Somewhat fuller proof of
Lemma on pages 126-127
in Hirsch, Smale, Devaney

lemma: For any $n \times n$ matrices A and B we have

$$\sum_{l=0}^{\infty}\left(\sum_{\substack{j,k\geq 0\\ j+k=l}} \frac{A^j}{j!}\frac{B^k}{k!}\right) = \left(\sum_{j=0}^{\infty}\frac{A^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right)$$

proof:
the left-hand side is e^{A+B}. The right hand side is $e^A e^B$. So we know that the series converge, it's just a question of showing that

$$\sum_{l=0}^{\infty}\left(\sum_{\substack{j,k\geq 0\\ j+k=l}} \frac{A^j}{j!}\frac{B^k}{k!}\right) \text{ and } \left(\sum_{j=0}^{\infty}\frac{A^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right)$$

have the same limit. Introduce partial sums.

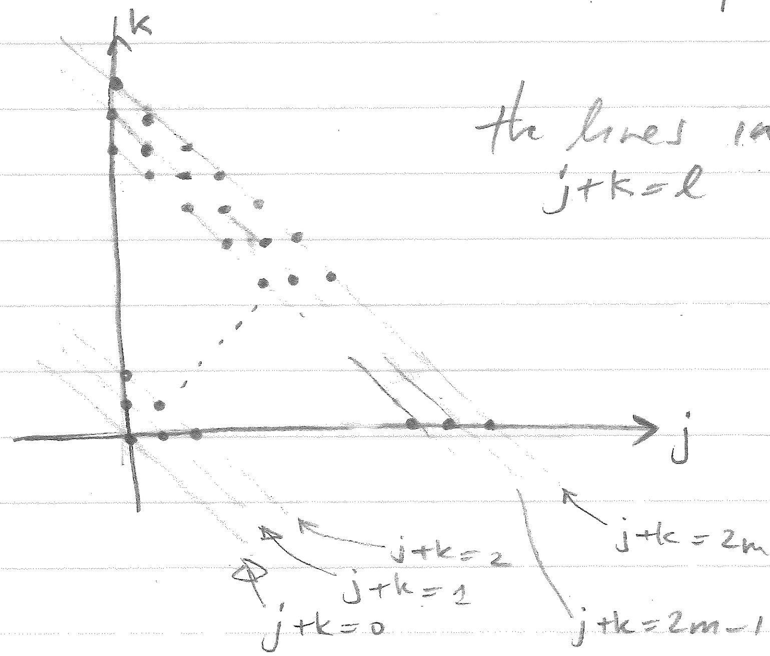
$$\alpha_m := \sum_{j=0}^{m}\frac{A^j}{j!} \qquad \beta_m := \sum_{k=0}^{m}\frac{B^k}{k!}$$

$$\gamma_{2m} := \sum_{l=0}^{2m}\left(\sum_{\substack{j,k\geq 0\\ j+k=l}} \frac{A^j}{j!}\frac{B^k}{k!}\right)$$

$$= \sum_{l=0}^{2m}\sum_{j=0}^{l}\frac{A^j}{j!}\frac{B^{l-j}}{(l-j)!}$$

the challenge: both terms in the sum depend on j; can't pull one of them out. Need to somehow trade the l for something manageable

Let's draw a picture in the j, k plane to better understand these partial sums



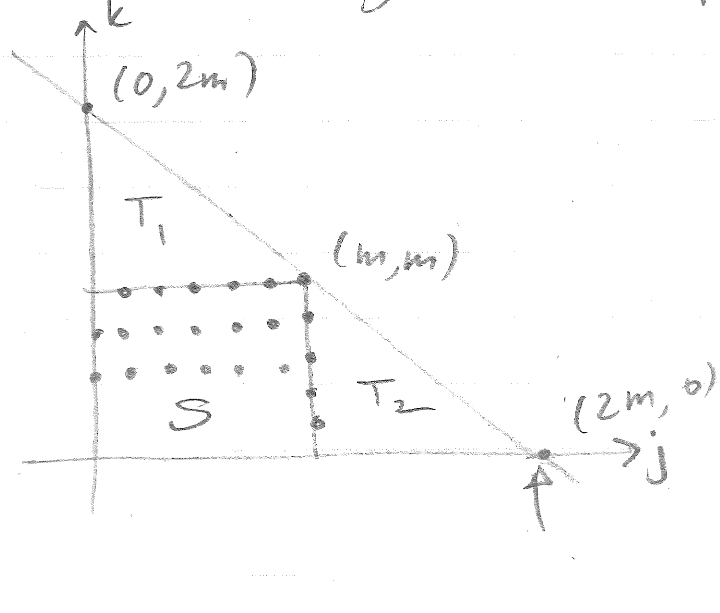
the lines indicate $j+k=\ell$ where $j,k \in \mathbb{R}$, $\ell \in \{0, 1, \dots, 2m\}$

the dots on the lines indicate $j, k \geq 0$, $j+k=\ell$, $j,k \in \mathbb{Z}$

That is γ_{2m} is adding up $\frac{A^j}{j!}\frac{B^k}{k!}$

where $(j,k) \in \mathbb{Z} \times \mathbb{Z}$ and (j,k) is in the "big triangle" (bounded by $j=0$, $k=0$, $j+k=2m$)

Idea: break the triangle into 3 pieces.



big triangle $= S \cup T_1 \cup T_2$

where $S = \{(j,k) \mid 0 \le j \le m,\ 0 \le k \le m\}$

$T_1 = \{(j,k) \mid 0 \le j < m,\ m+1 \le k \le 2m-j\}$

$T_2 = \{(j,k) \mid m+1 \le j \le 2m,\ 0 \le k \le 2m-j\}$

γ_{2m} = sum over big triangle

$$= \sum_S \frac{A^j}{j!}\frac{B^k}{k!} + \sum_{T_1} \frac{A^j}{j!}\frac{B^k}{k!} + \sum_{T_2} \frac{A^j}{j!}\frac{B^k}{k!}$$

$$= \underbrace{\sum_{j=0}^{m}\sum_{k=0}^{m} \frac{A^j}{j!}\frac{B^k}{k!}} + \sum_{j=0}^{m}\sum_{k=m+1}^{2m-j} \frac{A^j}{j!}\frac{B^k}{k!} + \sum_{j=m+1}^{2m}\sum_{k=0}^{2m-j} \frac{A^j}{j!}\frac{B^k}{k!}$$

$$= \sum_{j=0}^{m} \frac{A^j}{j!} \sum_{k=0}^{m} \frac{B^k}{k!} = \alpha_m \beta_m$$ $\boxed{\text{NICE!! ☺}}$

So

$$\gamma_{2m} - \alpha_m \beta_m = \sum_{j=0}^{m}\sum_{k=m+1}^{2m-j} \frac{A^j}{j!}\frac{B^k}{k!} + \sum_{j=m+1}^{2m}\sum_{k=0}^{2m-j} \frac{A^j}{j!}\frac{B^k}{k!}$$

we want to argue that the RHS goes to the zero matrix as $m \to \infty$. This would prove that

$$\gamma_{2m} - \alpha_m \beta_m \to 0 \quad \text{and}$$

therefore $e^{A+B} = e^A e^B$, as desired

First look at the sum over T_1

$$\sum_{j=0}^{m} \sum_{k=m+1}^{2m-j} \frac{A^j}{j!} \frac{B^k}{k!}$$

note there's a mistake in the book: the n is missing

use the triangle inequality and $\|\tilde{A}\tilde{B}\| \le n \|\tilde{A}\| \, \|\tilde{B}\|$

$$\left\| \sum_{j=0}^{m} \sum_{k=m+1}^{2m-j} \frac{A^j}{j!} \frac{B^k}{k!} \right\| \le n \sum_{j=0}^{m} \sum_{k=m+1}^{2m-j} \frac{\|A^j\|}{j!} \frac{\|B^k\|}{k!}$$

$$\le n \sum_{j=0}^{m} \sum_{k=m+1}^{\infty} \frac{\|A^j\|}{j!} \frac{\|B^k\|}{k!}$$ swapped $2m-j$ for ∞

$$\le n \sum_{j=0}^{\infty} \sum_{k=m+1}^{\infty} \frac{\|A^j\|}{j!} \frac{\|B^k\|}{k!}$$ swapped m for ∞

$$= n \sum_{j=0}^{\infty} \frac{\|A^j\|}{j!} \sum_{k=m+1}^{\infty} \frac{\|B^k\|}{k!}$$

$$\le n \sum_{j=0}^{\infty} \frac{n^{j-1}\|A\|^j}{j!} \sum_{k=m+1}^{\infty} \frac{n^{k-1}\|B\|^k}{k!}$$

$$= \frac{1}{n} \sum_{j=0}^{\infty} \frac{(n\|A\|)^j}{j!} \sum_{k=m+1}^{\infty} \frac{(n\|B\|)^k}{k!}$$

$$= \frac{1}{n} \left(e^{n\|A\|}\right) \sum_{k=m+1}^{\infty} \frac{(n\|B\|)^k}{k!}$$

So we've got

$$\left\| \sum_{T_1} \frac{A^j}{j!}\frac{B^k}{k!} \right\| \leq \frac{1}{n}\left(e^{n\|A\|}\right) \sum_{k=m+1}^{\infty} \frac{(n\|B\|)^k}{k!}$$

because $e^{n\|B\|} = \sum_{k=0}^{m} \frac{(n\|B\|)^k}{k!} + \sum_{m+1}^{\infty} \frac{(n\|B\|)^k}{k!}$

and the partial sum $\sum_{k=0}^{m} \frac{(n\|B\|)^k}{k!}$

converges to $e^{n\|B\|}$, we know that the "tail" converges to 0. And so by taking m large, we make $\sum_{k=m+1}^{\infty} \frac{(n\|B\|)^k}{k!}$ small. Hence $\left\| \sum_{T_1} \frac{A^j}{j!}\frac{B^k}{k!} \right\|$ can be taken arbitrarily small by taking m large.

Similarly, we show $\left\| \sum_{T_2} \frac{A^j}{j!}\frac{B^k}{k!} \right\| \leq n e^{n\|B\|} \sum_{j=m+1}^{\infty} \frac{(n\|A\|)^j}{j!}$

and we have $\sum_{j=m+1}^{\infty} \frac{(n\|A\|)^j}{j!}$ going to zero

as $m \to \infty$.

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