MAT267: HW5

Please do these problems and submit them by 11:59pm on Sunday (March 29).

This document was updated March 26 to clarify problem 1 and updated on March 27 to move two problems onto HW6 and to change the deadline to March 29. It was updated on March 29 to clarify the connection to the theorem on page 147.

1. Consider the two initial value problems

$$\begin{cases} x' = a(t)\,x \\ x(0) = x_0 \end{cases} \qquad \begin{cases} y' = b(t)\,y \\ y(0) = x_0 \end{cases}$$

   where $a, b \in C([0, t_1])$ for some $t_1 > 0$. Find the solutions $x(t)$ and $y(t)$ and prove that $\|x - y\|_\infty \leq M\|a - b\|_\infty$ for some finite value M. Conclude that solutions of the IVP $x' = a(t)x$ depends continuously on the coefficient in the function space $C([0, t_1])$. *That is, show that it's continuous at a; given $\epsilon > 0$, you're trying to find $\delta > 0$ so that $\|a - b\|_\infty < \delta$ implies $\|x - y\| < \epsilon$. It's fine for δ to depend on a.*

2. The theorem on page 147 of the book (Section 7.3) has to do with a solution $X(t)$ that's defined on a closed interval $[t_0, t_1]$. I.e. the solution's maximal interval of existence is (α, β) with $[t_0, t_1] \subset (\alpha, \beta)$.

   The theorem on page 394 of the book (Section 17.3) has to do with a *pair* of solutions Y and Z that are both defined on a closed interval $[t_0, t_1]$ where the closed interval is a subset of the respective maximal intervals of existence (α_Y, β_Y) and (α_Z, β_Z). This theorem gives you control of $|Y(t) - Z(t)|$ on $[t_0, t_1]$.

   Your job: After replacing the "F is C^1 in $\mathbb{R}^n$" condition in the theorem on page 147 with "F is locally Lipschitz in $\mathbb{R}^n$", prove the theorem. *All you need to do is prove that there is a ball $B_\rho(X_0)$ so that if $Y_0 \in B_\rho(X_0)$ then the initial value problem $Y' = F(Y)$ with $Y(t_0) = Y_0$ has a solution on $[t_0, t_1]$. After that, you should be able to get everything you need using the theorem on page 394 in the book.*

3. Solve the IVP $x' = x + \tau\, x^2$ with $x(0) = 1$. Call the solution $x_\tau(t)$ and its maximal interval of existence $(\alpha_\tau, \beta_\tau)$. What is the solution's maximal interval of existence if $\tau > 0$? If $-1 < \tau < 0$? If $\tau < -1$? If $\tau = 0$? If $\tau = -1$? Sketch five representative solutions ($\tau = -2, -1, -1/2, 0, 1$, for example).

   If you do a Taylor series expansion of $x_\tau(t)$ centred at $\tau = 0$, what are the first two terms? What are the first two terms if you do an expansion centred at $\tau = -1$? What are the first two terms if you do an expansion centred at $\tau = \tau_0$?

   Using the explicit solution, prove that $x_\tau(t)$ is continuous with respect to τ at $\tau = 1$ on $[0, 0.69] \subset (\alpha_1, \beta_1)$. Before even getting started, you need to take τ close enough to 1 so that that $[0, 0.69] \subset (\alpha_\tau, \beta_\tau)$. What constraint does this put on τ?

4. Chapter 8, problem 1 (on page 184). For the phase portraits, feel free to use `http://www.bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html` although you should first plot the equilibrium points and the behaviour of the linearized systems at the equilibrium points (as long as the eigenvalues have nonzero real parts).
5. **Moved to HW6!** Chapter 8, problem 3 (on page 184).
6. Chapter 8, problem 5 (on page 185).
7. **Moved to HW6!** Chapter 8, problem 6 (on page 185).
8. Chapter 8, problem 9 (on page 185).