MAT267: HW6

If you're choosing to do this assignment, please submit it by 11:59pm on Monday April 20

This document was updated April 7 to fix a typo in problem 3e. It was updated April 10 to fix problem 7d.

1. Chapter 8, problem 3 (on page 184).
2. Chapter 8, problem 6 (on page 185).
3. The arguments in section 8.2 relied on a differential inequality argument; it's important that you understand this argument fully.
   (a) Consider the ODE $x' = f(x) = \sin(x)$. What are the equilibrium solutions of this ODE?
   (b) Let x_{ss} be the smallest positive equilibrium solution of $x' = \sin(x)$. Define $y(t) = x(t) - x_{ss}$, the deviation from this equilibrium solution. If $x(t)$ is a solution of $x' = \sin(x)$, what ODE is $y(t)$ a solution of?
   (c) In part b), you found an ODE $y' = g(y)$. Write this as $y' = -\lambda y + h(y)$. Prove that if you choose λ well then
   $$\lim_{y \to 0} \frac{h(y)}{y} = 0.$$
   (d) Multiplying the rewritten ODE from part c) by y, you find a new ODE
   $$\left(\frac{1}{2} y^2\right)' = \psi(y) := y^2 \left(-\lambda + \frac{h(y)}{y}\right)$$
   Let $\widetilde{\psi}(y) = -(\lambda/2)\, y^2$. Prove there exists $\delta > 0$ so that if $|y| < \delta$ then $\psi(y) < \widetilde{\psi}(y)$.
   (e) Now to go from differential equations and differential inequalities to the behaviour of solutions. Assume $y(t)$ is a solution of the initial value problem $y' = g(y)$ and $y(0) = y_0$ where $y_0 \in B_\delta(0)$. Let (α, β) be the maximal interval of existence of the solution $y(t)$. Show that $y(t)$ is also a solution of $(y^2/2)' = \psi(y)$ on (α, β). Show that there's an interval $(c, d) \ni 0$ so that $y(t) \in B_\delta(0)$ for all $t \in (c, d)$. Show that for all $t \in (c, d)$, the solution $y(t)$ satisfies the differential inequality $(y^2/2)' < \widetilde{\psi}(y)$. Show that this implies that $y(t)^2 < y_0^2 \exp(-\lambda t)$ for all $t \in (0, d)$ and that it then follows that $d = \beta$ and that $\beta = \infty$ and that $y(t) \to 0$ as $t \to \infty$.

4. Consider the nonlinear system

$$X' = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix} X + H(X)$$

where $\alpha > 0$ and $|H(X)|/|X| \to 0$ as $|X| \to 0$. Assume that H is C^1, hence locally Lipschitz, in a neighbourhood of $\vec{0}$. Prove that there exists a $\delta > 0$ so that if $X_0 \in B_\delta(\vec{0})$ then the initial value problem has a solution $X(t)$ with interval of existence (a, ∞) and that $X(t) \to \vec{0}$ as $t \to \infty$.

5. Consider the nonlinear system

$$X' = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} X + H(X)$$

where $\lambda > 0$ and $|H(X)|/|X| \to 0$ as $|X| \to 0$. Assume that H is C^1, hence locally Lipschitz, in a neighbourhood of $\vec{0}$. Prove that there exists a $\delta > 0$ so that if $X_0 \in B_\delta(\vec{0})$ then the initial value problem has a solution $X(t)$ with interval of existence (a, ∞) and that $X(t) \to \vec{0}$ as $t \to \infty$.

6. Chapter 8, problem 11 (on page 186).

   Note: in doing this problem you're going to need to prove that a linear transformation is a continuous function. This will involve the matrix norm $\|P\|_\infty$ used in section 6.4 as well as the L^1 norm of a vector (which "plays nice" with the matrix norm) and the L^2 norm of a vector (which is the one used in this problem where it's denoted with absolute value signs).

$$\|X\|_1 = \sum_{i=1}^n |x_i|, \qquad |X| = \|X\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$$

7. In the previous problem, your change of coordinates transformed the linearized problem into one that's in Jordan canonical form. This allows you to use the structure of Jordan canonical form matrices to prove that if all of the eigenvalues have negative real parts then the equilibrium solution is asymptotically stable.

   In this problem, you'll figure out a different approach to asymptotic stability in which you define a new inner product on $\mathbb{R}^n$ and use its induced norm.

   (a) Given a real-valued matrix B, define

$$(X, Y)_B := \langle BX, BY \rangle = \sum_{i=1}^n (BX)_i (BY)_i$$

   That is, $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. What conditions on B will ensure that $(\cdot, \cdot)_B$ is an inner product? Prove that $(\cdot, \cdot)_B$ is an inner product under those conditions. ("Hah-hah very funny" to anyone whose condition is that $B = I \ldots$)

(b) The inner product $(\cdot,\cdot)_B$ induces a norm on $\mathbb{R}^n$ in the usual manner

$$\|X\|_B^2 := (X, X)_B.$$

Prove that this norm is equivalent to the usual L^2 norm generated by the usual $\langle\cdot,\cdot\rangle$ inner product by showing there exist $a, b \in (0,\infty)$ so that

$$a\|X\|_2 \leq \|X\|_B \leq b\|X\|_2.$$

Convince yourself that this implies that the topology generated by this norm is the same as the topology generated using the usual L^2 norm.

(c) Consider the nonlinear system $X' = F(X)$ where $F(\vec{0}) = 0$. Assume that F is C^2 in a neighbourhood $\mathcal{O}$ of $\vec{0}$. Writing $F(X) = DF(\vec{0})X + H(X)$, show that if $\delta > 0$ and $\overline{B_\delta(\vec{0})} \subset \mathcal{O}$ then there exists $C_\delta < \infty$ so that

$$\|H(X)\|_2 \leq C_\delta \|X\|_2^2, \qquad \text{for all } X \in B_\delta(\vec{0}).$$

(d) Assume that $DF(\vec{0})$ is diagonalizable and that all of its eigenvalues are negative with $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 < 0$. Find a choice of B for which you can prove

$$(DF(\vec{0})X, X)_B \leq \lambda_1\,(X, X)_B.$$

(e) Taking δ smaller, if necessary, show that if $Z(t)$ is a solution of $X' = F(X)$ with $Z(0) \in B_\delta(\vec{0})$ then

$$\frac{d}{dt}(Z(t), Z(t))_B \leq \lambda_1\,(Z(t), Z(t))_B$$

and conclude that the solution exists for all $t \geq 0$ and $Z(t) \to \vec{0}$ as $t \to \infty$.

In fact, the above approach generalizes if you prove the following linear algebra result: If all eigenvalues of $DF(\vec{0})$ have negative real part then there exists a (symmetric, positive definite) inner product $(\cdot,\cdot)$ and a positive constant K so that

$$(DF(\vec{0})X, X) \leq -K(X, X)$$

for all $X \in \mathbb{R}^n$.