Homework 1 solutions

1. (a) $x(0)=4$ $\quad t<5 \Rightarrow x' - x = -4 \Rightarrow (xe^{-t})' = -4e^{-t} \xRightarrow{\int_0^T} x(T)e^{-T} - 4 = 4e^{-T} - 4$

$\Rightarrow x(T) = 4 \quad (T<5)$.

Now if x is differentiable at 5, then we should have $\lim_{t\to 5^-}(x(t)-4) = 2 - x(5) \Rightarrow$

$x(5) = 2$ but then x will not be continuous at 5 which is absurd, so the interval of

definition (the domain) of the solution is $(-\infty, 5)$: $\boxed{\begin{array}{c} x(t) = 4 \\ t\in(-\infty,5)\end{array}}$

Qualitative behavior: $\lim_{T\to -\infty} x(T) = 4$.

(b) $t<5 \Rightarrow x(T)e^{-T} - x(0) = 4e^{-T} - 4 \Rightarrow x(T) = 4 - e^{T}$.

Now $\lim_{T\to 5^-} x(T) = 4 - e^5$ (So if we want x to be continuous, we need $x(5) = 4 - e^5$)

Also we need differentiability at 5, so $\lim_{T\to 5^-}(4 - x(T)) = x(5) - 2 \Rightarrow$

$x(5) = 2 + e^5$ But $2 + e^5 \neq 4 - e^5$ so there can't be a solution defined

after 5 (or on 5), so $\boxed{x(T) = 4 - e^T \quad T\in(-\infty,5)}$

Qualitative behavior: $\lim_{T\to -\infty} x(T) = 4$.

(c) First we have to see if we have a solution defined on some (t, ∞):

(and also at 0 because initial value is defined there)

Well as above we need $\lim_{T\to 5^-}(x(T) - 4) = 2 - x(5)$ (also we need continuity

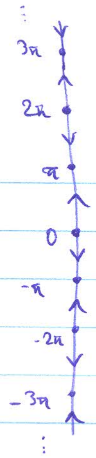
at 5) so $x(5) = 3$.

Now doing the same as above, we reach $x(t) = \begin{cases} 2 + e^{-(t-5)} & t\geq 5 \\ 4 - e^{t-5} & t<5 \end{cases}$

So $\boxed{\lim_{t\to\infty} x(t) = 2}$ (this is the solution to IVP $x(0) = 4 - e^{-5}$, $x' = \begin{cases} 4-x & t<5 \\ x-2 & t\geq 5 \end{cases}$)

So there's only one IVP based at $t=0$ which has solutions on $\mathbb{R}$. For that solution $x(t)\to 2$ as $t\to\infty$. The solution for the IVP with $x(0) = x_0$ with $x_0 \neq 4 - e^{-5}$ has interval of existence $(-\infty, 5)$ and so there's no limit as $t\to\infty$.

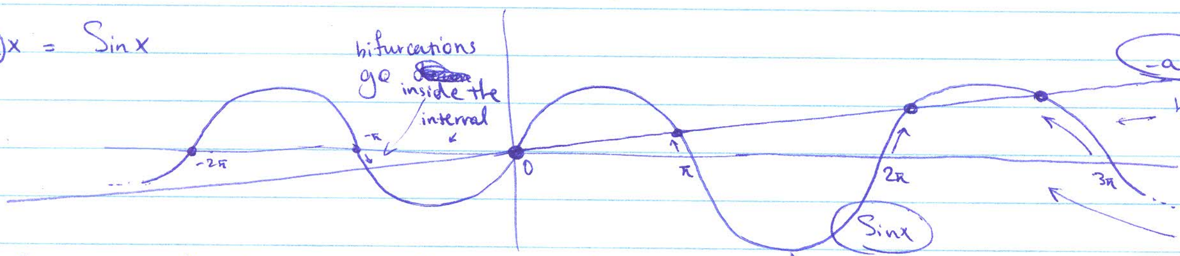
2. (a) $a=0 \Rightarrow x' = \sin x$



(b) When $a=0$ then as above shows $\{\dots,-2\pi,-\pi,0,\pi,2\pi,\dots\}$ are the bifurcations.

Now to find bifurcations for (let's say) $-1 \leq a < 0$ we should find solutions to $x'=0 \Rightarrow$

$(-a)x = \sin x$

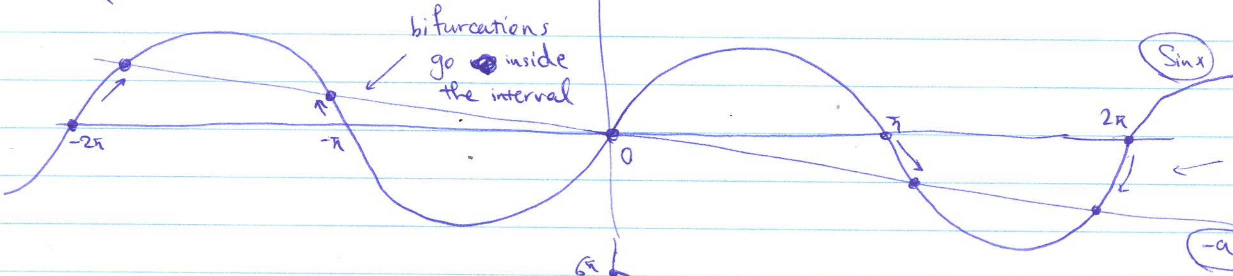


bifurcations go as ~~a goes from~~ 0 inside the interval. as a goes from 0 to -1.

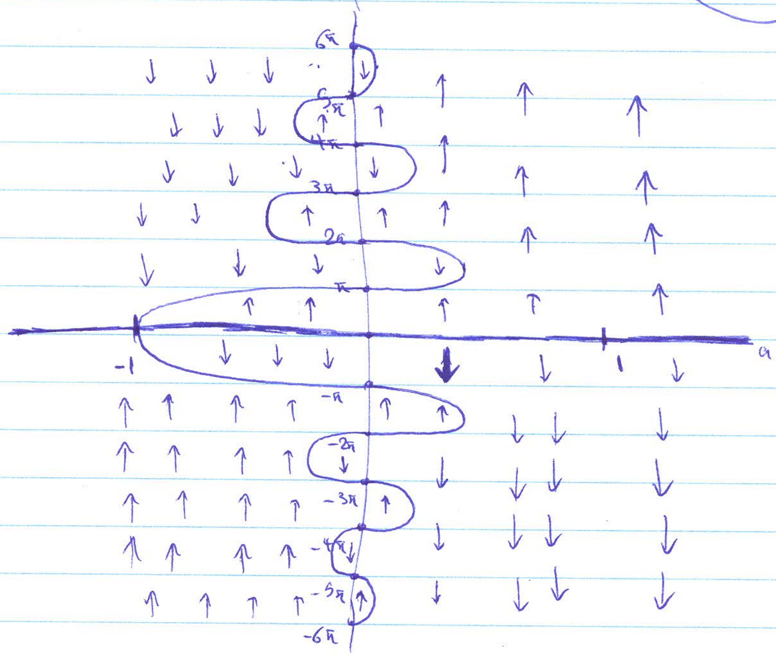
(there are finitely many bifurcations for $a \neq 0$).

also, as you can see at $a=-1$, 0 is the only bifurcation.

If $0 < a \leq 1$,

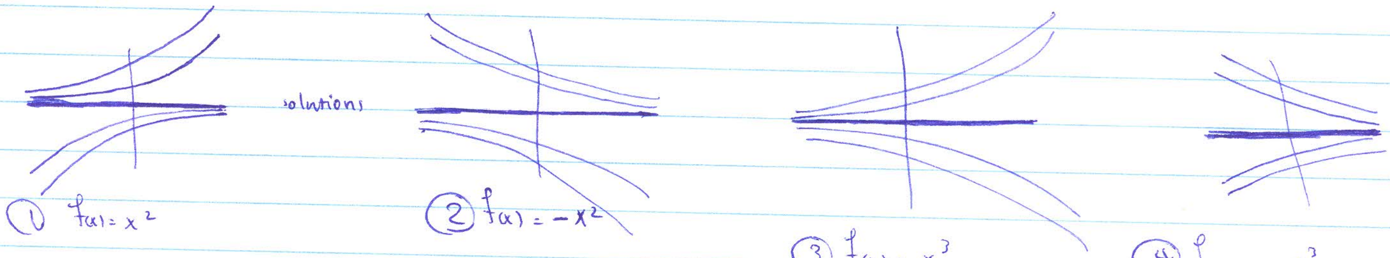


bifurcations go inside the interval as a goes from 0 to 1.



$x_0 = 0$

3. (a) Anything can happen! Cases: $f(x) = x^2$ $f(x) = -x^2$ $f(x) = x^3$ $f(x) = -x^3$



① $f(x) = x^2$ ② $f(x) = -x^2$ ③ $f(x) = x^3$ ④ $f(x) = -x^3$

(b) Using finite Taylor expansion, we get that around x_0,

$$f(x) = a(x - x_0)^2 + o((x - x_0)^2) \quad \text{near } x_0. \qquad a \neq 0$$

so the behaviour of solutions is like ① or ② depending on positivity or negativity of a.

(c) We have $f(x) = a(x - x_0)^3 + o((x - x_0)^3)$ near x_0. $a \neq 0$.

so the behaviour of solutions at x_0 is like ③ or ④ depending on positivity or negativity of a.

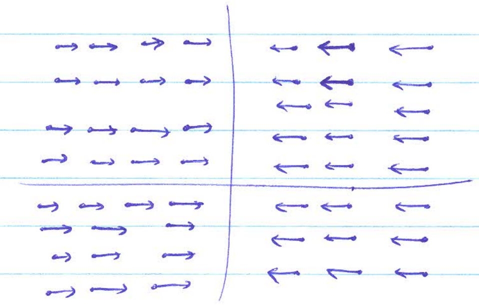
4. Eigenvalues are $a, 1$ so repeated $\Leftrightarrow a = 1$.

Eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1-a \end{pmatrix}$. As $a \to 0$ eigenvectors go to each other.

(another way to see this is saying that when eigenvalues are different, we have two different eigen-vectors but matrix $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ has only one eigenvector as it is not identity!).

50. The linear system $X' = A X$ with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

Direction field:



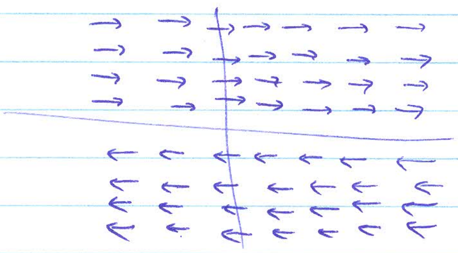
vector field: $(x \neq 0)$ $\begin{pmatrix} -x \\ 0 \end{pmatrix} \Big/ \left\| \begin{pmatrix} -x \\ 0 \end{pmatrix} \right\|$ (normalized).

General solution is $(A e^{-t}, C)$

A, C constants.

6. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \iff x' = y \; , \; y' = 0$ (has $(t, 1)$ as solution).

Direction field



vector field, $\begin{pmatrix} y \\ 0 \end{pmatrix} / \left\| \begin{pmatrix} y \\ 0 \end{pmatrix} \right\|$.

general solution $(At+B, A)$

A, B const.

7. Define $y(t) = \int_0^t \frac{x(s)}{1+s^2}\,ds$ so that $y'(t) = \frac{x(t)}{1+t^2} \Rightarrow$

$y'(t) = \lambda(1+y(t)) \Rightarrow (y(t)e^{-\lambda t})' = \lambda e^{-\lambda t} \Rightarrow y(t) = ce^{\lambda t} - 1$ with

$c = 1 + y(0) = 1$ $\left(y(0) = 0 \text{ by definition of } y(t)\right)$.

so $y(t) = e^{\lambda t} - 1 \Rightarrow \frac{x(t)}{1+t^2} = y'(t) = \lambda e^{\lambda t} \Rightarrow \boxed{x(t) = \lambda(e^{\lambda t})(1+t^2)}$.

8- First we prove that $x([t_0,\infty)) \subseteq [x_0, x_1)$ $(*)$

If $x(s) > x_1$ for some $s \in [t_0, \infty)$, then by intermediate value thm, $\exists s' \in [t_0,\infty)$ s.t. $x(s') = x_1$ which is contradiction to the fact that $x \equiv x_1$ is a solution and solutions are unique.

If $x(s) < x_0$ for some $s \in [t_0,\infty)$: First as $x'(t_0) = f(x_0) > 0$, $\exists t_0'$ s.t. $x(t) > x_0$ on $t \in [t_0, t_0']$. Now we know that $x(t_0') > x_0$ and $x(s) < x_0$

Now consider $x^{-1}(x_0) \cap [t_0', s]$ (which is nonempty by intermediate value thm) and consider its minimum (call it T). Now we know

$$x(T) = x_0, \quad x(t_0') > x_0$$

so by mean value thm, $\exists T' \in [t_0', T]$ s.t.

$$0 < f(x(T')) = x'(T') = \frac{x(T) - x(t_0')}{T - t_0'} < 0$$

$x(T') > 0$ because
T is minimum of $x^{-1}(x_0) \cap [t_0', s]$.

which is a contradiction. So claim $(*)$ is proved.

Now as $f(x(t)) = x'(t)$ and $f(x(t)) > 0$ (because $x(t) \in [x_0, x_1)$)
$t \in [t_0, \infty)$

we have that x is increasing in $[t_0, \infty)$.

Now suppose that $\lim_{t\to\infty} x(t) \neq x_1$, then there exists $\varepsilon > 0$ such that

$$x(t) \leq x_1 - \varepsilon \quad \forall t \in [t_0, \infty)$$

Then consider $\min_{x \in [x_0, x_1 - \varepsilon]} f(x) = L$. So we have for $t \in [t_0,\infty)$,

$x'(t) = f(x(t)) \geq L$ so that $x(s) - x_0 = \int_{t_0}^{s} x'(t) \geq \int_{t_0}^{s} L = L(s - t_0) \xrightarrow{s\to\infty} \infty$ Contradiction!

#8 Consider the initial value problem

$$\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$$

where the function f satisfies

$$f > 0 \text{ on } [x_0, x_1),\ f(x_1) = 0,\ f \text{ is } C^1$$

Assume the solution of the IVP is defined on some interval (t_1, ∞) where $t_1 < t_0$.
Prove that $\lim_{t \to \infty} x(t) = x_1$ and $\lim_{t \to \infty} x'(t) = 0$.

The following proof is from one of the MAT267 students.

proof:

We know $x(t) = x_1$ is an equilibrium solution. Because $x(0) = x_0 < x_1$ and $f > 0$ on $[x_0, x_1)$ we know that one of two things happens

- $x(t)$ is increasing and $x(t) < x_1 \ \forall t$
- $x(t)$ is increasing and $\exists \tilde{t}$ so that $x(\tilde{t}) = x_1$.

The second can't happen because this would violate the existence + uniqueness theorem. (Which will apply because f is C^1.)

Therefore $x(t)$ is increasing on $[t_0, \infty)$ and $x(t)$ is bounded above by x_1. It follows that $x(t)$ has a limit, call it x_∞

$$\lim_{t \to \infty} x(t) = x_\infty .$$

We'll now prove that $x_\infty = x_1$.

First of all, fix a $t \in [t_0, \infty)$. By the Mean Value theorem, $\exists\ \tau \in (t, t+1)$ so that

$$\frac{x(t+1)-x(t)}{(t+1)-t} = x'(\tau)$$

i.e. $x(t+1)-x(t) = x'(\tau) \overset{\text{by the ODE}}{=} f(x(\tau))$

for some $\tau \in (t, t+1)$.

This was for a fixed t. Now we're going to consider general t. So we'll remember that τ depended on t by denoting it $\tau(t)$.

By the MVT, for each $t \in [t_0, \infty)$ we have

$$\circledast \quad x(t+1)-x(t) = x'(\tau(t)) = f(x(\tau(t)))$$

where $t < \tau(t) < t+1$.

Now we're going to take $t \to \infty$ in $\circledast$.

$$\lim_{t\to\infty} x(t+1)-x(t) = \lim_{t\to\infty} x(t+1) - \lim_{t\to\infty} x(t)$$

$$= x_\infty - x_\infty = 0$$

$$\lim_{t\to\infty} f(x(\tau(t))) = f\left(\lim_{t\to\infty} x(\tau(t))\right) \quad \text{(f is contin.)}$$

$$= f(x_\infty) \quad (\tau(t) \to \infty \text{ as } t \to \infty)$$

Therefore $0 = f(x_\infty)$.

So we have $f(x_\infty)=0$ and we know $x_\infty \le x_1$ (because $\lim_{t\to\infty} x(t) \le x_1$)

We know $f>0$ on $[x_0, x_1)$. Therefore $x_\infty = x_1$.

. This proves $\lim_{t\to\infty} x(t) = x_1$, as desired.

Now to prove $\lim_{t\to\infty} x'(t) = 0$! We know $x(t)$ is a solution and so $x'(t) = f(x(t))$ for all $t \in (t_1, \infty)$. Therefore

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} f(x(t))$$

$$= f(x_\infty)$$ because $x(t) \to x_\infty$

$$= 0$$ — because we already figured out $f(x_\infty)=0$.

$\therefore //$

Here's a different proof. In the following proof, I first prove $\lim_{t \to \infty} x'(t) = 0$ and conclude that $\lim_{t \to \infty} x(t) = X_1$.

<u>proof</u>: By the previous argument, we know that $x(t) < x_1$ for all $t \in [t_0, \infty)$ and therefore $f(x(t)) > 0$ for all $t \in [t_0, \infty)$ and therefore $x'(t) > 0$ for all $t \in (t_0, \infty)$. (by the ODE)

Assume $\lim_{t \to \infty} x'(t) \neq 0$.

This means there's some $\varepsilon > 0$ and some sequence of times $\{t_k\}$, $t_k \to \infty$ so that $x'(t_k) > \varepsilon \quad \forall k$.

That is $f(x(t_k)) > \varepsilon \ \forall k$.

f is C^1 and therefore f' is continuous on $[X_0, X_1]$ $\Rightarrow$ f' is bounded on $[X_0, X_1]$ $\Rightarrow \exists M$ so that $|f'(x)| \leq M$ for all $x \in [X_0, X_1]$.

I'll now argue that $\exists \delta$ so that if $t \in (t_k - \delta, t_k + \delta)$ then $x'(t) > \frac{\varepsilon}{2}$. This mean's I'll have $x'(t) > \frac{\varepsilon}{2}$ on an infinite sequence of intervals and this is impossible. Why is it impossible?

$$x(T) = \int_{t_0}^{T} x'(t)\,dt$$

$$\geq \sum_{k=1}^{N} \int_{t_k-\delta}^{t_k+\delta} x'(t)\,dt$$ where N is the largest integer so that $t_k+\delta < T$

$$\geq \sum_{k=1}^{N} \int_{t_k-\delta}^{t_k+\delta} \varepsilon/2\,dt$$

$$= \sum_{k=1}^{N} \varepsilon\delta = N\varepsilon\delta.$$

as $T \to \infty$, N will go to ∞ and so however small ε and δ might be, for N large enough I'll have $N\varepsilon\delta > x_1$ and therefore $x(t) > x_1$. This means the solution has crossed paths with the equilibrium solution, violating existence + uniqueness!

So all I need to do is argue that

$$x'(t) > \frac{\varepsilon}{2}$$

if $t \in (t_k-\delta, t_k+\delta)$ where δ is chosen small enough.

By the mean value theorem,

$$x'(t) - x'(t_n) = x''(\xi)\,(t - t_n)$$

for some ξ between t & t_n.

Can I use the mean value theorem here? To do so, I need x' to be continuous on $[t, t_n]$ and x'' to be continuous on (t, t_n). <I assumed $t < t_n$ in writing that. If $t_n < t$ then reverse the intervals.> I'm going to want to use that x'' is bounded too, so let's do that as well.

- $x'(t) = f(x(t))$. We know $x(t)$ is continuous and we know f is continuous $\Rightarrow f(x(t))$ is as well. I claim x' is bounded on $[t_0, \infty)$. We know $x_0 \leq x(t) < x_1$ for all $t \in [t_0, \infty)$. We know f is continuous on $[x_0, x_1] \Rightarrow 0 \leq f(x) \leq \tilde{M}$ for some $\tilde{M}$.
  So we have x' continuous on $[t_0, \infty)$ and bounded on $[t_0, \infty)$.
- $x''(t) = \frac{d}{dt} x'(t) = \frac{d}{dt} f(x(t)) = f'(x(t))\,x'(t)$

  $$= f'(x(t))\,f(x(t))$$

  f & f' are continuous on $[x_0, x_1] \Rightarrow x''$ is continuous on $[t_0, \infty)$. We already had $|f'(x)| \leq M$ on $[x_0, x_1] \Rightarrow |x''(t)| \leq M\tilde{M}$.

So we have

$$|x'(t) - x'(t_k)| = |x''(\xi)|\,|t-t_k|$$
$$\le M\tilde{M}\,|t-t_k|$$

and we know $x'(t_k) > \varepsilon$.

So if we take $\delta = \frac{1}{M\tilde{M}}\frac{\varepsilon}{2}$ we'll have $|t-t_k| < \frac{1}{M\tilde{M}}\frac{\varepsilon}{2} \Rightarrow |x'(t) - x'(t_k)| < \frac{\varepsilon}{2}$

$$\Rightarrow x'(t) > x'(t_k) - \frac{\varepsilon}{2} > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

This finishes the proof: we know that there's a $\delta > 0$ so that $x' > \varepsilon/2$ on $(t_k-\delta, t_k+\delta)$ for $k \to \infty$, and this will force $x(t)$ to reach the equilibrium solution x_1 in finite time, which is impossible.

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Solution from another MAT267 student

PROBLEM 8

The continuous function $|f'|$ achieves a maximum $M \geq 0$ on the compact interval $[x_0, x_1]$. For any $\alpha \in [x_0, x_1)$, by the Mean Value Theorem we have[1]

$$\frac{f(\alpha)}{x_1 - \alpha} = \frac{f(\alpha) - f(x_1)}{x_1 - \alpha} = -f'(\alpha^*) \leq M$$

for some $\alpha^* \in (\alpha, x_1)$. It follows that $0 < f(\alpha) \leq M(x_1 - \alpha)$ for $\alpha \in [x_0, x_1)$.

We first show that $x(t) \geq x_0$ for all $t \in [t_0, \infty)$. Suppose not, so that there exists a maximal $t^* \in [t_0, \infty)$ such that the closed set $S := x^{-1}([x_0, \infty)) \cap [t_0, \infty)$ contains the interval $[t_0, t^*]$. Then by continuity $x(t^*) = x_0$,[2] so $x'(t^*) = f(x_0) > 0$. But then by Taylor's theorem it follows that $x > x_0$ on some half-open neighborhood $[t^*, t^{**})$ of t^*, contradicting the maximality of t^*.

Next, we show that $x(t) < x_1$ for all $t \in [t_0, \infty)$. For $t \in (t_1, \infty)$, define the function

$$y(t) := e^{Mt}(x_1 - x(t)).$$

Then $y(t_0) = x_1 - x_0 > 0$ and

$$\begin{aligned} y'(t) &= Me^{Mt}(x_1 - x(t)) - e^{Mt} f(x(t)) \\ &\geq Me^{Mt}(x_1 - x(t)) - e^{Mt} M(x_1 - x(t)) \\ &= 0 \end{aligned}$$

for all $t \in x^{-1}([x_0, x_1)) = y^{-1}((0, \infty))$. By a similar argument as before, it follows that $y(t) > 0$ for all $t \in [t_0, \infty)$. But by the construction of y, it follows that $x(t) < x_1$ for all $t \in [t_0, \infty)$.

Thus $x'(t) = f(x(t)) > 0$ for all $t \in [t_0, \infty)$, so $x(t)$ is an increasing bounded function, and hence has a limit $L := \lim_{t\to\infty} x(t) \in [x_0, x_1]$. By the continuity of f, it follows that $x'(t) = f(x(t)) \to f(L)$ as $t \to \infty$.

In particular, the function $x'(t)$ has a well-defined limit as $t \to \infty$. But

$$L = x_0 + \int_{t_0}^{\infty} x'(t)\,dt,$$

so in fact $x'(t) \to 0$ since the integral converges. Thus $f(L) = 0$, forcing $L = x_1$ as desired.