Sketches of HW Solutions

Homework 3 solutions

1. (a) Row reduction of $\left(A - \lambda_+ I \,\middle|\, V_+\right)$ is $\left(\begin{array}{cccc|c} 1 & & & -1 & -i \\ & 1 & & i & 1 \\ & & 1 & & 1 \end{array}\right)$, so we want $U_+ = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

that $a - d = -i$, $b + id = 1$, $c = 1$ so $U_+ = \begin{pmatrix} -i \\ 1 \\ 1 \\ 0 \end{pmatrix}$ works.

(b) We had $(A - \lambda_+ I)\, U_+ = V_+ \Rightarrow \overline{(A - \lambda_+ I)}\, \overline{U_+} = \overline{V_+} \Rightarrow (A - \lambda_- I)\, \overline{U_+} = V_- \Rightarrow U_- := \overline{U_+}$.

(c) We get $P = \left(V_+ \,\middle|\, U_+ \,\middle|\, V_- \,\middle|\, U_- \right)$.

(d) $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$AV_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} = 2V_1 - V_2 = (\mathrm{Re}\,\lambda_+) V_1 - (\mathrm{Im}\,\lambda_+) V_2$ $AV_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} = V_1 + 2V_2 = (\mathrm{Im}\,\lambda_+) V_1 + (\mathrm{Re}\,\lambda_+) V_2$ $\mathrm{Re}\,\lambda_+ = 2$, $\mathrm{Im}\,\lambda_+ = 1$.

(e) $V_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $V_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$AV_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix} = V_1 + 2V_3 - V_4 = V_1 + (\mathrm{Re}\,\lambda_+) V_3 - (\mathrm{Im}\,\lambda_+) V_4$ $AV_4 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = V_2 + V_3 + 2V_4 = V_2 + (\mathrm{Im}\,\lambda_+) V_3 + (\mathrm{Re}\,V_3) V_4$

(f) $P = \left(V_1 \,|\, V_2 \,|\, V_3 \,|\, V_4 \right)$.

(g) Do the same as above to check it.

2- Call $A = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Now we see that $v = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors for A.

Now let's find U_v with $(A-2I)U_v = v \xrightarrow{\text{by solving}} U_v$ can be chosen to be $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Now let's find U_{U_v} with $(A-2I)U_{U_v} = U_v \xrightarrow{\text{by solving}} U_{U_v}$ can be chosen to be $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

So we can choose $P = \left(v \middle| U_v \middle| U_{U_v} \middle| w \right)$ to get

$$P^{-1} A P = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{2} \end{pmatrix}.$$

Now let $X = A(t)\,X_0$ then $A'(t)\,X_0 = A \cdot A(t)\,X_0$, this is true for all initial X_0, so

$$A'(t) = A\,A(t) \Rightarrow \underbrace{\left(P^{-1}A(t)P\right)'}_{B(t)'} = \left(P^{-1}AP\right)\underbrace{\left(P^{-1}A(t)P\right)}_{B(t)} \Rightarrow B(t)' = \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{pmatrix} B(t), \quad B(0) = I$$

First let's solve this generally for a vector $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$:

$f_4 = Ce^{2t}$ $\quad f_3 = De^{2t}$ $\quad f_2' = 2f_2 + f_3 \Rightarrow \left(f_2 e^{-2t}\right)' = e^{-2t} f_3 = D \Rightarrow f_2 = (Dt + E)\,e^{2t}$.

$f_1' = 2f_1 + f_2 \Rightarrow \left(f_1 e^{-2t}\right)' = e^{-2t} f_2 = Dt + E \Rightarrow f_1 = \left(\frac{Dt^2}{2} + Et + F\right)e^{2t}$.

Now we put the initial $B(0) = I$ (because we needed $X(0) = A(0)\,X_0$ for all X_0) to get:

$$B(t) = \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} & \\ & e^{2t} & te^{2t} & \\ & & e^{2t} & \\ & & & e^{2t} \end{pmatrix}$$

(It could be also taken by $\exp(Jt)$ where J is the Jordan form).

so $A(t) = P\,B(t)\,P^{-1} = \ldots$ you can compute it!

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3. We know that flow of $X' = AX$ is $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto e^{At}x$ and χ_t for B, then we have

$\forall x \quad \varphi_t(Px) = P(\chi_t x) \iff \forall x \quad e^{At}Px = Pe^{Bt}x \iff P^{-1}e^{At}P = e^{Bt}$ true by assumption

$\left(P^{-1}e^{At}P = P^{-1}\left(\sum \frac{(At)^n}{n!}\right)P = \sum \frac{(P^{-1}AP)^n t^n}{n!} = \sum \frac{B^n t^n}{n!} = e^{Bt} \right).$

We seek a homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\phi_t^B(H(X)) = H(\phi_t^A(X)).$$

Let $H(X) := P^{-1}X$. Then $\phi_t^B(H(X)) = \phi_t^B(P^{-1}X) = e^{Bt}P^{-1}X$

and $H(\phi_t^A(X)) = P^{-1}\phi_t^A(X) = P^{-1}e^{tA}X$

Because $e^{At} = Pe^{Bt}P^{-1} \Rightarrow P^{-1}e^{At} = e^{Bt}P^{-1}$ and so

$$\phi_t^B(H(X)) = H(\phi_t^B(X)).$$

To show it's a homeomorphism, we need to prove that $H : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and invertible and $H^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

$$\|H(X) - H(Y)\| = \|P^{-1}X - P^{-1}Y\| = \|P^{-1}(X - Y)\|$$

Can we make this small by taking $\|X - Y\|$ small?

Is matrix multiplication continuous with respect to the Euclidean norm?

We have the matrix norm $\|C\|_\infty = \max_{i,j} |C_{ij}|$

$$(CX)_i = \sum_{j=1}^n C_{ij}X_j \Rightarrow |(CX)_i| = \left|\sum_{j=1}^n C_{ij}X_j\right| \le \sum_{j=1}^n |C_{ij}||X_j|$$

$$\le \max_{1 \le j \le n} |C_{ij}| \sum_{j=1}^n |X_j|$$

If we define the L^1 norm on $\mathbb{R}^n$ by

$$\|X\|_1 = \sum_{i=1}^n |X_i| \text{ then } \|CX\|_1 \le \|C\|_\infty \|X\|_1$$

This proves that matrix multiplication is continuous on $\mathbb{R}^n$ using the L^1 norm topology: Given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{\|C\|_\infty}$. Then

if $\|X-Y\|_1 < \delta$ we have

$$\|CX - CY\|_1 = \|C(X-Y)\|_1 \leq \|C\|_\infty \|X-Y\|_1$$
$$< \|C\|_\infty \frac{\varepsilon}{\|C\|_\infty} = \varepsilon$$

If we want continuity with respect to the L^2 (or "Euclidean") norm we use the inequalities

- $X \in \mathbb{R}^n \Rightarrow X = \sum_{i=1}^n x_i E_i$

$$\|X\|_1 \leq \sum_{i=1}^n |x_i| \, \|E_i\|_1 \leq \sqrt{\sum_{i=1}^n |x_i|^2} \sqrt{\sum_{i=1}^n \|E_i\|_1^2}$$
$$= \|X\|_2 \sqrt{n}$$

- $$\|X\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2} \leq \sqrt{\sum_{i=1}^n \|X\|_1^2} = \sqrt{n \|X\|_1^2}$$
$$= \sqrt{n} \, \|X\|_1$$

So

$$\frac{1}{\sqrt{n}} \|X\|_1 \leq \|X\|_2 \leq \sqrt{n} \, \|X\|_1 \quad \forall X \in \mathbb{R}^n$$

Given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{n\|C\|_\infty}$

If $\|X-Y\|_2 < \delta$ then $\frac{1}{\sqrt{n}}\|X-Y\|_1 \leq \|X-Y\|_2$

$$\Rightarrow \|X-Y\|_1 \leq \sqrt{n}\|X-Y\|_2 < \sqrt{n}\,\delta$$

$$\Rightarrow \|CX-CY\|_1 \leq \|C\|_\infty \|X-Y\|_1 < \|C\|_\infty \sqrt{n}\,\delta$$

On the other hand $\|Z\|_2 \leq \sqrt{n}\|Z\|_1$

$$\Rightarrow \|CX-CY\|_2 \leq \sqrt{n}\|CX-CY\|_1 \leq n\|C\|_\infty \delta < \varepsilon$$

because $\delta = \dfrac{\varepsilon}{n\|C\|_\infty}$

So $\|X-Y\|_2 < \delta \Rightarrow \|CX-CY\|_2 < \varepsilon$ as desired.

$H(x) = P^{-1}X$ is continuous on $\mathbb{R}^n$ wrt the L^2 norm. And $H^{-1}(x) = PX$ is as well.

4. A can be put in Jordan normal form and the number of blocks is equal to number of indep. eigenvectors, so the result follows.

(The TA's referring you to your MAT247 book... :-))

5. $\boxed{14}$ (a) open and dense : It is an open condition obviously, density follows from the fact that if A is $\det A = 0$, then you can find small ε such that $\det(A - \varepsilon I) \neq 0$ as A can't have so many eigenvalues (i.e. it has finitely many).

(b) not open and dense : $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is dense so take rational diagonal and arbitrary other components, not open as irrationals $\subseteq \mathbb{R}$ are dense.

(c) open and dense : $\mathbb{Z} \subseteq \mathbb{R}$ is closed, so $\mathbb{Z} \times \mathbb{R}^{n^2-1} \subseteq \mathbb{R}^{n^2}$ is closed and union of finitely many (n^2-many) closed is closed. Density follows from the fact that near integers there are nonintegers.

(d) not open and not dense : It has boundary $\det A = 3$ (which is nonempty) and not dense as det is a continuous function and we have matrices of $\det = 1$. (or any other number not in $[3, 4]$).

(e) open and not dense : $L(\mathbb{R}^n) \longrightarrow \mathbb{C}^n / S_n$ (symmetric matrix acting on components) is continuous as coefficients

$$A \longmapsto \underbrace{(\lambda_1, \lambda_2, \dots, \lambda_n)}_{\text{Eigenvalues of } A}$$

of char. pol is given in terms of components of A (continuously). so it is open. Not dense by the same fact above and that we have matrice with some eigenvalue with norm not in $[-1, 1]$.

(f) open and $\begin{cases} \text{not dense} & n \text{ odd} \\ \text{not dense} & n \text{ even} \end{cases}$: If n is odd, as every polynomial with odd degree has a real root, it is empty. If n is even, then it is open because if we perturb coefficients of a polynomial with no real root, then again it doesn't have real roots (if the perturbation is small) and not dense by perturbing coefficients of a polynomial p (monic and even degree) that has a solution of $p' = 0$ for some x_0 with $p(x_0) < 0$.

(g) open and dense : condition of a polynomial p having a repeated root is that $\gcd(p, p') \neq 1$ and can be given by vanishing of a polynomial condition in terms of the coefficients (cf. resultant), so it is open (actually it is the complement of an algebraic set). Density follows from the same fact above.

$\boxed{15}$ (a) ✓ by the same reasoning as (e) before.

(b) ✗ by (f) before.

(c) ✗ This is determined by sign of discriminant of the cubic characteristic polynomial.

(d) ✓ If $e^{At_0} = e^{\cancel{A}0} = I$ for some $t_0 \neq 0$ but this means that all eigenvalues of A are zero.

(e) ✓ Condition of having same imaginary parts is at least inside polynomial condition (consider $x = a + ib$ in a characteristic polynomial and find two polynomials in a, b then fix b and write the resultant in terms of a for one of these polynomials, vanishing of this resultant is a necessary condition and sending a matrix to n-tuple (up to permutation) of real parts of eigenvalues is continuous).

(f) ✓ It is $\{\det(A - I) \neq 0\} \cap \{\det(A + I) \neq 0\}$.

6. (a) For the matrix $A = \begin{pmatrix} & & a \\ & b & \\ a & & \end{pmatrix}$ we have

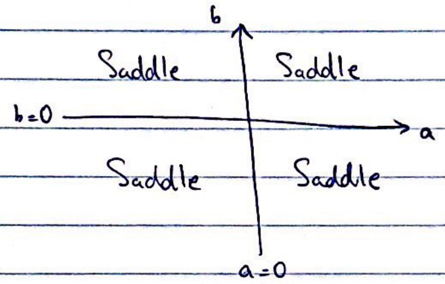
$$A\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}a\\0\\a\end{pmatrix} \qquad A\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-a\\0\\a\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\b\\0\end{pmatrix} \text{ so if } P = \begin{pmatrix}1&0&1\\0&1&0\\1&0&-1\end{pmatrix}$$

then $B := P^{-1}AP = \begin{pmatrix} a & & \\ & b & \\ & & -a\end{pmatrix}$ so we have $e^B = \begin{pmatrix} e^a & & \\ & e^b & \\ & & e^{-a}\end{pmatrix}$

so we have $A = PBP^{-1}$ where $P^{-1} = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-1}{2}\end{pmatrix}$ so

$$e^A = Pe^BP^{-1} = \begin{pmatrix} \frac{e^a+e^{-a}}{2} & 0 & \frac{e^a-e^{-a}}{2} \\ 0 & e^b & 0 \\ \frac{e^a-e^{-a}}{2} & 0 & \frac{e^a+e^{-a}}{2}\end{pmatrix}.$$

(b) Eigenvalues are $\pm a, b$, so



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7. $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad X'' = \overbrace{\begin{pmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_1+k_2) \end{pmatrix}}^{A} X$

Define $Y = X'$, then the first-order linear system is

$$\begin{cases} X' = 0X + 1Y \\ Y' = AX + 0Y \end{cases}$$

so let $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$, then $Z' = \overbrace{\begin{pmatrix} & I \\ A & \end{pmatrix}}^{B} Z$

Now $\det\left(\begin{pmatrix} & I \\ A & \end{pmatrix} - \lambda I_4\right) = \det\begin{pmatrix} -\lambda I & I \\ A & -\lambda I \end{pmatrix} = \det(\lambda^2 I - A)$

Eigenvalues of A are $-k_1$, $-(k_1+2k_2)$ so λ is the squares of these.

Eigenvectors are

$a = -(k_1+k_2)$ $\quad v_1 = \begin{pmatrix} \frac{1}{\sqrt{a-b}} \\ \frac{-1}{\sqrt{a-b}} \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{-1}{\sqrt{a-b}} \\ \frac{1}{\sqrt{a-b}} \\ -1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} \frac{-1}{\sqrt{a+b}} \\ \frac{-1}{\sqrt{a+b}} \\ 1 \\ 1 \end{pmatrix}$, $v_4 = \begin{pmatrix} \frac{1}{\sqrt{a+b}} \\ \frac{1}{\sqrt{a+b}} \\ 1 \\ 1 \end{pmatrix}$.

$b = k_2$

eigenvectors: $-\sqrt{a-b}$, $\sqrt{a-b}$, $-\sqrt{a+b}$, $\sqrt{a+b}$

Let $P = (v_1 | v_2 | v_3 | v_4)$ then $P^{-1} B P = \mathrm{diag}(-\sqrt{a-b}, \sqrt{a-b}, -\sqrt{a+b}, \sqrt{a+b})$.

so the general solution for Z is $\left(P \begin{pmatrix} e^{-\sqrt{a-b}\,t} & & & \\ & e^{\sqrt{a-b}\,t} & & \\ & & e^{-\sqrt{a+b}\,t} & \\ & & & e^{\sqrt{a+b}\,t} \end{pmatrix} P^{-1}\right) z_0$ and you can easily find solutions of X.

~~X is periodic iff~~ $\cancel{Z = \begin{pmatrix} X \\ X' \end{pmatrix}}$ ~~is and Z is iff~~ ~~is.~~

~~we have~~ We want z_0 to be an eigenvector with eigenvalue 1 for some t. We have $i\omega_1 = \sqrt{a+b}$ and $\sqrt{a-b} = i\omega_2$,

$\omega_2 = \sqrt{b-a} = \sqrt{k_2 + k_1 + k_2} = \sqrt{k_1 + 2k_2}$

$\omega_1 = \sqrt{-b-a} = \sqrt{k_1 + k_2 - k_2} = \sqrt{k_1}$

Need $\frac{\omega_2}{\omega_1}$ rational to get periodic solution

$\sqrt{1 + \frac{2k_2}{k_1}}$ rational $\Rightarrow$ periodic solution

8- (a) $\lambda^2 - \lambda + (-2) = 0 \Rightarrow \lambda = 2, -1$ $\quad P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} 2 & \\ & -1 \end{pmatrix} \Rightarrow P^{-1}e^A P = \begin{pmatrix} e^2 & \\ & e^{-1} \end{pmatrix} \Rightarrow e^A = P \begin{pmatrix} e^2 & \\ & e^{-1} \end{pmatrix} P^{-1} = \dots$

(e) This has $A^3 = 0$, so $\exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} & 1 & 2 \\ & & 3 \\ & & \end{pmatrix} + \begin{pmatrix} & & \frac{3}{2} \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3.5 \\ & 1 & 3 \\ & & 1 \end{pmatrix}$.

(f) $\exp \begin{pmatrix} 2 & & \\ & 3 & \\ & 1 & 3 \end{pmatrix} = \begin{pmatrix} e^2 & \\ & \exp\begin{pmatrix} 3 & \\ 1 & 3 \end{pmatrix} \end{pmatrix}$

$\exp \begin{pmatrix} 3t & \\ t & 3t \end{pmatrix} = \exp \begin{pmatrix} 3t & \\ & 3t \end{pmatrix} \exp \begin{pmatrix} & \\ t & \end{pmatrix} = \begin{pmatrix} e^{3t} & \\ & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & \\ te^{3t} & e^{3t} \end{pmatrix}$ put $t=1$ to get $\exp A = \begin{pmatrix} e^2 & & \\ & e^{3t} & \\ & te^{3t} & e^{3t} \end{pmatrix}$.

(j) $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{pmatrix} + \begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{pmatrix} / 2 + \begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{pmatrix} / 3! + \dots = \begin{pmatrix} e & & & \\ e-1 & 1 & & \\ e-1 & & 1 & \\ e-1 & & & 1 \end{pmatrix}$.

9- $A = \begin{pmatrix} & 1 \\ & \end{pmatrix}$ $\quad B = \begin{pmatrix} & \\ 1 & \end{pmatrix}$ then $\exp A = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $\quad \exp B = \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$

$\exp(A+B) = \exp \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} + \frac{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}{2} + \frac{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}{3!} + \frac{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}{4!} + \dots$

$= I\left(1 + \frac{1}{2!} + \frac{1}{4!} + \dots\right) + \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\left(1 + \frac{1}{3!} + \frac{1}{5!} + \dots\right)$

but $\exp A \exp B = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

10- (a) $\exp A \exp B = \sum A^n/n! \sum B^n/n! = \sum \frac{A^m B^r}{m!\,r!} = \sum \frac{B^r A^m}{r!\,m!} = \sum B^n/n! \sum A^n/n! = \exp B \exp A$.

(b) $(\exp A) B = \sum A^n/n! \; B = \sum \frac{A^n B}{n!} = \sum \frac{B A^n}{n!} = B \sum A^n/n! = B \exp A$.

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