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NOTES ON BLOWUP AND LONG WAVE UNSTABLE THIN FILM EQUATIONS

M. C. PUGH

1. Introduction

We provide a gentle introduction to a body of work done in collaboration with Andrea Bertozzi, Richard S. Laugesen, and Dejan Slepcev and independently by Elena Beretta, Andrew Bernoff, and Thomas Witelski.

We consider the evolution equation

$$u_t = -(u^n u_{xxx})_x - (u^m u_x)_x. \tag{1}$$

This is the one dimensional version of $u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u)$, with $f(u) = u^n$ and $g(u) = u^n$. Such equations have been used to model the dynamics of a thin film of viscous liquid spreading on a flat solid surface. The air/liquid interface is at height $z = u(x, y, t) \geq 0$ and the liquid/solid interface is at $z = 0$. The one dimensional equation (1) applies if the liquid film is uniform in the y direction. We refer to [16], [17] for reviews of the physical and modeling literature.

Bertozzi and Pugh [4] introduced three regimes for the equation: subcritical ($m < n + 2$), critical ($m = n + 2$), and supercritical ($m > n + 2$). In these notes, we give an introduction to these dynamical regimes. Before addressing why the balance between m and $n+2$ is crucial, we first consider a more classical PDE which has analogous regimes. In addition, we refer the reader to Levine's survey article on the role of critical exponents in blowup theorems [13].

1.1. The semilinear heat equation and its blowup regimes

Recall the semilinear heat equation in one dimension:

$$u_t = u_{xx} + u^p. \tag{2}$$

Without the lower-order "reaction" term, the PDE is simply the heat equation, $u_t = u_{xx}$, which has solutions that are smooth and exist for all time. Absent the diffusion term, the PDE is the ODE, $u_t = u^p$. If $p \leq 1$, the ODE's solutions exist for all time, whatever the initial data. If $p > 1$, there exist initial data that yield solutions that blow up in finite time: $u(t) \to \infty$ as $t \to T^* < \infty$.

In fact, the critical exponent $p = 1$ suggested by the ODE is also critical for the PDE: if $p \leq 1$, then all initial data yield solutions that exist for all time and if $p > 1$, there are initial data that yield solutions that blow up in finite time. More specifically, if $1 < p \leq 3$, then any nontrivial solution must blow up in finite time [1], [7], [20]. Whereas, if $3 < p$, some solutions exist for all time, while other solutions blow up in finite time [1]. Fine information is known about the set of points at which blowup occurs and about the spatiotemporal structure of solutions near blowup points. In one dimension, the blowup is a focussing type, with solutions forming "peaks" that grow taller and narrower in a selfsimilar manner, centered at isolated blowup points [11], [6], [14].

1.2. The long wave unstable thin Film equation and its blowup regimes

Bertozzi and Pugh [4] conjectured the following regimes for nonnegative solutions of (1) that are periodic in space:

$$\begin{cases} m < n+2 & \text{subcritical, solutions exist and are bounded for all time;} \\ m = n+2 & \text{critical, no behavior conjectured;} \\ m > n+2 & \text{supercritical, solutions may blow up in finite or infinite time.} \end{cases}$$

For the supercritical case, it is further conjectured that nonnegative solutions can blow up in finite time only if $2m > n$; otherwise, they can grow at most exponentially in time. The conjecture is also made for nonnegative solutions on the line that have compact support [5].

Nonnegative periodic solutions have been proven to exist for nonnegative initial data for a range of exponents (n, m); positive initial data allow a larger range of exponents [4]. Nonnegative, compactly supported solutions on the line have also been proven to exist [5].

The existence theory assumes, at the very least, $n > 0$ and $m > n$. Hence, for the resulting solutions, if $m > n+2$, then any blowup must occur in finite time (since $2m > n$ holds automatically). The degeneracy of the coefficient, $u^n \to 0$ as $u \to 0$, is used to ensure that the constructed solutions cannot be nonnegative in some region of space at one time and negative there at some later time — the degeneracy ensures that nonnegative initial data yield nonnegative solutions. If $n = 0$, there is no degeneracy; the fourth-order term is linear. In this case, nonnegative initial data are not expected to yield nonnegative solutions. Indeed, the PDE with no second-order term, $u_t = -u_{xxxx}$, has explicit solutions that do not preserve sign.[1] The cases with $n < 0$ have not been addressed analytically or computationally, as far as we know. The condition $m > n$ is a technical condition used in proving

[1] Unlike for second-order parabolic problems, there is no comparison principle.

that the approximate solutions in the construction converge to a solution of (1).

2. Methods by which the conjectured regimes can be predicted

We now explain two methods by which one could predict the blowup regimes given above. The first is the "hard" method of exact solutions; the second is the "soft" method of functional analysis.

2.1. Exact solutions

Numerical simulations of the periodic initial value problem suggest that two types of behaviors can emerge. In the subcritical and critical cases, the solution may exist for all time, broadening in width and decreasing in amplitude in a selfsimilar manner [21]. In the critical and supercritical cases, the solution may cease to exist in finite time: in some region of the solution, a peak emerges, growing taller and thinner as the blowup time approaches. Near the blowup point, the solution is more and more selfsimilar [4]. For this reason, it is natural to seek selfsimilar solutions of (1):

$$u_{ss}(x,t) = (T+\sigma t)^{\alpha}\, U(\eta) \qquad \text{where} \quad \eta = \frac{x-x_0}{(T+\sigma t)^{\beta}} \tag{3}$$

and $\sigma = \pm 1$. One seeks exact solutions partly to find if they could explain observed emergent structures in solutions of the initial value problem. Could there be regions in space in which a solution's dynamics are well-modelled by an exact solution like u_{ss}?

If $\sigma = 1$, then the solution u_{ss} is global; it exists for all $t > -T$. If $\sigma = -1$, then the solution will cease to exist in finite time; it exists for all $t < T$. If

$$\alpha = -\frac{1}{2m-n} \qquad \text{and} \qquad \beta = \frac{m-n}{2(2m-n)}, \tag{4}$$

then u_{ss} will be a solution of the PDE (1) if the "shape function" U is a solution of a particular ODE. In the global existence case ($\sigma = 1$), if $\alpha < 0$, the amplitude of u_{ss} will decrease as $t \to \infty$ and if $\beta > 0$, then the solution will broaden. We call such solutions "broadening." In the finite time blowup case ($\sigma = -1$), if $\alpha < 0$, then the amplitude of u_{ss} will increase as $t \to T$ and if $\beta > 0$, then the solution will focus at $x = x_0$. The signs of α and β are the same for broadening and blowup solutions. This is natural because the difference between these two types of solutions is simply time reversal — which is precisely what σ captures.

The values of σ, α, and β appear in the ODE that the shape function U satisfies. If this ODE has solutions with "droplet" profiles (nonnegative, compactly supported on $[-a,a]$ with $a < \infty$, increasing on $[-a,0]$, and decreasing on $[0,a]$), then the solutions (3) may bear upon the numerically observed dynamics described above. We note that for some values of (n,m) there may be additional emergent behaviors. For example, it can happen that, in finite time, the solution “pinches” (goes to zero at some point in space and time) at one point while blowing up at another [4]. In this case, we would seek a blowup solution of the form (3) to describe the blowup while also seeking a solution of the form (3) to describe the pinching. For the pinching, we would seek a solution with $\sigma = -1$ and $\alpha > 0$ and $\beta > 0$. In terms of the shape function U, one would hope that the ODE has a positive solution U that has a minimum at $\eta = 0$ and has $U(\eta) \to \infty$ as $|\eta| \to \infty$.

We first consider the $\sigma = -1$ case, seeking solutions u_{ss} that blow up in finite time, focussing at a point. The requirements $\alpha < 0$ and $\beta > 0$ imply that both $2m > n$ and $m > n$ must hold. This reduces to $m > n$ if $n > 0$ and $2m > n$ if $n < 0$. It remains to see how the m versus $n+2$ balance arises.

In this direction, if solutions of the initial value problem conserve mass and the shape function U has a finite integral, then one can use the mass of u_{ss} to study the initial value problem.[2] The mass of u_{ss} is

$$M_{ss}(t) := \int u_{ss}(x,t)\,dx = (T-t)^{\alpha-\beta} \int U(\eta)\,d\eta = (T-t)^{\frac{m-n-2}{2(2m-n)}} \int U(\eta)\,d\eta.$$

Since $2m > n$, it follows that

$$\begin{cases} m > n+2 & \Longrightarrow M_{ss}(t) \to 0 \quad \text{as} \quad t \to T, \\ m = n+2 & \Longrightarrow M_{ss}(t) = M_{ss}(0) \quad \forall t < T, \\ m < n+2 & \Longrightarrow M_{ss}(t) \to \infty \quad \text{as} \quad t \to T. \end{cases}$$

And so, one sees that in the subcritical regime ($m < n+2$), if the solution of the initial value problem has finite mass, then a finite-time focussing blowup cannot emerge: such a blowup would require infinite mass. In this way, we see how the subcritical,“no blow up,” regime could be conjectured.

In the supercritical regime ($m > n+2$), if the initial data u_0 has nonzero mass, then one can choose a time T such that

$$\int u_{ss}(x,0)\,dx < \int u_0(x)\,dx = \int u(x,t)\,dx.$$

Since the mass $M_{ss}(t)$ is decreasing in time, it follows that a finite-time focussing blowup could emerge in the solution u. This doesn’t imply that all nonzero initial data result in solutions that blow up in finite time — the supercritical regime has nonzero steady state solutions [12].

In the critical regime ($m = n+2$), the mass $M_{ss}(t)$ is constant in time and equals $\int U$. It follows that

$\int u_0(x)\,dx < \int U(\eta)\,d\eta \Longrightarrow$
finite time, focussing blowup cannot emerge in u,
$\int u_0(x)\,dx > \int U(\eta)\,d\eta \Longrightarrow$
finite time, focussing blowup might emerge in u.

[2]In the following arguments, we assume that a “droplet” shape function exists.

For $n > 0$, the ODE for the shape function has been carefully studied [18]. For the blowup ($\sigma = -1$) case, the authors find that if $0 < n < 3/2$, then there are infinitely many nonnegative, compactly supported solutions of the form (3) that have zero contact angles.[3] Denoting this family of shape functions by $\mathcal{U}_{-1}$, they prove

$$\inf_{U \in \mathcal{U}_{-1}} \int U(\eta)\, d\eta = \frac{2\sqrt{2}}{3}\pi =: M_c\,. \tag{5}$$

Earlier, Beretta [2] had proven the existence of infinitely many nonnegative, compactly supported, zero-contact-angle solutions for the spreading ($\sigma = 1$) case. We refer to these solutions as "source-type" because they are self-similar, they preserve mass, and they tend to a delta function as $t \to -T$. Denoting the corresponding family of shape functions by $\mathcal{U}_{+1}$, the methods of [18] imply

$$\sup_{U \in \mathcal{U}_{+1}} \int U(\eta)\, d\eta = M_c\,. \tag{6}$$

And so the mass M_c appears to be critical, in some way, in the critical regime ($m = n + 2$) of the evolution equation.

Witelski, Bernoff, and Bertozzi considered the PDE (1) in the critical regime, demonstrating that the mass M_c is indeed critical [21]. They took $n = 1$ and $m = 3$, exponents for which one can prove that given compactly supported nonnegative initial data there exists a nonnegative, compactly supported solution on the line that has finite speed of propagation [5]. This allowed the authors to make an elegant choice of initial data, one that would explore the mass M_c. They took two separated, nonnegative "drops," each of which has mass less than M_c but with joint mass greater than M_c. Because of the finite speed of propagation, at first these separated drops evolve independently of one another. Since their mass is less than M_c, they initially spread and one expects that as $t \to \infty$, each one would (in the absence of the other) converge to one of Beretta's selfsimilar source-type ($\sigma = 1$) solutions. This early evolution is shown in Figure 1. In fact, the droplets run into one another in finite time. As soon as the droplets run into one another, resulting in a profile with mass greater than M_c, the solution begins to evolve to

[3] One can prove that if $[a, b]$ is the support of the shape function U, then U is smooth on (a, b). The contact angle is defined via the limit of $U'(\eta)$ taken from within the support. The existence theory for the initial value problem results in solutions that have zero contact angles for almost all times. It is this reason that shape functions with zero contact angles are considered in [18]. The authors also prove that if $n \geq 3/2$, then nonnegative, compactly supported, zero-contact-angle solutions of the form (3) cannot exist. However, this does not mean that the initial value problem cannot exhibit finite time focussing blowup of a selfsimilar type; it simply means that if there were such a blowup, then near the blowup point the solution would be close to (asymptotically matched onto) a selfsimilar solution that has nonzero contact angles. This asymptotic matching needn't reflect the behavior of u_{ss} near its contact line.

blow up in finite time with a focussing singularity. This later evolution is shown in Figure 2.

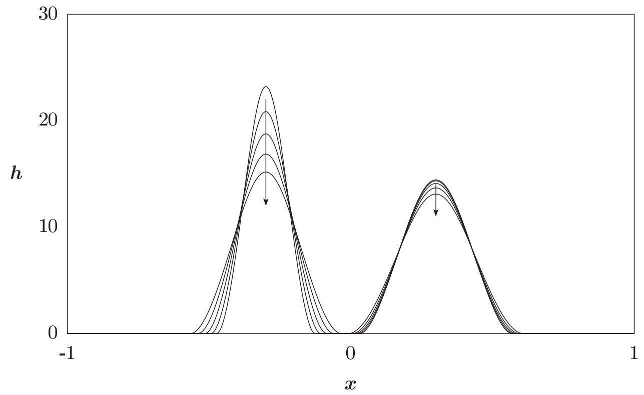

FIGURE 1. A simulation of a solution of the evolution equation (1) in the critical regime ($m = n + 2$), specifically with $n = 1$ and $m = 3$. The initial data are two separated droplets, each of which has mass less than M_c but whose joint mass is greater than M_c. Initially there is short-time, approximately selfsimilar spreading. Figure courtesy of Witelski, Bernoff, and Bertozzi [21].

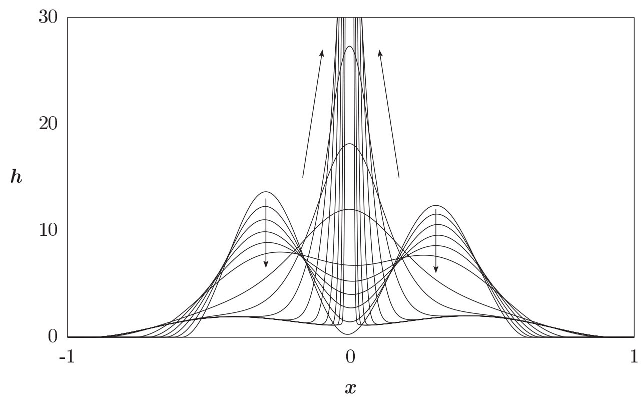

FIGURE 2. This is the sequel to Figure 1. There is subsequent merging and eventual finite-time blowup. Figure courtesy [21].

2.2. ENERGY DISSIPATION

Positive, periodic solutions of the PDE (1) dissipate the energy

$$\mathcal{E}(u(\cdot,t)) := \int \frac{1}{2}u_x^2(x,t) - \frac{u^{m-n+2}(x,t)}{(m-n+2)(m-n+1)}\,dx, \tag{7}$$

where the integral is over one period of the solution. This dissipation holds in all regimes: subcritical, critical, and supercritical. The integrand in (7) has two terms, one positive and one possibly negative, and so it isn't immediately obvious what this dissipation means. It could be that the energy diverges to $-\infty$ in finite time.

In the subcritical regime ($m < n+2$), one can prove that the dissipated energy is directly related to the H^1 norm of the solution. There are constants c_1, c_2, and q such that a positive periodic solution of (1) will satisfy, at each moment in time,

$$\text{(8)} \quad \frac{1}{4}|u|_{H^1}^2 < \mathcal{E}(u) + c_2\overline{u}^q + c_1 + \frac{1}{4}\,\overline{u}^2 < \mathcal{E}(u_0) + c_2\overline{u}^q + c_1 + \frac{1}{4}\,\overline{u}^2 < \infty\,,$$

where $\overline{u} = \int u(x,t)\,dx = \int u_0(x)\,dx$. And so the dissipated energy gives H^1 control of the solution. The construction of nonnegative periodic solutions [4] and of nonnegative, compactly supported solutions on the line [5] involves approximation by positive periodic solutions that solve a problem similar to (1), dissipate an energy similar to (7), and satisfy the inequalities (8). These inequalities for the approximate problem are key in proving the global-in-time existence of nonnegative solutions.

In the critical regime ($m = n+2$), Witelski, Bernoff, and Bertozzi made an observation involving a sharp Sz.-Nagy inequality. In the critical regime, the dissipated energy is

$$\text{(9)} \qquad \mathcal{E}(u(\cdot,t)) = \int \frac{1}{2}u_x^2(x,t) - \frac{u^4(x,t)}{12}\,dx.$$

A sharp Sz.-Nagy inequality for H^1 functions [15], [19] is

$$\int v^4(x)\,dx \le \frac{9}{4\pi^2}\left(\int |v(x)|\,dx\right)^2 \int v_x^2(x)\,dx.$$

For nonnegative functions $\int |v| = \int v$. This allows one to use the Sz.-Nagy inequality to find a lower bound on the dissipated energy (9). Taking $v(x) = u(x,\cdot)$, one finds that at each moment in time, the positive periodic solutions satisfy

$$\left[\frac{1}{2} - \frac{3}{16\pi^2}\left(\int u_0(x)\,dx\right)^2\right]\int u(x,t)^2\,dx \le \mathcal{E}(u(\cdot,t)).$$

On the lefthand side is a coefficient multiplying $\int u^2$. If this coefficient is positive, then the lower bound on $\mathcal{E}(u(\cdot,t))$ is sufficient to control the approximating problems and to construct both nonnegative periodic solutions and nonnegative, compactly supported solutions on the line that exist for all time. The sign of that coefficient is determined by the mass of the initial data. It is positive if

$$\int u_0(x)\,dx < \frac{2\sqrt{2}}{3}\pi = M_c\,.$$

In this way, we see that if the initial data have mass less than M_c, then the existence theory will result in nonnegative solutions that exist for all time. If the initial mass is greater than M_c, then we don't have any obvious control of the dissipated energy, and finite-time blowup is not ruled out.

This shows that in the critical regime ($m = n + 2$) of the evolution equation (1), a geometric constraint on the initial data ensures global-in-time existence of a solution. It's not sufficient to know that the "usual" norms and energies are initially finite; one needs additional information about the initial data, in this case its mass. This type of situation commonly arises in the critical regimes of those evolution equations that have subcritical, critical, and supercritical regimes. This is one reason why the critical regimes can be so analytically challenging.

These soft methods don't give information on whether M_c is sharp for the evolution. To answer this, one would like to know the answers to two questions:

- Given $0 < \epsilon \ll 1$, can one find initial datum with mass $M_c + \epsilon$ that yields a solution that blows up in finite time?
- Given $0 < \epsilon \ll 1$, can one find initial datum with mass $M_c - \epsilon$ that yields a solution that exists for all time?

The results on the selfsimilar solutions u_{ss} are useful here: the result (5) on the focussing, finite-time blowup solutions makes the answer to the first question, "Yes." Similarly, the result (6) on the source-type shape functions makes the answer to the second question, "Yes." Hence, the mass M_c is sharp.[4]

We close by noting that the compactly supported, zero-contact-angle droplet steady state of [12] has mass M_c and that the shape functions of the source-type and blowup solutions (3) converge to this steady state as $\epsilon \to 0$ [18]. Also, having initial data with mass greater than M_c does not ensure finite-time blowup. A counterexample would be two nonoverlapping, compactly supported, zero-contact-angle droplet steady states.

3. Closing Comments on Long Wave Unstable Thin Film Equations

These notes address the three regimes of the long wave unstable thin film equation (1). While blow-up is not possible in the subcritical cases, this does not mean that interesting behavior isn't present. We turn to the larger class of equations,

$$u_t = -(f(u)u_{xxx})_x - (g(u)u_x)_x,$$

where $f, g \geq 0$ are chosen for physical situations in which surface tension effects compete with intermolecular forces. There are interesting phenomena in the subcritical regime, such as the formation of ultrathin films (regions

[4]One use of exact solutions is that they could yield such a sharpness result even if there were no general existence theory.

in which the liquid film is so very thin it looks like a dry spot), spinodal dewetting, and coarsening [3], [8], [9], [10].

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Department of Mathematics; University of Toronto; Toronto, ON M5S 2E4 Canada

E-mail address: mpugh@math.utoronto.ca