MAT267: HW2

Please do these problems and submit them by 11:59pm on Saturday (Feb 8).

This document last updated Feb 8 to clarify the interval of existence language in question 1.

1. Consider the autonomous ODE $x' = f(x)$.

   (a) time translation Let $x(t)$ be a solution of the ODE. Define the function $y(t)$ by $y(t) = x(t - t_0)$. Demonstrate that $y(t)$ is a solution of the ODE. If $x(t)$ has interval of existence (a, b), what is the interval of existence of the solution $y(t)$?

   (b) time reversal Let $x(t)$ be a solution of the ODE. Define the function $y(t)$ by $y(t) = x(-t)$. Demonstrate that $y(t)$ is a solution of the ODE $x' = -f(x)$. If $x(t)$ has interval of existence (a, b), what is the interval of existence of the solution $y(t)$?

   (c) time rescaling Let $x(t)$ be a solution of the ODE. Define the function $y(t)$ by $y(t) = x(\sigma t)$ for some $\sigma > 0$ Demonstrate that $y(t)$ is a solution of the ODE $x' = \sigma f(x)$. If $x(t)$ has interval of existence (a, b), what is the interval of existence of the solution $y(t)$?

   You can do the above at the level of symbolic manipulation without getting a visceral understanding of what's going on. To get a more practical understanding you can play around with some simple problems, graphing the solutions using desmos.

2. The initial value problem

$$X' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} X, \qquad \text{with} \qquad X(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

   has the solution

$$X(t) = e^{\alpha t} \left(x_1 \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + x_2 \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} \right).$$

   The solution is written as "an exponential term times a linear combination of circular solutions". But is a linear combination of circular solutions a circular solution? Answer this question by demonstrating that "a linear combination of circular solutions" can be written as a single "circular solution" by finding $R > 0$ and ϕ so that

$$X(t) = e^{\alpha t} \left(x_1 \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + x_2 \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} \right) = e^{\alpha t} R \begin{pmatrix} \cos(\beta t - \phi) \\ -\sin(\beta t - \phi) \end{pmatrix}.$$

3. Chapter 3, problem 5.

4. Chapter 3, problem 6.
5. Chapter 3, problem 9.
6. Chapter 3, problem 13.
7. Chapter 3, problem 14.
8. Chapter 4, problem 1.
9. Chapter 4, problem 2.
10. Chapter 4, problem 5.
11. Chapter 4, problem 6.
12. Let $x(t)$ be a solution of

$$x'' - 2x' + x = 2\,e^t$$

   (a) If $x(t) > 0$ for all $t \in \mathbb{R}$, must $x'(t) > 0$ for all $t \in \mathbb{R}$? Explain.

   (b) If $x'(t) > 0$ for all $t \in \mathbb{R}$, must $x(t) > 0$ for all $t \in \mathbb{R}$? Explain.

   Hint: Start by finding the general solution of $x'' - 2x' + x = 2\,e^t$.

   First, you're going to have to find the general solution of the homogenous, linear, second-order ODE $x'' - 2x' + x = 0$. *You can do this by writing down the system of two linear first-order equations, finding the general solution of the system, and then extracting* $x(t)$ *from that solution. If you view the ODE as a linear operator* ($\mathcal{L} : C^2 \to C^0$) *where* $\mathcal{L}(x) := x'' - 2x' + x$, *you've just found all functions in its kernel (a two-dimensional subspace of* C^2*).*

   Second, to find the general solution of $\mathcal{L}(x) = 2\,e^t$ *you just need to find one solution* $x_p(t)$ *and you then create the general solution by adding the kernel to that one solution:*

   the general solution of $x'' - 2x' + x = 2\,e^t$
   is the same thing as
   $x_p(t)$ + *the general solution of* $x'' - 2x' + x = 0$

   How do you find $x_p(t)$*? Try guessing. You want to guess a function that will produce something that has "the right shape" after it's been acted upon by* $\mathcal{L}$. *If your guess is too simple, it'll be in the kernel of* $\mathcal{L}$ *and you'll get nowhere. If your guess is too complicated, you'll work too hard. There is a way of constructing* $x_p(t)$ *without guessing but this particular problem is simple enough that guessing's the fastest way.*