# MAT267: HW1

Please do these problems and submit them by 11:59pm on Saturday (Jan 25).

1. Chapter 1, problem 7.
2. Chapter 1, problem 5.
3. Chapter 1, problem 13. (Assume $f'$, $f''$, and $f'''$ all exist and are continuous.)
4. Chapter 2, problem 7.
5. Chapter 2, problem 9.
6. Chapter 2, problem 10.
7. Find all continuous functions $x : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$x(t) = \lambda\,(1+t^2)\left(1 + \int_0^t \frac{x(s)}{1+s^2}\,ds\right)$$
for all $t \in \mathbb{R}$. Here $\lambda$ is a fixed real number.
8. In class, we studied the logistic equation
$$x' = ax(1-x).$$
We know that if $x(t)$ is a solution and $x(t) \in (0,1)$ then $x'(t) > 0$. I then said, "If the solution is defined for all time, we know it's increasing and it's bounded above and so $\lim_{t\to\infty} x(t) = x_\infty$. It then follows that $x'(t) \to 0$ as $t \to \infty$. Because $x'(t) = 0$ if and only if $x(t) = 0$ or $x(t) = 1$, it follows that $x_\infty = 1$." Please prove this for the following generalized situation:

   *Consider the initial value problem*
$$\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$$
   *where the function* $f$ *satisfies*
$$f > 0 \text{ on } [x_0, x_1), \quad f(x_1) = 0, \quad f \text{ is } C^1.$$
   *Assume the solution of the IVP is defined on some interval* $(t_1, \infty)$ *where* $t_1 < t_0$. *It follows that* $\lim_{t\to\infty} x(t) = x_1$.

   In terms of what's meant for "$f$ is $C^1$"... it's sufficient to assume that $f : [a,b] \to \mathbb{R}$ and $f'$ is continuous on $(a,b)$ for some interval $(a,b)$ that contains $[x_0, x_1]$. The hard part of this problem is *proving* $x'(t) \to 0$ as $t \to \infty$. You may assume that solutions are unique.

   If you like analysis and have spare time, consider this question for the nonautonomous ODE $x' = f(x,t)$. What additional conditions on $f$ would ensure the result still holds?