x \le e$ and $\ln x \le y \le x$.

Compute the volume of the solid generated when the region W is rotated about the line $x = -1$.

Solution:

We use cylindrical shells to compute this volume.

Then, $V = \int_1^e 2\pi(1+x)(x - \ln x)\,dx = 2\pi \int_1^e (x^2 + x - \ln x - x\ln x)\,dx$.

That is: $V = 2\pi \left[\frac{1}{3}x^3 + \frac{3}{4}x^2 + x - \frac{1}{2}x^2 \ln x - x\ln x \right]_1^e = \frac{\pi}{6} \, (4e^3 + 3e^2 - 25)$.

3. Calculus with parametric curves.

Consider the curve C with parametric equations $x = 2\int_1^t \sqrt{2+u^2}\,du$ and $y = t + \frac{1}{3}t^3$.

a) Find the slope of the tangent line to the curve C at the point where $t = 2$.

Solution:

Notice that $\frac{dx}{dt} = 2\sqrt{2+t^2}$ (Fundamental Theorem of Calculus), then $\frac{dy}{dx} = \frac{1+t^2}{2\sqrt{2+t^2}}$.

The slope of the tangent at $t = 2$ is $m = \frac{5\sqrt{6}}{12}$.

b) Find the values of t corresponding to each of the inflection points, if any, of the curve C .

Solution:

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$, where $\frac{dx}{dt} > 0$ for all values of t .

So, the condition $\frac{d}{dt}(\frac{dy}{dx}) = 0$ is enough for $\frac{d^2y}{dx^2} = 0$.

Now $\frac{d^2y}{dx^2} = \frac{t(3+t^2)}{2(2+t^2)^{3/2}}$, which is $= 0$ only when $t = 0$, is < 0 when $t < 0$ and is > 0 when $t > 0$.

The curve C has only one inflection point and it occurs at the point where $t = 0$.

c) Find the area of the surface obtained when the portion of C with $0 \le t \le \sqrt{2}$ is rotated about the x-axis.

Solution:

$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 4(2+t^2) + (1+t^2)^2 = t^4 + 6t^2 + 9$.

The area of the surface of revolution is:

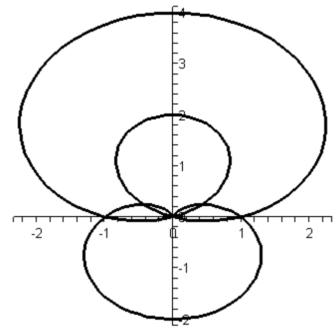
$$A = \int_0^{\sqrt{2}} 2\pi(t + \frac{1}{3}t^3)\sqrt{t^4+6t^2+9}\,dt = \frac{\pi}{6}\int_0^{\sqrt{2}} (4t^3+12t)\sqrt{t^4+6t^2+9}\,dt .$$

That is: $A = \frac{\pi}{9}\left[(t^4+6t^2+9)^{3/2}\right]_0^{\sqrt{2}} = \frac{98\pi}{9}$.

4. Polar coordinates.

a) Find the coordinates of each of the points of intersection of the polar curves $r = 1 - \sin\theta$ and $r = 1 + 3\sin\theta$.

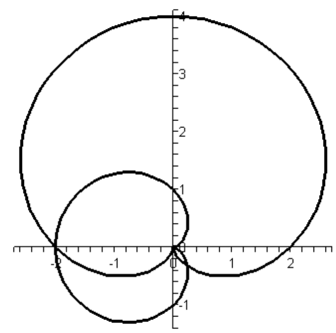
Solution:



A rough sketch of the two polar curves reveals that they have exactly five points of intersection. We can easily find two of these five points by solving the equation $1 - \sin\theta = 1 + 3\sin\theta$. The points $(1, 0)$ and $(1, \pi)$ are the only ones where the values of r and θ used to give the position of the intersection point are the same for both curves. The origin is also a point of intersection of the two curves. It is reached when $\theta = \pi/2$ on the curve $r = 1 - \sin\theta$ but is reached when $\theta = \pi + \arcsin(1/3)$ and also when $\theta = 2\pi - \arcsin(1/3)$ on the curve $r = 1 + 3\sin\theta$. The other points of intersection are the points described as $(1/2, \pi/6)$ and $(1/2, 5\pi/6)$ on the curve $r = 1 - \sin\theta$, which are respectively the same as the points described as $(-1/2, 7\pi/6)$ and $(-1/2, 11\pi/6)$ on the curve $r = 1 + 3\sin\theta$.

b) Compute the area of the region enclosed by the polar curve $r = \min(1 - \cos\theta, 2 + 2\sin\theta)$, where $0 \le \theta \le 3\pi/2$.

Solution:



Again here, a rough sketch of $r = 1 - \cos\theta$ and $r = 2 + 2\sin\theta$ reveals that the given curve $r = \min(1 - \cos\theta, 2 + 2\sin\theta)$, $0 \le \theta \le 3\pi/2$, is a continuous and closed curve, with $r = \begin{cases} 1-\cos\theta & \text{if} \quad 0 \le \theta \le \pi \\ 2+2\sin\theta & \text{if} \quad \pi < \theta \le 3\pi/2 \end{cases}$.

The area enclosed by this polar curve is $A = A_1 + A_2$, where $A_1 = \frac{1}{2}\int_0^{\pi}(1-\cos\theta)^2 d\theta$ and $A_2 = \frac{1}{2}(\int_{\pi}^{3\pi/2}(2+2\sin\theta)^2 d\theta)$.

Now, $A_1 = \frac{1}{2}\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta = \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{\pi}$ $= \frac{3\pi}{4}$. Similarly, $A_2 = \frac{3\pi - 8}{2}$ and $A = \frac{9\pi - 16}{4}$.

c) Compute the length of the polar curve $r = \max(1 - \cos\theta, 2 + 2\sin\theta)$, where $0 \le \theta \le 3\pi/2$.

Solution:

Notice that in this case, $\max(1 - \cos\theta, 2 + 2\sin\theta) = \begin{cases} 2+2\sin\theta & \text{if} \quad 0 \le \theta \le \pi \\ 1-\cos\theta & \text{if} \quad \pi < \theta \le 3\pi/2 \end{cases}$.

The length of this polar curve is $L = L_1 + L_2$, where $L_1 = \int_0^{\pi}\sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\,d\theta$ and $L_2 = \int_{\pi}^{3\pi/2}\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\,d\theta$. Now, $L_1 = \int_0^{\pi}\sqrt{8(1+\sin\theta)}\,d\theta = \int_0^{\pi}\frac{2\sqrt{2}\,|\cos\theta|}{\sqrt{1-\sin\theta}}\,d\theta$ $= \int_0^{\pi/2}\frac{4\sqrt{2}\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta = \left[-8\sqrt{2}\sqrt{1-\sin\theta}\right]_0^{\pi/2} = 8\sqrt{2}$. Similarly, $L_2 = 4 - 2\sqrt{2}$. Then, $L = 4 + 6\sqrt{2}$.

5. Three dimensional coordinate systems. Vectors.

a) Find an equation of the largest sphere that passes through the point $(-1, 1, 4)$ and is such that each of its points (x, y, z) satisfies the condition $x^2 + y^2 + z^2 \le 136 + 2(x + 2y + 3z)$.

Solution:

The given condition can be expressed as $(x-1)^2 + (y-2)^2 + (z-3)^2 \le 150$, which describes the points (x, y, z) whose distance to the point $(1, 2, 3)$ is $\le 5\sqrt{6}$. Notice now that the distance between the points $A(-1, 1, 4)$ and $B(1, 2, 3)$ is $\sqrt{6}$. Therefore, the largest sphere that satisfies the conditions of the problem is the sphere with radius $\frac{1}{2}(\sqrt{6} + 5\sqrt{6}) = 3\sqrt{6}$, centred at the point with coordinates

$(-1, 1, 4) + 3((1, 2, 3) - (-1, 1, 4)) = (5, 4, 1)$.

An equation of such sphere is $(x-5)^2 + (y-4)^2 + (z-1)^2 = 54$.

b) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any pair of three-dimensional vectors such that $|\mathbf{v}_1| = 2$, $|\mathbf{v}_2| = 3$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 5$.

Let $\mathbf{v}_3 = \operatorname{proj}_{\mathbf{v}_1}\mathbf{v}_2$, $\mathbf{v}_4 = \operatorname{proj}_{\mathbf{v}_2}\mathbf{v}_3$, $\mathbf{v}_5 = \operatorname{proj}_{\mathbf{v}_3}\mathbf{v}_4$, and so on. Compute $\sum_{i=1}^{\infty} |\mathbf{v}_i|$.

Solution:

$$\mathbf{v}_3 = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|^2}\mathbf{v}_1 = \frac{5}{2^2}\mathbf{v}_1, \quad \mathbf{v}_4 = \frac{\mathbf{v}_2 \cdot \mathbf{v}_3}{|\mathbf{v}_2|^2}\mathbf{v}_2 = \left(\frac{5}{2^2}\right)\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_2|^2}\mathbf{v}_2 = \frac{5^2}{2^2 3^2}\mathbf{v}_2, \ldots$$

After a few more computations, it is possible to observe that:

$$|\mathbf{v}_2| = 3, \quad |\mathbf{v}_3| = \frac{5}{2} = 3\left(\frac{5}{6}\right), \quad |\mathbf{v}_4| = \frac{5^2}{2^2 3} = 3\left(\frac{5}{6}\right)^2, \quad |\mathbf{v}_5| = \frac{5^3}{2^3 3^2} = 3\left(\frac{5}{6}\right)^3, \ldots$$

Then, $\sum_{i=1}^{\infty} |\mathbf{v}_i| = 2 + 3\sum_{i=0}^{\infty}\left(\frac{5}{6}\right)^i = 2 + 3\left(\frac{1}{1-\frac{5}{6}}\right) = 20$.

c) Let L_1 be the line that passes through the points $(1, 2, 6)$ and $(2, 4, 8)$ and let L_2 be the line of intersection of the planes Π_1 and Π_2, where Π_1 is the plane with equation $x - y + 2z = -1$ and Π_2 is the plane that passes through the points $(3, 2, -1)$, $(0, 0, 1)$ and $(1, 2, 1)$.
Compute the distance between the lines L_1 and L_2.

Solution:

$P_1(1, 2, 6)$ is a point on L_1 and the vector $\mathbf{v}_1 = (1, 2, 2)$ is a vector in the direction of L_1.

Using any of the procedures to obtain an equation for a plane that passes through three given (non collinear points) we can conclude that $2x - y + 2z = 2$ is an equation of the plane Π_2.

Now we can find points on the line $L_2 = \Pi_1 \cap \Pi_2$, as well as vectors in its direction.

For example $P_2(3, 4, 0)$ is a point on L_2 and $\mathbf{v}_2 = (1, -1, 2) \times (2, -1, 2) = (0, 2, 1)$ is a vector in the direction of L_2.

Let now $\mathbf{v} = (3-1, 4-2, 0-6) = (2, 2, -6)$ be the vector that has initial point P_1 and terminal point P_2 and let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-2, -1, 2)$, which is a vector perpendicular to both lines L_1 and L_2.

Notice that the distance d between the two lines L_1 and L_2 can be computed as $d = |\operatorname{proj}_{\mathbf{n}}\mathbf{v}|$.

Finally, $d = \dfrac{|(2,2,-6)\cdot(-2,-1,2)|}{|(-2,-1,2)|} = 6$.