HW 5 Solutions

1 - Solutions are $x(t) = x_0 e^{\int_0^t a(s)ds}$ $y(t) = x_0 e^{\int_0^t b(s)ds}$.

If we mean $\| \|_\infty$ to be the sup norm and M not dependent on $a-b$, then the inequality is not true. (Take $b(t) = 0$ and $a(t)$ increasing constant).

Nonetheless, $x_0 e^{\int_0^t a(s)ds}$ is continuous wrt a.

$$|x(t) - y(t)| = |x_0| \left| e^{\int_0^t a(s)ds} - e^{\int_0^t b(s)ds} \right|$$

use MVT $f(z) = e^z$ $f(w) - f(z) = f'(c)(w-z)$

where $w = \int_0^t a(s)ds$ and $z = \int_0^t b(s)ds$ and

So

$$|x(t) - y(t)| = |x_0| e^{c(t)} \left| \int_0^t a(s)ds - \int_0^t b(s)ds \right|$$

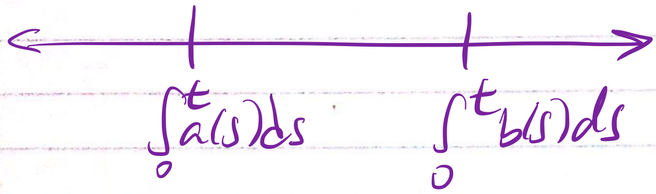
$$\leq |x_0| e^{c(t)} \int_0^t |a(s) - b(s)| ds$$

$$\leq |x_0| t \|a-b\|_\infty e^{c(t)}$$

$$\leq |x_0| t_1 \|a-b\|_\infty e^{c(t)}$$

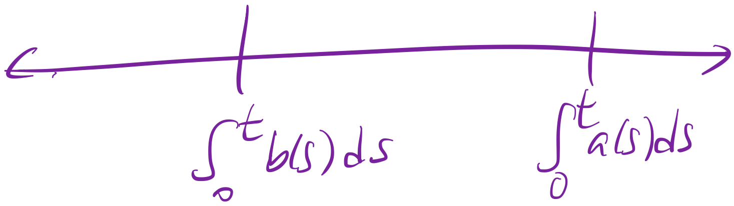
Above, $c(t)$ is "some value between $\int_0^t a(s)ds$ and $\int_0^t b(s)ds$". That's all we know about it. we want to bound $e^{c(t)}$ in some way.

case 1:



$$\Rightarrow c(t) \leq \int_0^t b(s)ds \leq \|b\|_\infty t_1 \Rightarrow e^{c(t)} \leq e^{\|b\|_\infty t_1}$$

Case 2:



$$\Rightarrow \quad c(t) \le \int_0^t a(s)\,ds \le \|a\|_\infty t_1 \Rightarrow e^{c(t)} \le e^{\|a\|_\infty t_1}$$

And so we know

$$e^{c(t)} \le \max\{e^{\|a\|_\infty t_1}, e^{\|b\|_\infty t_1}\}$$

but that's not helpful because we want a bound that doesn't depend on b! And we only care about b "close to" a in any case...

Assume $\|b-a\|_\infty \le 1$ Then $|b(t)-a(t)| \le 1$

$\forall t \in [0, t_1] \Rightarrow |b(t)| \le |a(t)| + |b(t)-a(t)| \le |a(t)|+1$

$$\le \|a\|_\infty + 1$$

This is true for all $t \in [0, t_1]$. Taking the supremum over t, $\|b\|_\infty \le \|a\|_\infty + 1$

Hence $\|b-a\|_\infty \le 1 \Rightarrow e^{c(t)} \le e^{(\|a\|_\infty+1)t_1}$

and so $\|b-a\|_\infty \le 1 \Rightarrow$

$$|x(t)-y(t)| \le |x_0|\, t_1 \|a-b\|_\infty e^{(\|a\|_\infty+1)t_1}$$

This is true $\forall t \in [0, t_1]$ so take the supremum over t in $[0, t_1]$ to conclude

If $\|b-a\|_\infty \le 1$ then $\|x-y\|_\infty \le |x_0| t_1 \|a-b\|_\infty e^{(\|a\|_\infty+1)t_1}$

Now given $\varepsilon > 0$ let $\delta = \min\{1, \frac{\varepsilon}{|x_0|t_1} e^{-(\|a\|_\infty+1)t_1}\}$.

If $\|b-a\|_\infty < \delta$ then $\|b-a\|_\infty < 1$ hence

$$\|x-y\|_\infty \le |x_0| t_1 \|a-b\|_\infty e^{(\|a\|_\infty+1)t_1}$$

But $\|b-a\|_\infty < \delta$ implies $\|b-a\|_\infty < \frac{\varepsilon}{|x_0|t_1} e^{-(\|a\|_\infty+1)t_1}$

which implies $\|x-y\|_\infty < \varepsilon$, as desired. //

[Note: there was nothing magical about the control $\|b-a\|_\infty \le 1$. I just needed something to exclude b from being arbitrarily large. I could have chosen

$$\delta = \min\{10, \frac{\varepsilon}{|x_0|t_1} e^{-(\|a\|_\infty+10)t_1}\}$$

just as easily. ☺]

#2 Prove the following theorem.

Theorem Consider the differential equation $X' = F(X)$ where $F: \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. Suppose that $X(t)$ is a solution of this equation that is defined on the closed interval $[t_0, t_1]$ with $X(t_0) = X_0$. Then there is a neighborhood U of X_0 and a constant K such that if $Y_0 \in U$ then there is a unique solution

$Y(t)$ also defined on $[t_0, t_1]$ with $Y(t_0) = Y_1$.
Moreover, $Y(t)$ satisfies

$$|Y(t) - X(t)| \le |Y_0 - X_0| e^{K(t-t_0)}$$

for all $t \in [t_0, t_1]$.

Proof: Let $\varepsilon > 0$.

X is continuous and $[t_0, t_1]$ is compact so $\Gamma = \{X(t) \mid t \in [t_0, t_1]\}$ is compact. Because Γ is compact,

$$\mathrm{dist}(Z, \Gamma) = \min\{\|Z - W\| \mid W \in \Gamma\}$$

is defined and so

$$\Gamma_\varepsilon = \{Z \in \mathbb{R}^n \mid \mathrm{dist}(Z, \Gamma) \le \varepsilon\}$$

is a closed and bounded, hence compact, subset of $\mathbb{R}^n$. F is locally Lipschitz so $\exists\ K < \infty$ so that

$$\|F(X) - F(Y)\| \le K\|X - Y\| \quad \forall X, Y \in \Gamma_\varepsilon.$$

Take $\delta = \varepsilon e^{-K(t_1 - t_0)} < \varepsilon$.

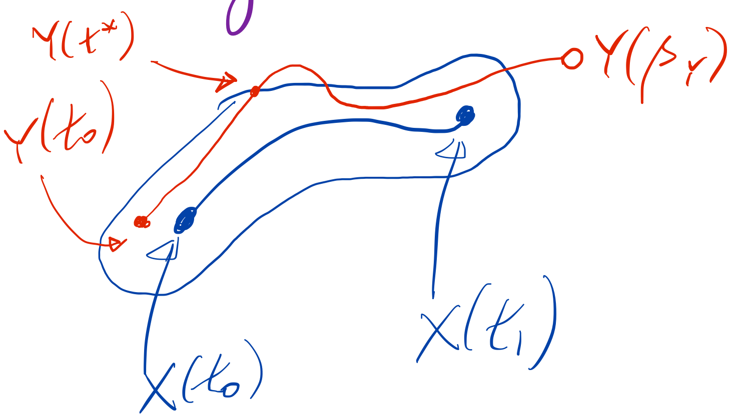
If $Y_0 \in B_\delta(X_0)$ then $\|X_0 - Y_0\| < \varepsilon$

$\Rightarrow Y_0 \in \Gamma_\varepsilon$. F is continuous on

Γ_ε and Lipschitz on Γ_ε $\Rightarrow$ there's a unique solution of $X' = F(X)$ that satisfies $Y(t_0) = Y_0$. Let (α_Y, β_Y) be the maximal interval of existence of this solution.

We can control $\|X(t) - Y(t)\|$ if both X and Y are a) defined up to time t and b) both are in Γ_ε up to time t.

We don't know how long the solution is is Γ_ε though



Maybe this could happen?

Let $t^* = \sup\{\tau \mid Y(t) \in \Gamma_\varepsilon$ for all $t \in [t_0, \tau]\}$

By construction, $Y(t) \in \Gamma_\varepsilon$ for all $t \in [t_0, t^*)$. By the same argument as in the proof of the theorem on page 398, because Γ_ε is compact the solution can be extended to (and beyond) t^*. i.e. $t^* < \beta_Y$.

[It follows that $\text{dist}(Y(t^*), \Gamma) = \varepsilon$ because if the distance were less than ε, t^* wouldn't be maximal.]

- $X(t) \in \Gamma_\varepsilon$ for $t \in [t_0, t_1]$
- $Y(t) \in \Gamma_\varepsilon$ for $t \in [t_0, t^*]$

So for $t \in [t_0, \min\{t_1, t^*\}]$

we have

$$\|X(t) - Y(t)\| \leq \|X_0 - Y_0\| e^{K(t-t_0)} \quad (*)$$

(it was key that both solutions are in Γ_ε (where we have K-control up to time t to get $(*)$)

Hence

$$\|X(t)-Y(t)\| < \varepsilon e^{-K(t_1-t_0)} e^{K(t-t_0)}$$

$$= \varepsilon e^{-K(t_1-t)}$$

is true for all t in $[t_0, \min\{t_1, t^*\}]$.

I claim that $t^* > t_1$. That is, $Y(t)$ cannot leave Γ_ε until after time t_1.

Assume not. If $t^* \le t_1$ then the bound holds at time t^*:

$$\|X(t^*)-Y(t^*)\| < \varepsilon e^{-K(t_1-t^*)} \le \varepsilon$$

$$\Rightarrow \text{dist}(Y(t^*), \Gamma) \le \|X(t^*)-Y(t^*)\| < \varepsilon$$

which is impossible. Therefore $t^* > t_1$ and we have a unique solution Y on $[t_0, t_1]$ and $\forall\, t \in [t_0, t_1]$ we have the desired bound

$$\|X(t)-Y(t)\| \le \|X_0 - Y_0\| e^{K(t-t_0)}.$$

Integrating near 0 (I am assuming that $\tau \neq -1$).

3. $x' = x + \tau x^2 \Rightarrow \int \frac{dx}{x+\tau x^2} + C = \int dt \Rightarrow \int \frac{dx}{x} + \int \frac{-\tau\, dx}{1+\tau x} + C = t \Rightarrow$

$\log x - \log(1+\tau x) + C = t \Rightarrow \frac{x}{1+\tau x} = e^{t-C} \Rightarrow x - \tau e^{t-C} x = e^{t-C} \Rightarrow$

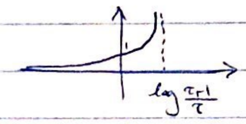
$x = \frac{e^{t-C}}{1-\tau e^{t-C}}$ $\quad x(0) = 1 \Rightarrow \frac{\boxed{e^{-C}}\, D}{1-\tau e^{-C}} = 1 \Rightarrow D = 1 - \tau D \Rightarrow D = \frac{1}{1+\tau}$.

so $x(t) = \left(\left(\frac{1}{1+\tau}\right) e^t \Big/ 1 - \frac{\tau e^t}{1+\tau} \right)$ for $\tau \neq -1$.

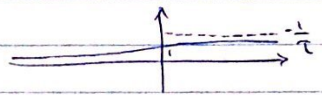
Interval for this $x(t)$ is ~~state $(-\infty, \infty)$~~ obtained:

$1 - \frac{\tau e^t}{1+\tau} = 0 \Leftrightarrow 1+\tau-\tau e^t = 0 \underset{\text{Assuming } \tau \neq 0}{\Rightarrow} e^t = \frac{\tau+1}{\tau} \Rightarrow t = \log\left(\frac{\tau+1}{\tau}\right)$ (Assuming $\tau > 0$).

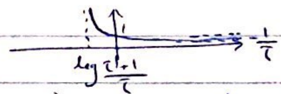
So $\tau > 0 \Rightarrow$ Interval $\left(-\infty, \log\left(\frac{\tau+1}{\tau}\right)\right)$.



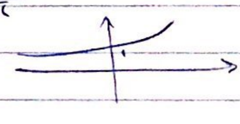
$0 > \tau > -1 \Rightarrow$ Interval $(-\infty, \infty)$.



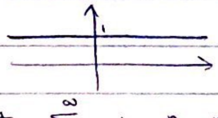
$-1 > \tau \Rightarrow$ Interval $\left(\log\left(\frac{\tau+1}{\tau}\right), \infty\right)$.



$\tau = 0 \Rightarrow$ Interval $(-\infty, \infty)$ (solution is e^t).



$\tau = -1 \Rightarrow x \equiv 1$ $(-\infty, \infty)$.



$x(t) = \frac{e^t}{1+\tau(1-e^t)} = e^t \sum_{n=0}^{\infty} (-1)^n (1-e^t)^n \tau^n = e^t \sum_{n=0}^{\infty} (e^t-1)^n \tau^n = e^t + (e^{2t} - e^t)\tau + \cdots$

$x(t) = \frac{e^t}{1+(\tau+1)(1-e^t) - 1 + e^t} = \frac{e^t}{e^t + (\tau+1)(1-e^t)} = \frac{1}{1+(\tau+1)\left(\frac{1-e^t}{e^t}\right)} = \sum \left(\frac{e^t-1}{e^t}\right)^n (\tau+1)^n =$

$1 + \left(\frac{e^t-1}{e^t}\right)\tau + \cdots$

Interval at $\tau = 1$ is $(-\infty, \log 2) \approx (-\infty, 0.69\ldots)$

So because $\log \frac{\tau+1}{\tau}$ is continuous near $\tau = 1$, interval changes continuously.

$x(t) = \dfrac{e^t}{1+\tau-\tau e^t}$ is obviously continuous wrt τ (near $\tau = 1$ on $[0, 0.69]$).

Constraint of having 0.69 near $\tau=1$ is $\tau \in \left(-\infty, \dfrac{1}{e^{0.69}-1}\right)$.

#4

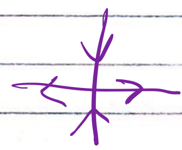
a) $\begin{cases} x' = \sin(x) \\ y' = \cos(y) \end{cases}$ $\qquad x' = 0 \iff x = n\pi \quad n \in \mathbb{N}$

$\qquad\qquad\qquad\qquad\qquad y' = 0 \iff y = \frac{\pi}{2} + m\pi \quad m \in \mathbb{N}$

$$DF(x,y) = \begin{pmatrix} \cos(x) & 0 \\ 0 & -\sin(y) \end{pmatrix} \qquad X_{eq}(n,m) = \begin{pmatrix} n\pi \\ \frac{\pi}{2} + m\pi \end{pmatrix}$$

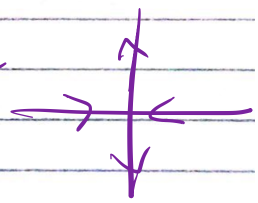
i) both n and m are even

$\rightarrow DF(X_{eq}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ saddle



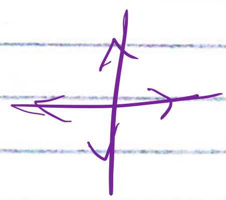
ii) both n and m are odd

$\rightarrow DF(X_{eq}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ saddle



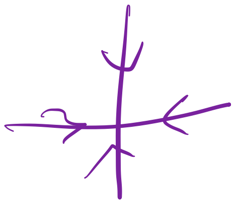
iii) n is even and m is odd

$\rightarrow DF(X_{eq}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ source

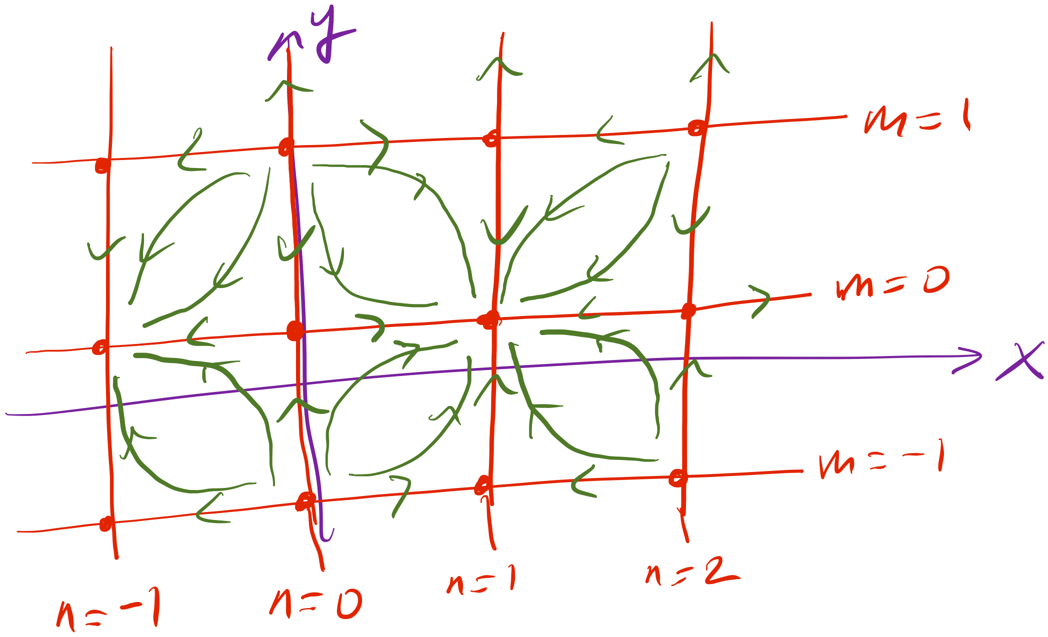


iv) n is odd and m is even

$$\rightarrow DF(X_{eq}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ sink



Rough sketch of phase portrait



b) $\begin{cases} x' = x(x^2+y^2) \\ y' = y(x^2+y^2) \end{cases}$

$x' = 0 \Leftrightarrow x = 0$ or $x^2+y^2=0$

$y' = 0 \Leftrightarrow y = 0$ or $x^2+y^2=0$

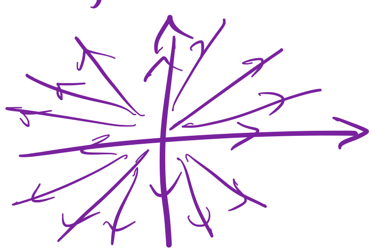
$$X_{eq} = (0,0)$$

$$DF(x,y) = \begin{pmatrix} 3x^2+y^2 & 2xy \\ 2xy & x^2+3y^2 \end{pmatrix}$$

$$DF(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

No useful info from linearization

Rough sketch of phase portrait:



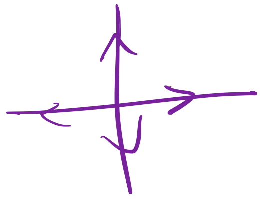
c) $\begin{cases} x' = x + y^2 \\ y' = 2y \end{cases}$ $\quad$ $x' = 0 \iff x = y^2$

$y' = 0 \iff y = 0$

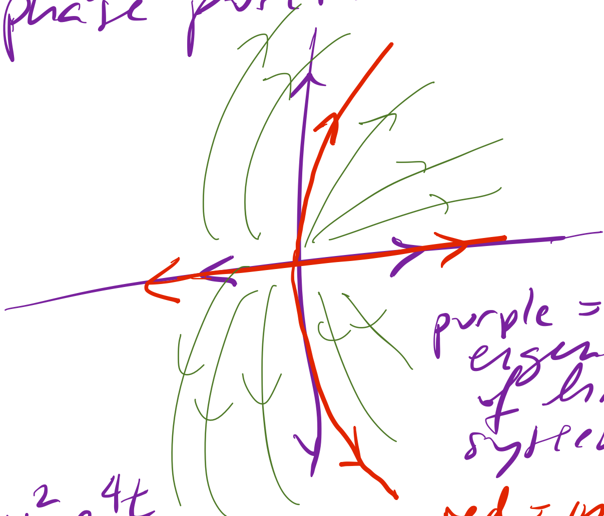
$\Rightarrow X_{eq} = (0,0)$

$$DF = \begin{pmatrix} 1 & 2y \\ 0 & 2 \end{pmatrix}$$

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$



Rough sketch of phase portrait:



exact solution is

$$\begin{cases} x(t) = (x_0 - \frac{1}{3}y_0^2)e^t + \frac{1}{3}y_0^2 e^{4t} \\ y(t) = y_0 e^{2t} \end{cases}$$

the curve $y = 0$ is invariant

if $(x_0, y_0) = (x_0, 0)$ then $(x(t), y(t)) = (x_0 e^t, 0)$

the curve $x - \frac{1}{3}y^2$ is invariant

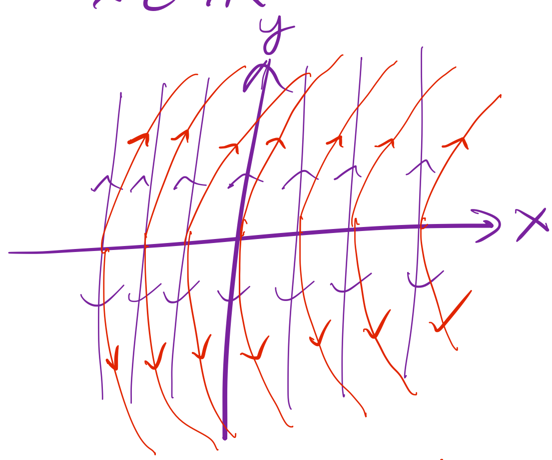
because $x(t) - \frac{1}{3}y(t)^2 = e^t\left(x_0 - \frac{1}{3}y_0^2\right)$

so if $x_0 - \frac{1}{3}y_0^2 = 0$ then this will hold true $\forall t$.

d) $\begin{cases} x' = y^2 \\ y' = y \end{cases}$ $x' = 0 \iff y = 0$

$y' = 0 \iff y = 0.$

there's a line of equilibrium solutions $X_{eq} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $x \in \mathbb{R}$



$$DF(x,y) = \begin{pmatrix} 0 & 2y \\ 0 & 1 \end{pmatrix}$$

$$DF(X_{eq}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

red = invariant curves of nonlinear problem

purple = unstable eigenspace of linearized problem

exact solution?

$$\begin{cases} x(t) = (x_0 - \frac{1}{2}y_0^2) + \frac{1}{2}y_0^2 e^{2t} \\ y(t) = y_0 e^t \end{cases}$$

Again the curve $y = 0$ is invariant if $(x_0, y_0) = (x_0, 0)$ then $(x(t), y(t)) = (x_0, 0)$

The curve $x - y^2/2 = C$ is invariant because $x(t) - \frac{y(t)^2}{2} = x_0 - \frac{y_0^2}{2} = C$

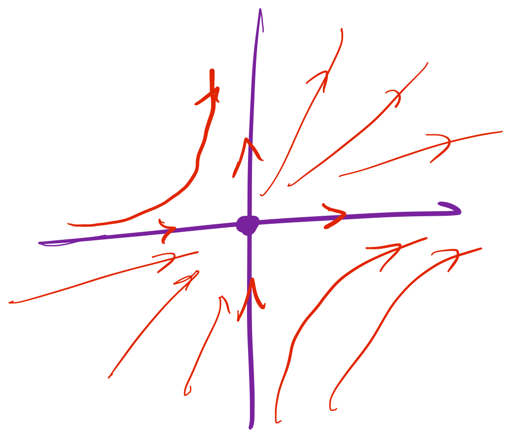
e) $\begin{cases} x' = x^2 \\ y' = y^2 \end{cases}$ $\quad x' = 0 \iff x = 0$

$y' = 0 \iff y = 0$

$X_{eq} = \vec{0}$ is the only equilibrium solution.

$DF(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$

$DF(X_{eq}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ no information



$x(t) = \dfrac{x_0}{1 - x_0 t}$

& exact solution

$y(t) = \dfrac{y_0}{1 - y_0 t}$

the curve $x = 0$ is invariant to the flow, as is the curve $y = 0$.

#5 $\begin{cases} x' = x^2 + y \\ y' = x - y + a \end{cases}$

a) find all equilibrium points and compute the linearized equation at each.

$y' = 0 \iff x - y + a = 0$
$\iff y = x + a$

$x' = 0 \iff x^2 + y = 0$

$x' = 0$ and $y' = 0$ Need $x^2 + x + a = 0$
$x = \frac{1}{2}(-1 \pm \sqrt{1-4a})$

$a = 0$
$\frac{1}{2}(-1 \pm 1)$
0
-1
$(0,0)$
$(-1,-1)$

- So if $a < \frac{1}{4}$ then there are two equilibrium solutions

$$X_+ = \begin{pmatrix} \frac{1}{2}(-1+\sqrt{1-4a}) \\ a + \frac{1}{2}(-1+\sqrt{1-4a}) \end{pmatrix}$$

$$X_- = \begin{pmatrix} \frac{1}{2}(-1-\sqrt{1-4a}) \\ a + \frac{1}{2}(-1-\sqrt{1-4a}) \end{pmatrix}$$

- if $a = 1/4$ Then there is one equilibrium $X_{eq} = \begin{pmatrix} -1/2 \\ -1/4 \end{pmatrix}$

- if $a > 1/4$ then there are no equilibrium solutions.

$$F(x,y) = \begin{pmatrix} x^2 + y \\ x - y + a \end{pmatrix}$$

$$DF(x,y) = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$$

$$DF(X_{eq}) = \begin{pmatrix} 2x_\pm & 1 \\ 1 & -1 \end{pmatrix}$$

$x_\pm(a) = \frac{1}{2}\left(-1 \pm \sqrt{1-4a}\right)$

find characteristic polynomial and eigenvectors

The eigenvalues are

$$\lambda = \frac{1}{2}\left(-1 + 2x \pm \sqrt{5+4x+4x^2}\right)$$

with eigenvectors

$$\vec{v}_\pm = \begin{pmatrix} 1 + 2x \pm \sqrt{5+4x+4x^2} \\ 2 \end{pmatrix}$$

eigensystem when $x = \frac{1}{2}\left(-1 + \sqrt{1-4a}\right)$?

$\lambda_1 = -1 + \frac{1}{2}\sqrt{1-4a} - \frac{1}{2}\sqrt{5-4a}$

with eigenvector $\vec{V}_1 = \begin{pmatrix} \sqrt{1-4a} - \sqrt{5-4a} \\ 2 \end{pmatrix}$

$\lambda_2 = -1 + \frac{1}{2}\sqrt{1-4a} + \frac{1}{2}\sqrt{5-4a}$

with eigenvector $\vec{V}_2 = \begin{pmatrix} \sqrt{1-4a} + \sqrt{5-4a} \\ 2 \end{pmatrix}$

as $a \uparrow 1/4$, $\lambda_1 \to -2$ and $\lambda_2 \to 0$

$\vec{V}_1 \to \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ $\quad \vec{V}_2 \to \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

for $a < 1/4$ $\quad \lambda_1 < 0$ and $\lambda_2 > 0$

eigensystem when $x = \frac{1}{2}(-1 - \sqrt{1-4a})$?

$\lambda_1 = -1 - \frac{1}{2}\sqrt{1-4a} - \frac{1}{2}\sqrt{5-4a}$

$\vec{V}_1 = \begin{pmatrix} -\sqrt{1-4a} - \sqrt{5-4a} \\ 2 \end{pmatrix}$

and

$$\lambda_2 = -1 - \tfrac{1}{2}\sqrt{1-4a} + \tfrac{1}{2}\sqrt{5-4a}$$

$$\vec{V}_2 = \begin{pmatrix} -\sqrt{1-4a}+\sqrt{5-4a} \\ 2 \end{pmatrix}$$

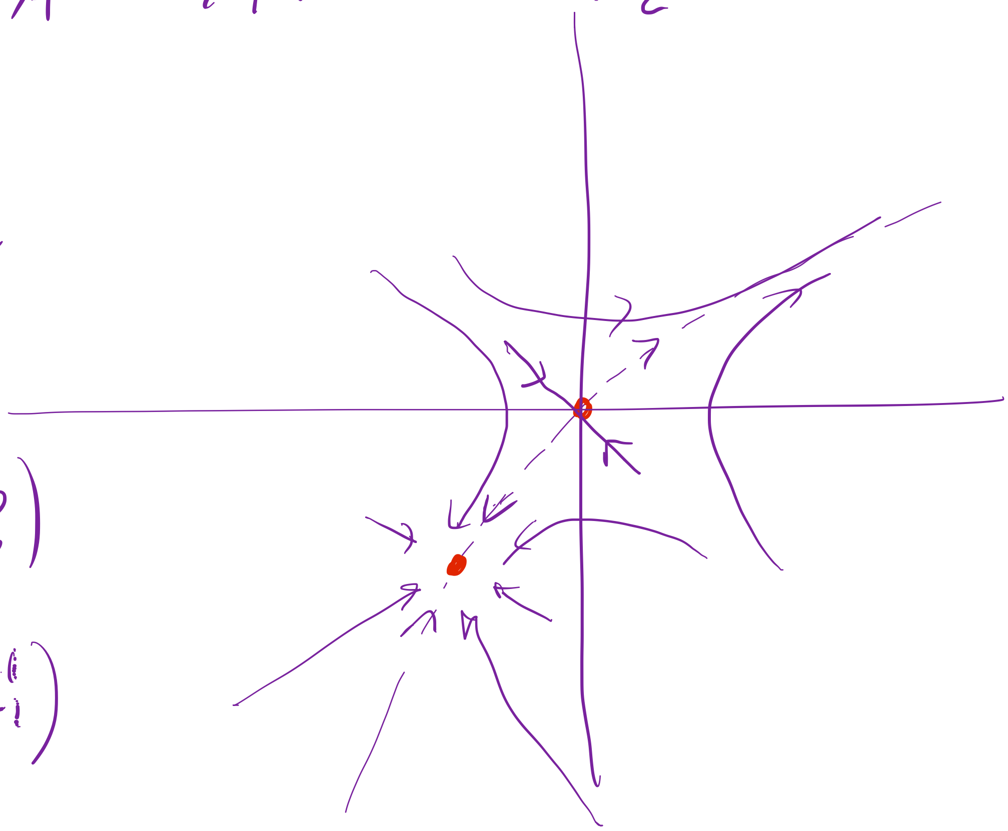
as $a \uparrow 1/4$ $\quad \lambda_1 \to -2$ and $\lambda_2 \to 0$

$\vec{V}_1 \to \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and $\vec{V}_2 \to \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

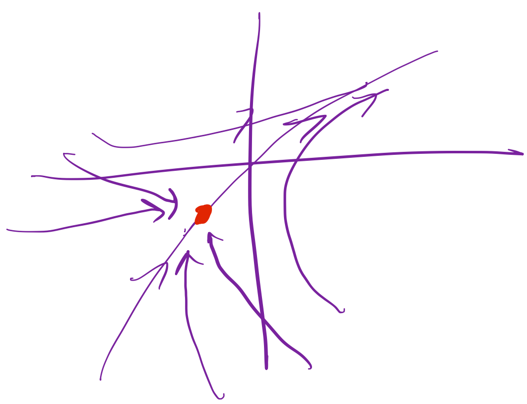
for $a < 1/4$ $\quad \lambda_1 < 0$ and $\lambda_2 < 0$.

sample phase portrait $a = 0$

$X_{eq} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$



$a = 1/4$

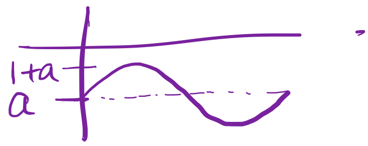


Ch 8, #9

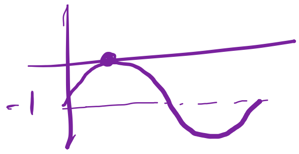
$$\begin{cases} r' = r - r^2 = r(1-r) \\ \theta' = \sin\theta + a \end{cases}$$

a) For what values of a does the system undergo a bifurcation?

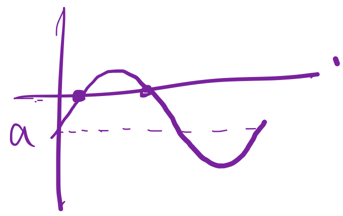
$\theta' < 0 \quad \forall \theta$ if $a < -1$



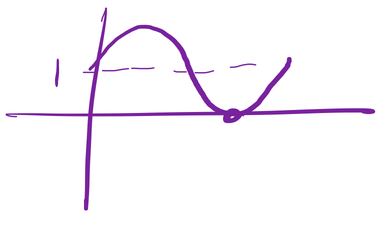
$\theta = 0$ at $\theta = \frac{\pi}{2}$ if $a = -1$



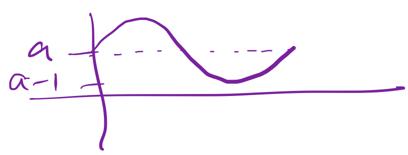
$\theta = 0$ at 2 values of θ if $-1 < a < 1$



$\theta = 0$ at $\theta = \frac{3\pi}{2}$ if $a = 1$



$\theta' > 0 \quad \forall \theta$ if $a > 1$



Bifurcations occur at $a = -1$ and $a = +1$

b) describe the local behaviour of solutions near the bifurcation values at, before, and after the bifurcation.

$$\begin{cases} r' = r(1-r) \\ \theta' = \sin(\theta) + a \end{cases}$$

equilibrium solutions have $r=0$ or $r=1$.

$r' = f(r)$ linearizes to

$$r' = f'(r_{eq})(r - r_{eq})$$

$$f(r) = r - r^2 \Rightarrow f'(r) = 1 - 2r$$

$r_{eq} = 0$ has $f'(r_{eq}) = 1$

linearized system is $r' = r$

$\Rightarrow r_{eq} = 0$ is <u>unstable</u>

$r_{eq} = 1$ has $f'(r_{eq}) = -1$

linearized system is $r' = -(r-1)$

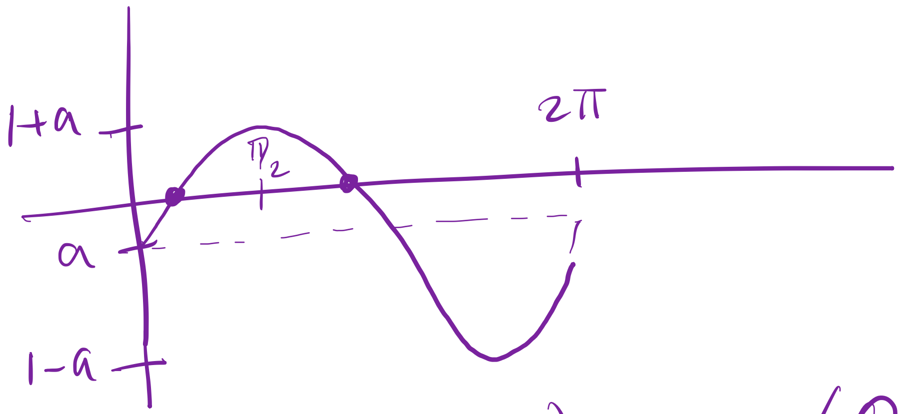
$\Rightarrow r_{eq} = 1$ is asymptotically stable in the r direction.

$\theta' = \sin(\theta) + a = f(\theta)$

this linearizes to

$$\theta' = f'(\theta_{eq})(\theta - \theta_{eq})$$

$$\theta' = \cos(\theta_{eq})(\theta - \theta_{eq}).$$



θ_{eq}^{-} is in $(0, \pi/2) \Rightarrow \cos(\theta_{eq}^{-}) > 0$

θ_{eq}^{+} is in $(\pi/2, \pi) \Rightarrow \cos(\theta_{eq}^{+}) < 0$

$\therefore$ θ_{eq}^{-} is unstable in the θ direction because the linearized θ equation is $\theta' = \underbrace{\cos(\theta_{eq}^{-})}_{\text{negative}}(\theta - \theta_{eq}^{-})$

and by the same logic, θ_{eq}^{+} is stable in the θ direction

So if $-1 < a < 1$ there are 3 equilibrium solutions

$(0, *)$ unstable (* could be any angle bec. $r = 0$)

$(1, \theta_{eq}^{-})$ saddle

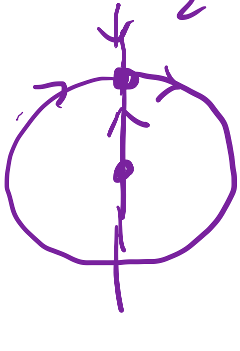
$(1, \theta_{eq}^{+})$ sink

If $a = -1$

then $\theta' = \sin(\theta) - 1$ has one equilibrium $\theta = \pi/2$ and the ODE is approximated by

$$\theta' = -\frac{1}{2}(\theta - \pi/2)^2$$

(Taylor poly centred at $\theta = \pi/2$)

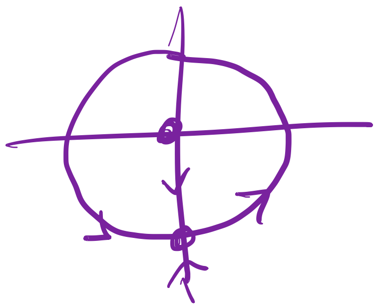


both equilibrium solutions $(0, *)$ and $(1, \pi/2)$ are unstable

If $a=1$

then $\theta' = \sin(\theta)+1$ has one equilibrium $\theta_{eq} = 3\pi/2$ and is approximated by

$$\theta' = \tfrac{1}{2}(\theta - \pi/2)^2$$

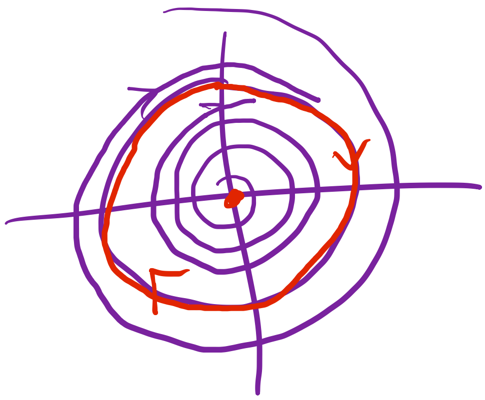


both equil. solutions $(0,*)$ and $(1, 3\pi/2)$ are unstable

c) Sketch the phase portrait of the system for all possible different cases.

$a<-1$

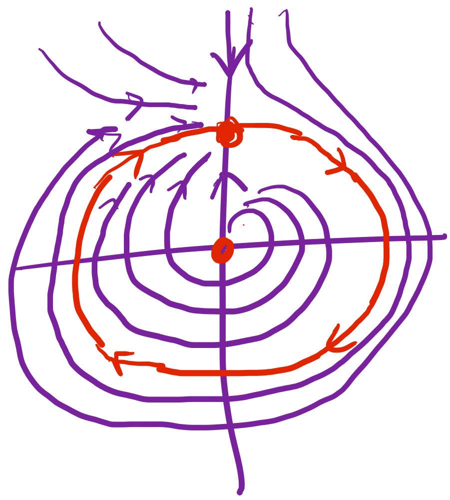
$\theta'<0$ for all θ



$r=1$ is an attracting limit cycle, all solutions except $(0,t)$ converge to this

periodic solution as $t \to \infty$.
motion is clockwise

$a = -1$



$\theta' < 0$ except at $\theta = \pi/2$

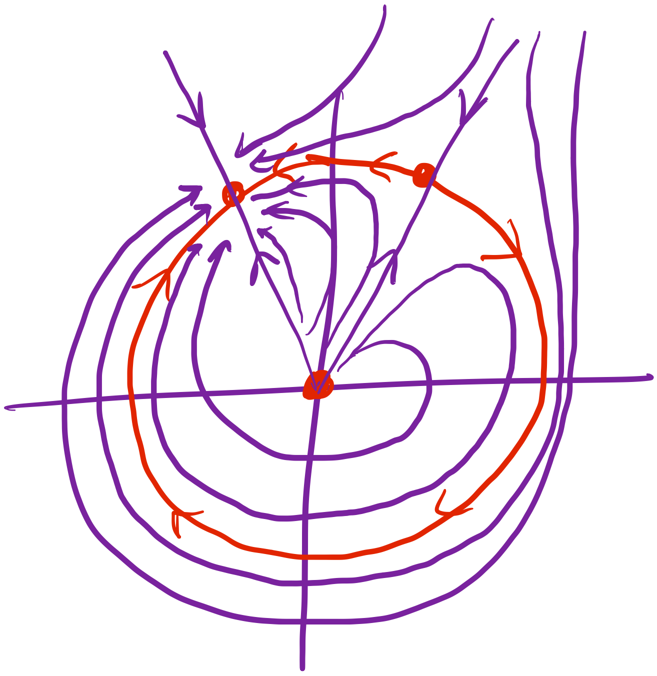
$$\begin{cases} x' = x(1-r) - y\left(\frac{y}{r} - a\right) \\ y' = y(1-r) + x\left(\frac{y}{r} - a\right) \end{cases}$$

use this for the phase portrait plotter if you want...

As a increases through $a = -1$ the equilibrium solution comes into existence (at $a = -1$) and then splits into two equilibrium solutions. One is stable (in the θ direction) and the other is unstable (in the θ direction. The ODE for θ goes through a saddle node bifurcation.

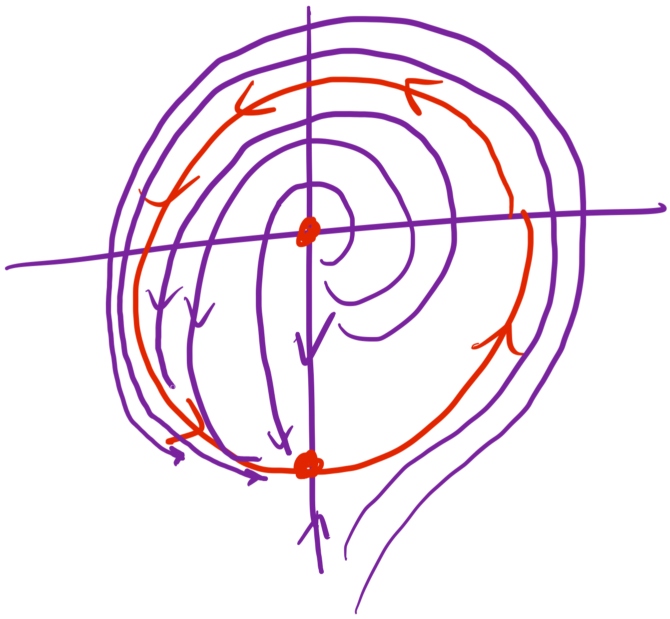
In 2d, this means the one equilibrium $(1, \pi/2)$ splits into two equil. solutions, one is a sink and the other is a saddle (because $r=1$ is stable)

$-1 < a < 1$



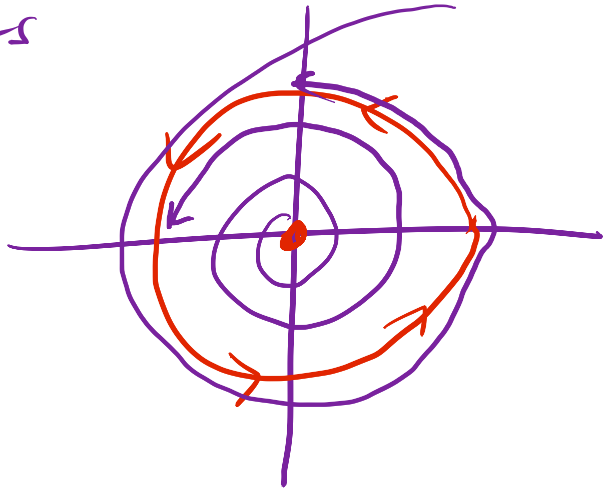
$a = 1$

the two equilibrium solutions on the unit circle have coalesced into one at $\theta = \frac{3\pi}{2}$



$a > 1$

There's another saddle node bifurcation in the θ equation and the equilibrium solution disappears



Back to having an attractive limit cycle, but it's counterclockwise now.