1) Sketch of solutions to HW2.

1. a) $y'(t) = (x(t-t_0))' = x'(t-t_0)\cdot 1 = x'(t-t_0) = f(x(t-t_0)) = f(y(t))$.

Interval is $(a+t_0, b+t_0)$.

b) $y'(t) = (x(-t))' = -x'(-t) = -f(x(-t)) = -f(y(t))$.

Interval is $(-b, -a)$.

c) $y'(t) = (x(\sigma t))' = \sigma x'(\sigma t) = \sigma f(x(\sigma t)) = \sigma f(y(t))$.

Interval is $\left(\frac{a}{\sigma}, \frac{b}{\sigma}\right)$.

2) We have $\cos(\beta t - \varphi) = \cos(\beta t)\overset{x_1/R}{\boxed{\cos(+\varphi)}} + \sin(\beta t)\overset{x_2/R}{\boxed{\sin(\varphi)}}$ (We assumed $(x_1, x_2) \neq (0,0)$)

$-\sin(\beta t - \varphi) = \cos(\beta t)\underset{x_2/R}{\boxed{(+\sin\varphi)}} + \sin(\beta t)\underset{-x_1/R}{\boxed{(-\cos\varphi)}}$

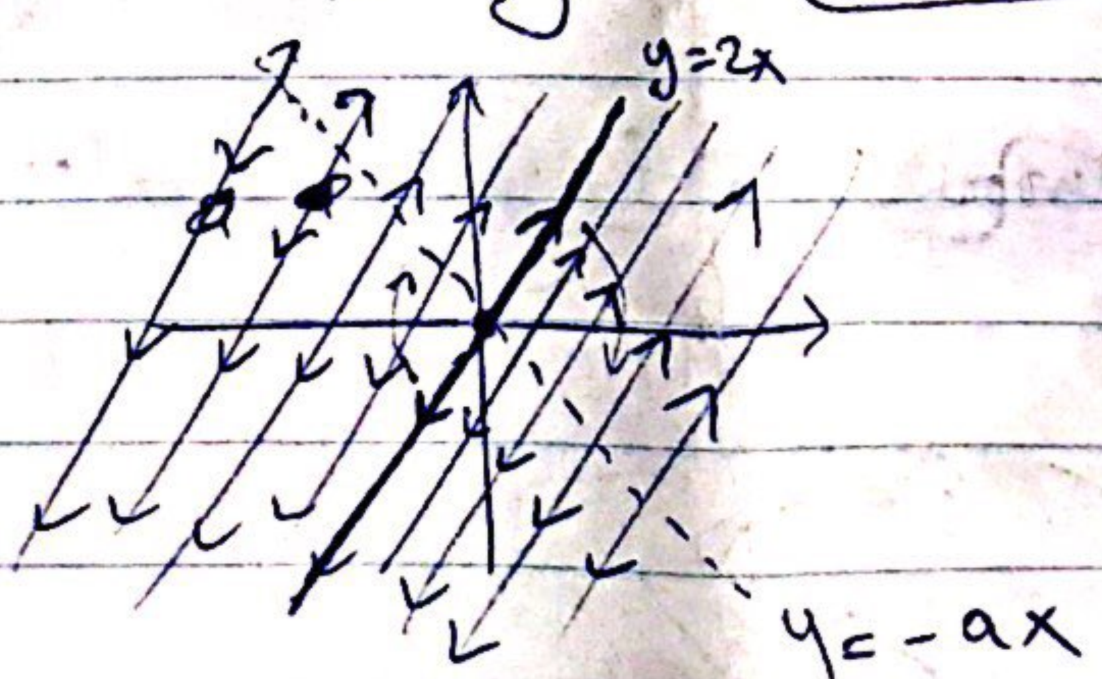
So we have to find a solution to $x_1 = R\cos\varphi$ $x_2 = R\sin\varphi$

which we can do it! Choose $R = \sqrt{x_1^2 + x_2^2}$ and $\varphi = \arccos\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right)$.

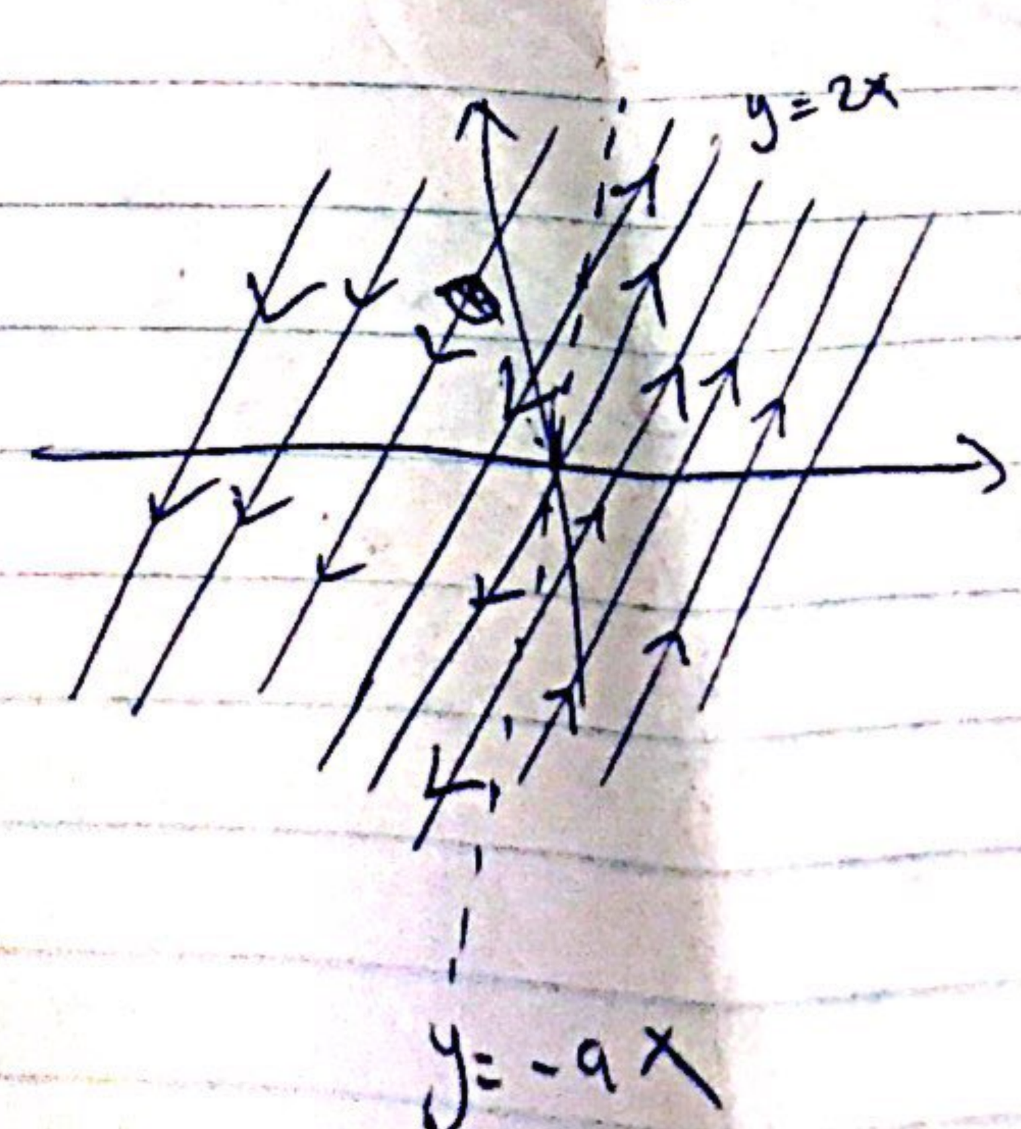
3) $\begin{pmatrix} a & 1 \\ 2a & 2 \end{pmatrix} \Rightarrow$ charpol $= \lambda^2 - (a+2)\lambda + 0 = 0 \Rightarrow \lambda = a+2,\ \lambda = 0$

so eigenvalue sign changes are happening at $\boxed{a = -2}$.

For $a > -2$:



For $a < -2$



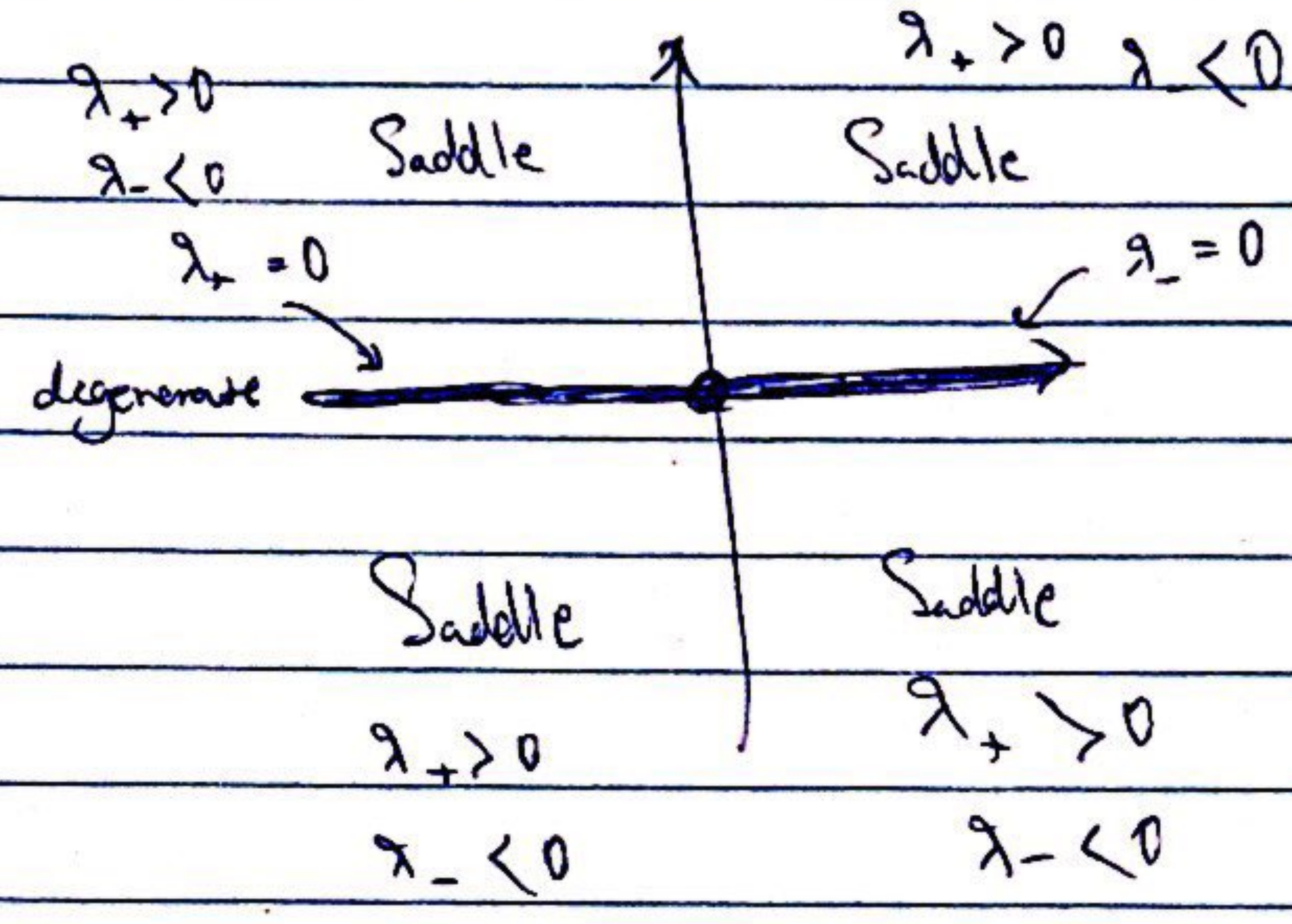
4) Eigenvalues: $\lambda^2 - 2a\lambda + (-b^2) = 0$

$$\lambda_{\pm} = a \pm \sqrt{a^2+b^2}$$

so they are real and $\sqrt{a^2+b^2} \geq |a|$ so

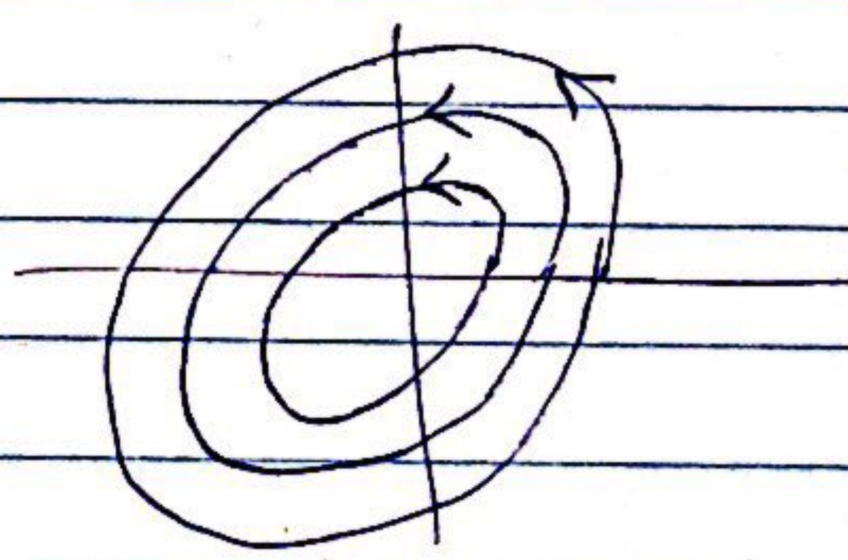
$a + \sqrt{a^2+b^2} \geq 0$ and $\underline{0}$ exactly when $a \leq 0, b = 0$.

$a - \sqrt{a^2+b^2} \leq 0$ and $\underline{0}$ exactly when $a \geq 0, b = 0$.



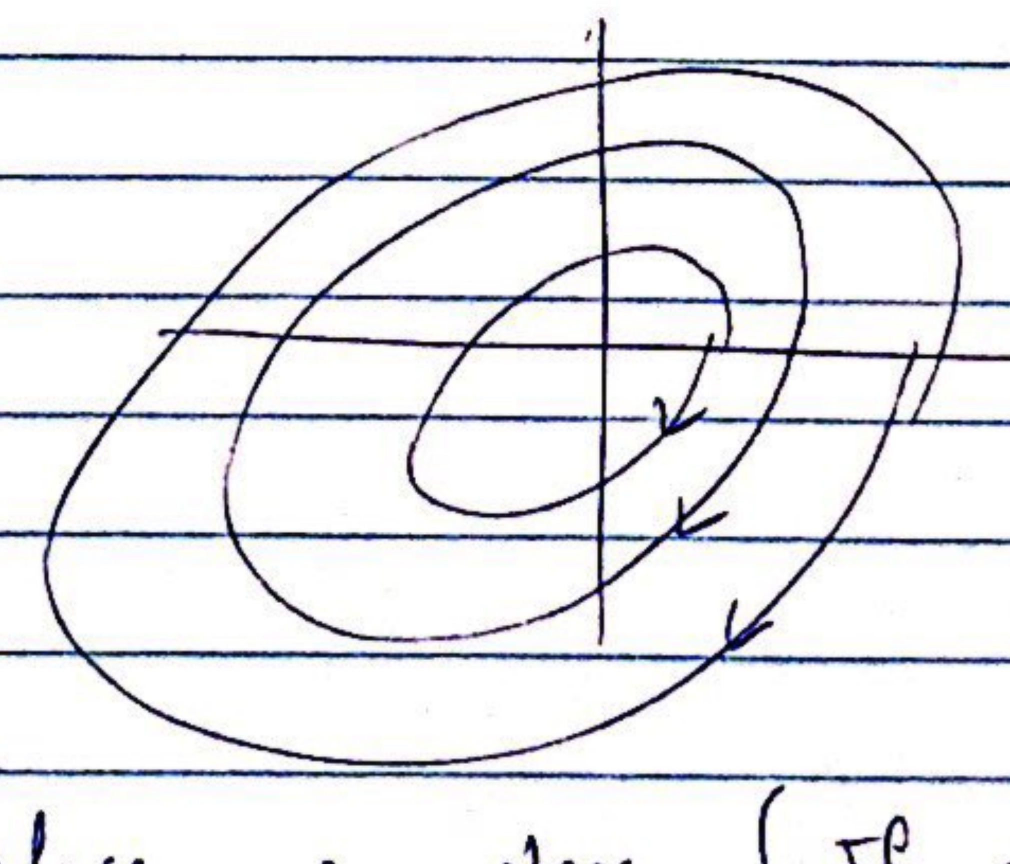
5) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Computing at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have $\begin{pmatrix} a \\ c \end{pmatrix}$ so if $c > 0$

then $\uparrow$ $\Rightarrow$ phase portrait



If $c < 0$ then $\downarrow$ phase portrait

Note: the solution shows a sketch for the case of purely imaginary eigen-values



but the check using the sign of c works for all complex eigen-values with

c can't be zero as eigenvalues are complex (If $c=0$ then $a, d \in \mathbb{R}$ are eigenvalues). nonzero imaginary part.)

6) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix}$

$-(a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix}$

$(ad-bc)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} ad-bc & \\ & ad-bc \end{pmatrix}$

Sum : $\underline{0}$

The sum is $A^2 - (\mathrm{tr}\, A) A + (\det A) I$ which is charpol.

7) Suppose that v is not an eigenvector, then

$(A - \lambda I)v \neq 0$ and

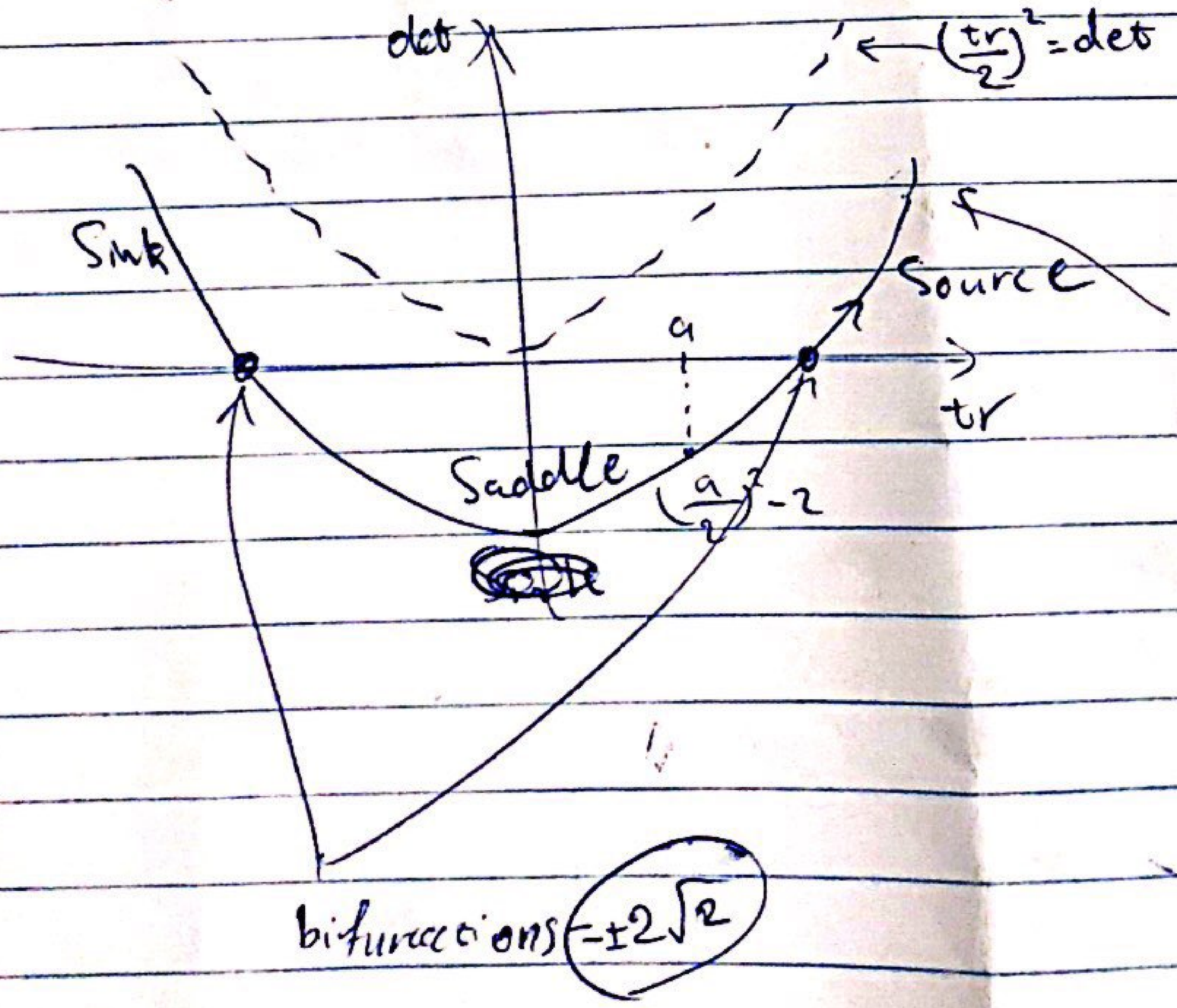
Because A has repeated roots we have $\det A = \lambda^2$, $\mathrm{tr}\, A = 2\lambda$

$A\left(\boxed{(A-\lambda I)v}\right) = (A^2 - \lambda A)v = \Big(\underbrace{(\mathrm{tr}\, A)}_{2\lambda} A - \underbrace{(\det A)}_{\lambda^2} I - \lambda A\Big)v$

$= (2\lambda A - \lambda^2 I - \lambda A)v = (\lambda A - \lambda^2 I)v = \lambda \boxed{(A - \lambda I)v}$.

because trace is sum of eigenvalues and det is product of eigenvalues

8) a)



trace $= a$

det $= -\left(2 - \frac{a^2}{4}\right) = \left(\frac{a}{2}\right)^2 - 2$

$\det = \left(\frac{tr}{2}\right)^2 - 2$

So these two graphs don't intersect.

9) $\det = a^2 - b^2 \quad tr = 2a$

Circle $tr = 0 \quad \det > 0 \iff$ nothing!

Spiral clockwise $tr^2 < 4\det \iff 4a^2 < 4(a^2 - b^2) \iff 4b^2 < 0 \iff$ nothing!
$tr < 0$

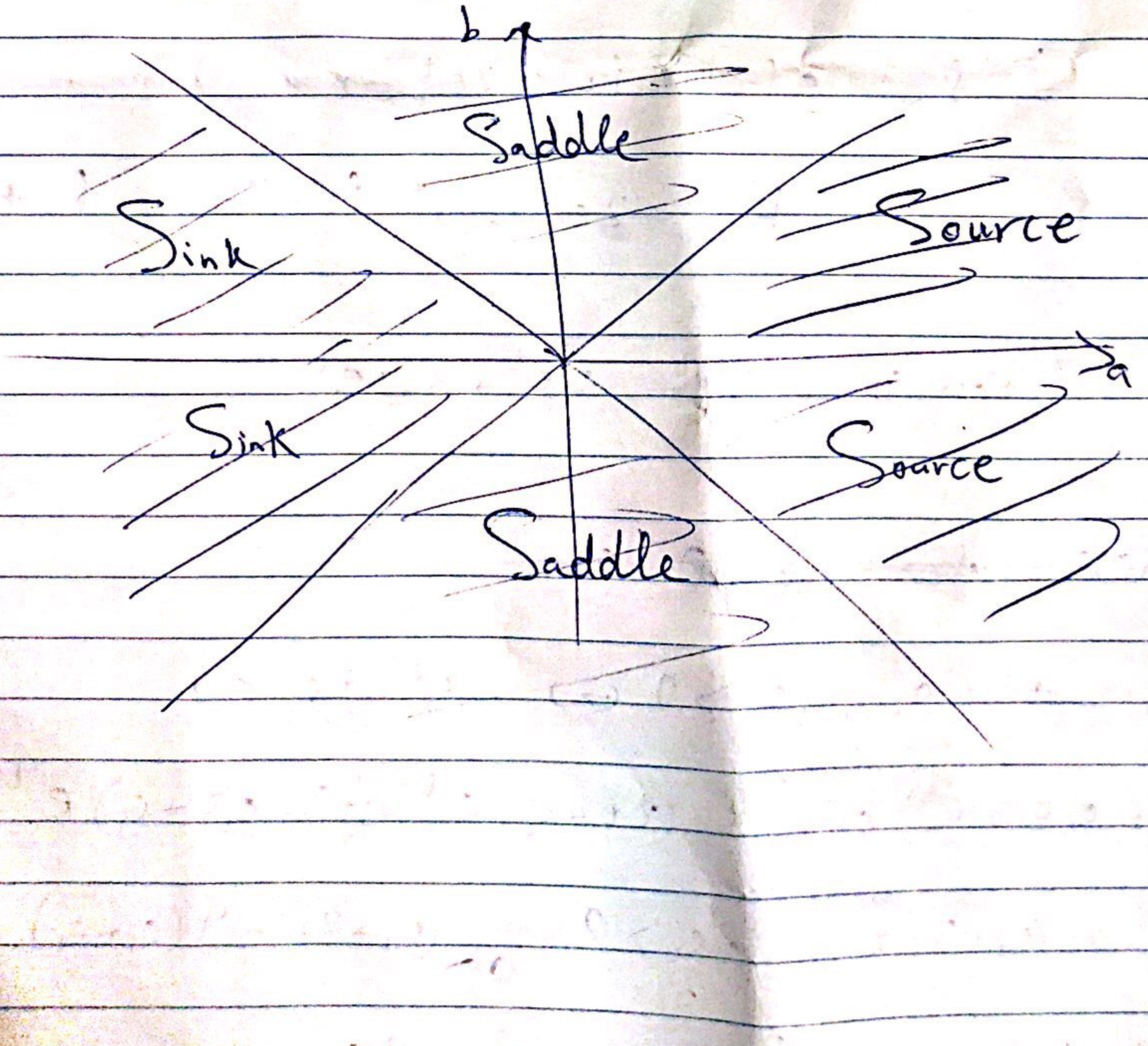
Spiral counterclockwise $tr^2 < 4\det \iff$ nothing.
$tr > 0$

Sink $\left\{ \begin{array}{l} tr^2 > 4\det \iff (2a)^2 > 4(a^2 - b^2) \iff 4b^2 > 0 \iff \boxed{\text{Anything}} \\ tr < 0 \quad \det > 0 \iff a^2 - b^2 > 0 \iff \boxed{|a| > |b|} \qquad tr < 0 \iff \boxed{a < 0} \end{array} \right\}$

$-a =$

Source $\left\{ \begin{array}{l} tr^2 > 4\det \iff \\ tr > 0 \quad \det > 0 \end{array} \right. \quad \boxed{\begin{array}{l} b > 0 \\ a > 0 \end{array} \quad a > b} \Big\}$

Saddle $\det < 0 \iff a^2 < b^2 \iff |a| < |b|$



Ch 4, problem 5.

5a) $X' = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} X \quad Y' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} Y$

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = P \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} P^{-1} = PJP^{-1}$$

where $P = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} = Q \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} Q^{-1} = Q\hat{J}Q^{-1}$$

where $Q = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$ and $\hat{J} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

Seek 3 homeomorphisms

H_1 is the homeomorphism that makes $X' = AX$ and $X' = JX$ conjugate.

H_2 is the homeomorphism that makes $X' = JX$ and $X' = \hat{J}X$ conjugate

H_3 is the homeomorphism that makes $X' = \hat{J}X$ and $X' = BX$ conjugate.

we know $\phi_t^A(X) = P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} X$

$$\phi_t^J(x) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} X$$

$$\phi_t^{\hat{J}}(x) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} X$$

$$\phi_t^B(x) = Q\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} Q^{-1} X.$$

For $H_1 : \mathbb{R}^2 \to \mathbb{R}^2$ we want

$$H_1(\phi_t^J(X)) = \phi_t^A(H_1(X))$$

that is

$$H_1\left(\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} X\right) = P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} H_1(X)$$

take $H_1(X) = PX$ then

$$H_1\left(\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} X\right) = P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} X$$

$$P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} H_1(X) = P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} P X$$

$$= P\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} X$$

So we've found H_1. it's 1:1 and onto, a continuous linear map.

For H_2. We want

$$H_2(\phi_t^{J}(x)) = \phi_t^{J}(H_2(x))$$

$$H_2\left(\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} X\right) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} H_2(X).$$

$$H_2(X) = \begin{pmatrix} h_1(x_1, x_2) \\ h_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(x_1)\,|x_1|^{1/2} \\ \operatorname{sgn}(x_2)\,|x_2|^2 \end{pmatrix}$$

$$H_2\left(\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} X\right) = H_2\begin{pmatrix} e^{-2t}x_1 \\ e^{t}x_2 \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(x_1)e^{-t}|x_1|^{1/2} \\ \operatorname{sgn}(x_2)e^{2t}(x_2)^2 \end{pmatrix}$$

$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} H_2(X) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} \operatorname{sgn}(x_1)|x_1|^{1/2} \\ \operatorname{sgn}(x_2)\,x_2^2 \end{pmatrix}$$ same.

As discussed in class and in the book, $H_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is 1:1, onto, continuous and H_2^{-1} is as well.

For H_3, we want

$$H_3(\phi_t^{B}(x)) = \phi_t^{J}(H_3(x))$$

$$H_3\left(Q\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix}Q^{-1}X\right) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} H_3 X$$

take $H_3(x) = Q^{-1}X$. then

$$H_3\left(Q\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^t \end{pmatrix}Q^{-1}X\right) = Q^{-1}\left(Q\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^t \end{pmatrix}Q^{-1}X\right)$$

$$= \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^t \end{pmatrix}Q^{-1}X$$

$$= \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^t \end{pmatrix}H_3 X \text{ as desired}$$

H_3 is a linear map and Q is invertible so H_3 is a homeomorphism.

Now to construct $H : \mathbb{R}^2 \to \mathbb{R}^2$ so that

$$H(\phi_t^B(x)) = \phi_t^A(H(x))$$

$$H_1(\phi_t^J(x)) = \phi_t^A(H_1(x)) \Rightarrow \phi_t^J(x) = H_1^{-1}(\phi_t^A(H_1(x)))$$

Similarly, $\phi_t^{\hat{J}}(x) = H_2^{-1}(\phi_t^J(H_2(x)))$ and

$$\phi_t^B(x) = H_3^{-1}(\phi_t^{\hat{J}}(H_3(x)))$$

Putting these together...

$$\phi_t^B(x) = H_3^{-1}(H_2^{-1}(\phi_t^J(H_2(H_3(x))))) \text{ and}$$

so
$$\phi_t^B(X) = H_3^{-1}(H_2^{-1}(H_1^{-1}(\phi_t^A(H_1(H_2(H_3(X)))))))$$

i.e.
$$\phi_t^B(X) = H^{-1}(\phi_t^A(H(X)))$$

where $H = H_1 \circ H_2 \circ H_3$.

Now for part b.
$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X \quad Y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} Y$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} = PJP^{-1} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} Y$$

$$\phi_t^A(X) = P\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} P^{-1} X$$

$$\phi_t^B(X) = \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} X$$

$$H(\phi_t^B(X)) = \phi_t^A(H(X)) \quad ? \quad \text{take}$$

$$H(X) = PX.$$ This is a homeomorphism

and
$$H(\phi_t^B(X)) = P\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} X$$
$$= P\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} P^{-1} PX = \phi_t^A(H(X)).$$

Ch 4 problem 6.

- Prove that any two linear systems with the same eigenvalues $\pm i\beta$ ($\beta \neq 0$) are conjugate. (Implicitly we're talking about 2x2 system because the entire chapter is about 2x2 systems

$$A = P\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} P^{-1} \qquad B = Q\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} Q^{-1}$$

then ϕ_t^A and ϕ_t^J are conjugate using

$$H_1(X) = PX \qquad H_1(\phi_t^J(X)) = \phi_t^A(H_1(X))$$

and ϕ_t^J and ϕ_t^B are conjugate using

$$H_2(X) = Q^{-1}X \qquad H_2(\phi_t^B(X)) = \phi_t^J(H_2(X)).$$

$\Rightarrow \phi_t^B(X)$ and $\phi_t^A(X)$ are conjugate using

$$H(X) = H_2(H_1(X)).$$

- What happens if

$$A \sim \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \text{ and } B \sim \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \text{ where } \gamma \neq 0?$$

We just need to find out if it's possible to find a homeomorphism H so that

$$H\left(\phi_t^{\hat{J}}(X)\right) = \phi_t^{J}\left(H(X)\right) \quad \text{where } J = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$\hat{J} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}$$

$$\phi_t^{J}(X) = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} X$$

$$= \begin{pmatrix} X_1 \cos(\beta t) + X_2 \sin(\beta t) \\ -X_1 \sin(\beta t) + X_2 \cos(\beta t) \end{pmatrix}$$

$$\phi_t^{\hat{J}}(X) = \begin{pmatrix} X_1 \cos(\gamma t) + X_2 \sin(\gamma t) \\ -X_2 \sin(\gamma t) + X_2 \cos(\gamma t) \end{pmatrix}$$

Note: $\phi^{J}_{\frac{2\pi}{\beta}}(X) = X$ the flow is periodic with period $\frac{2\pi}{\beta}$

$\phi^{\hat{J}}_{\frac{2\pi}{\gamma}}(X) = X$ the other flow is periodic with period $\frac{2\pi}{\gamma}$

if $\exists$ H so that

$H\left(\phi_t^{J}(X)\right) = \phi_t^{\hat{J}}\left(H(X)\right)$ is true $\forall t$ then it's true at $t = \frac{2\pi}{\beta}$

$$H\left(\phi^{J}_{\frac{2\pi}{\beta}}(X)\right) = \phi^{\hat{J}}_{\frac{2\pi}{\beta}}\left(H(X)\right)$$

ie. $H(X) = \phi^{\hat{J}}_{\frac{2\pi}{\beta}}\left(H(X)\right)$ if this is true for all X

then this means $\phi^{\hat{J}}_{\frac{2\pi}{\beta}}(X) =$ the identity map.

but this is only true if $\frac{2\pi}{\beta} = k\frac{2\pi}{\gamma}$ for some $k \in \mathbb{N}$

we can repeat the argument using H^{-1}...

$$H^{-1}(\phi^{\hat{J}}_t(x)) = \phi^{J}_t(H^{-1}(x)) \quad \forall t$$ so it's

true when $t = 2\pi/\gamma$

$$H^{-1}(\phi^{\hat{J}}_{\frac{2\pi}{\gamma}}(X)) = \phi^{J}_{\frac{2\pi}{\gamma}}(H^{-1}(X)) \quad \forall X$$

and so

$$H^{-1}(X) = \phi^{J}_{\frac{2\pi}{\gamma}}(H^{-1}(X)) \quad \forall X.$$

$\Rightarrow$ because H^{-1} is a homeomorphism we must have $\phi^{J}_{\frac{2\pi}{\gamma}}(X) = X \Rightarrow \frac{2\pi}{\gamma} = \hat{k}\frac{2\pi}{\beta}$ for some $\hat{k} \in \mathbb{N}$

putting it all together,

$$\frac{2\pi}{\beta} = k\left(\frac{2\pi}{\gamma}\right) = k\left(\hat{k}\left(\frac{2\pi}{\beta}\right)\right)$$ for some $k, \hat{k} \in \mathbb{N}$

$\Rightarrow k\hat{k} = 1 \Rightarrow k = 1$ and $\hat{k} = 1 \Rightarrow \gamma = \beta$

which is impossible. $\Rightarrow \nexists$ a homeomorphism and therefore if $A \sim \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, $B \sim \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}$

where $\beta \neq \gamma$ then the flows are not conjugate.

What if $A \sim \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ and $B \sim \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$?

then the flows will be conjugate because

if $J = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ and $\hat{J} = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$ then

$$\phi_t^J(x) = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} X$$

$$\phi_t^{\hat{J}}(X) = \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} X$$

$$H(X) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X$$

$$H(\phi_t^{\hat{J}}(x)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} X$$

$$= \begin{pmatrix} \cos & -\sin \\ -\sin & -\cos \end{pmatrix} X$$

$$\phi_t^J(H(x)) = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X$$

$$= \begin{pmatrix} \cos & -\sin \\ -\sin & -\cos \end{pmatrix} X$$

So $H(\hat{\phi}_t^J(X)) = \phi_t^J(X)$ because the solutions have the same period. One's going clockwise and the other's going counterclockwise. So the linear transformation that swaps X_2 for $-X_2$ is all we need.

12) $x'' - 2x' + x = 0 \Rightarrow x_1 = e^t \quad x_2 = te^t$

$x_2'' = (e^t + te^t)' = e^t + e^t + te^t = \cancel{2e^t}$ $(2+t)e^t$

$x_2' = e^t + te^t$

$x_2 = te^t$

$x_2'' - 2x_2' + x_2 = (2+t)e^t - 2(1+t)e^t + te^t = \cancel{2e^t} + \cancel{te^t} - \cancel{2e^t} - \cancel{2te^t}$
$+ \cancel{te^t} = 0$

So general solution is $c_1e^t + c_2te^t$ (for $x'' - 2x' + x = 0$).

Now we need a solution of $x'' - 2x' + x = 2e^t$

$x = fe^t \Rightarrow f''e^t + \cancel{2f'e^t} + fe^t - 2(\cancel{f'e^t} + fe^t) + fe^t = 2e^t \Leftrightarrow$

$= f''e^t \cancel{fe^t} + \cancel{fe^t} = 2e^t \Leftrightarrow$

$f''e^t = 2e^t \Leftrightarrow f'' = 2 \Leftrightarrow f = t^2 + c.$

So the general solution is $t^2e^t + c_1e^t + c_2te^t$.

a) ~~If $x(t) > 0$ $\forall t \in \mathbb{R}$ then~~

$x(t) > 0 \Leftrightarrow t^2 + c_1 + c_2t > 0 \Leftrightarrow c_2^2 - 4c_1 < 0$

$x'(t) = \left((t^2 + c_2t + c_1)e^t\right)' = (2t + c_2)e^t + (t^2 + c_2t + c_1)e^t > 0 \Leftrightarrow$

$t^2 + (c_2+2)t + c_1 + c_2 > 0 \Leftrightarrow 0 > (c_2+2)^2 - 4(c_1+c_2) =$

$c_2^2 + 4 + \cancel{4c_2} - \cancel{4c_2} - 4c_1 = c_2^2 - 4c_1 + 4 \Leftrightarrow$

$c_2^2 - 4c_1 < -4$

So $x'(t) > 0 \Rightarrow x(t) > 0$ but not the other way.