

LECTURES ON PURE SPINORS AND MOMENT MAPS

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1. INTRODUCTION

This article is an expanded version of notes for my lectures at the summer school on ‘Poisson geometry in mathematics and physics’ at Keio University, Yokohama, June 5–9 2006. The plan of these lectures was to give an elementary introduction to the theory of Dirac structures, with applications to Lie group valued moment maps. Special emphasis was given to the *pure spinor* approach to Dirac structures, developed in Alekseev-Xu [7] and Gualtieri [20]. (See [11, 12, 16] for the more standard approach.) The connection to moment maps was made in the work of Bursztyn-Crainic [10]. Parts of these lecture notes are based on a forthcoming joint paper [1] with Anton Alekseev and Henrique Bursztyn.

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2. VOLUME FORMS ON CONJUGACY CLASSES

We will begin with the following **FACT**, which at first sight may seem quite unrelated to the theme of these lectures:

FACT. Let G be a simply connected semi-simple real Lie group. Then every conjugacy class in G carries a canonical invariant volume form.

By definition, a conjugacy class \mathcal{C} is an orbit for the conjugation action,

$$\mathrm{Ad}: G \rightarrow \mathrm{Diff}(G), \quad \mathrm{Ad}(g).a = gag^{-1}.$$

It is thus a smooth Ad-invariant submanifold of G . By the existence of a ‘canonical’ volume form, we mean that there exists an explicit construction, not depending on any further choices.

More generally, the above FACT holds for simply connected Lie groups with a bi-invariant pseudo-Riemannian metric. In the semi-simple case, such a metric is provided by the Killing form. The assumption that G is simply connected may be relaxed as well – the precise condition will be given below. Without any assumption on $\pi_1(G)$, the conjugacy classes may be non-orientable, but they still carry canonical invariant measures.

Exercise 2.1. (a) Show that $\mathrm{SO}(3)$ has a conjugacy class diffeomorphic to $\mathbb{R}P(2)$. This is the simplest example of a non-orientable conjugacy class.

(b) Let G be the conformal group of the real line \mathbb{R} (the group generated by dilations and translations). Show that G does not admit a bi-invariant pseudo-Riemannian metric, and that G has conjugacy classes not admitting invariant measures.

The above FACT does not appear to be well-known. Indeed, it is not obvious how to use the pseudo-Riemannian metric to produce a measure on \mathcal{C} , since the restriction of this metric to \mathcal{C} may be degenerate or even zero. Recall on the other hand that any co-adjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ carries a canonical volume form, the *Liouville form* for the Kirillov-Kostant-Souriau symplectic form ω on \mathcal{O} :

$$(1) \quad \omega(\xi_1^\sharp, \xi_2^\sharp)|_\mu = \langle \mu, [\xi_1, \xi_2] \rangle, \quad \mu \in \mathcal{O}.$$

Here $\xi^\sharp \in \mathfrak{X}(\mathcal{O})$ denotes the vector field generated by $\xi \in \mathfrak{g}$ under the co-adjoint action. Letting $n = \frac{1}{2} \dim \mathcal{O}$, the Liouville form is $\frac{1}{n!} \omega^n$, or equivalently the top degree part of the differential form $\exp \omega$. One is tempted to try something similar for conjugacy classes. Unfortunately, conjugacy classes need not admit symplectic forms, in general:

Exercise 2.2. Show that the group $\mathrm{Spin}(5)$ (the connected double cover of $\mathrm{SO}(5)$) has a conjugacy class isomorphic to S^4 . The 4-sphere does not admit an almost complex structure, hence also no non-degenerate 2-form.

Nevertheless, our construction of the volume form on \mathcal{C} will be similar to that of the Liouville form on coadjoint orbits. Let

$$B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$$

be the Ad-invariant inner product on \mathfrak{g} , corresponding to the bi-invariant pseudo-Riemannian metric on G . There is an Ad-invariant 2-form $\omega \in \Omega^2(\mathcal{C})$, given by the formula

$$(2) \quad \omega(\xi_1^\sharp, \xi_2^\sharp)|_g = B\left(\frac{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}{2} \xi_1, \xi_2\right), \quad g \in \mathcal{C}.$$

This 2-form was introduced by Guruprasad-Huebschmann-Jeffrey-Weinstein in their paper [21] on moduli spaces of flat connections, and plays a key role in the theory of group valued moment maps [4]. Its similarity to the KKS formula (1) becomes evident if we use B to identify \mathfrak{g}^* with \mathfrak{g} : The KKS 2-form is defined by the skew-adjoint operator $\mathrm{ad}_\mu = [\mu, \cdot]$, while the GHJW 2-form is defined by the skew-adjoint operator $\frac{1}{2}(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})$. An important difference is that the GHJW 2-form may well be degenerate. It can even be zero:

Exercise 2.3. Show that the GHJW 2-form vanishes on the conjugacy class \mathcal{C} if and only if the elements of \mathcal{C} square to elements of the center of G . For $G = \mathrm{SU}(2)$, there is one such conjugacy class (besides the central elements themselves): $\mathcal{C} = \{A \in \mathrm{SU}(2) \mid \mathrm{tr}(A) = 0\}$.

To proceed we need a certain differential form on the group G . Let $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ denote the left-, right-invariant Maurer-Cartan forms. Thus $\theta^L = g^{-1}dg$ and $\theta^R = dgg^{-1}$ in matrix representations of G .

Theorem 2.4. *Suppose G is a simply connected Lie group with a bi-invariant pseudo Riemannian metric, corresponding to the scalar product B on \mathfrak{g} . Then there is a well-defined smooth, Ad-invariant differential form $\psi \in \Omega(G)$ such that*

$$(3) \quad \psi_g = \det^{1/2}\left(\frac{\text{Ad}_g + 1}{2}\right) \exp\left(\frac{1}{4}B\left(\frac{1 - \text{Ad}_g}{1 + \text{Ad}_g}\theta^L, \theta^L\right)\right)$$

at elements $g \in G$ such that $\text{Ad}_g + 1$ is invertible.

Note that the 2-form in the exponential becomes singular at points where $\text{Ad}_g + 1$ fails to be invertible. The Theorem ensures that these singularities are compensated by the zeroes of the determinant factor. We can now write down our formula for the volume form on conjugacy classes.

Theorem 2.5. *With the assumptions of Theorem 2.4, the top degree part of the differential form*

$$(4) \quad e^\omega \iota_{\mathcal{C}}^* \psi$$

defines a volume form on \mathcal{C} . Here $\omega \in \Omega^2(\mathcal{C})$ is the GHJW 2-form on \mathcal{C} , and $\iota_{\mathcal{C}}: \mathcal{C} \hookrightarrow G$ denotes the inclusion.

Since ψ is an even form, the Theorem says in particular that $\dim \mathcal{C}$ is even. Although Formula (4) is very explicit, it is not very easy to evaluate in practice. In particular, it is a non-trivial task to work out the top degree part ‘by hand’, and to verify that it is indeed non-vanishing! Also, the complicated formula for ψ may seem rather mysterious at this point.

What I would like to explain, in the first part of these lectures, is that the differential form ψ is a *pure spinor* on G , and that Theorem 2.5 may be understood as the non-degeneracy of a pairing between two pure spinors, $e^{-\omega}$ and $\iota_{\mathcal{C}}^* \psi$ on \mathcal{C} .

Remark 2.6. For the case of a *compact* Lie group, with B positive definite, Theorem 2.5 was first proved in [6], using a cumbersome evaluation of the top degree part of (4). The general case was obtained in [1].

The volume forms on conjugacy classes are not only similar to the Liouville volume form on coadjoint orbits, but are actually generalizations of the latter:

Exercise 2.7. Let K be any Lie group. The semi-direct product $G = \mathfrak{k}^* \rtimes K$ (where K acts on \mathfrak{k}^* by the co-adjoint action) carries a bi-invariant pseudo-Riemannian metric, with associated bilinear form B on $\mathfrak{g} = \mathfrak{k}^* \rtimes \mathfrak{k}$ given by the pairing between \mathfrak{k} and \mathfrak{k}^* . Show that the inclusion $\mathfrak{k}^* \hookrightarrow G$ restricts to a diffeomorphism from any K -coadjoint orbit \mathcal{O} onto a G -conjugacy class \mathcal{C} . Furthermore, the GHJW 2-form on \mathcal{C} equals the KKS 2-form on \mathcal{O} , and the volume form on \mathcal{C} constructed above is just the ordinary Liouville form on \mathcal{O} .

3. CLIFFORD ALGEBRAS AND SPINORS

This Section summarizes a number of standard facts about Clifford algebras and spinors. Further details may be found in the classic monograph [15].

3.1. The Clifford algebra. Let W be a vector space, equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $m = \dim W$. The *Clifford algebra* over $(W, \langle \cdot, \cdot \rangle)$ is the associative unital algebra, linearly generated by the elements $w \in W$ subject to relations

$$w_1 w_2 + w_2 w_1 = \langle w_1, w_2 \rangle 1, \quad w_i \in W.$$

Elements of the Clifford algebra $\text{Cl}(W)$ may be written as linear combinations of products of elements $w_i \in W$. There is a canonical filtration,

$$\text{Cl}(W) = \text{Cl}^{(m)}(W) \supset \cdots \supset \text{Cl}^{(1)}(W) \supset \text{Cl}^{(0)}(W) = \mathbb{R}$$

with $\text{Cl}^{(k)}(W)$ the subspace spanned by products of $\leq k$ generators. The associated graded algebra $\text{gr}(\text{Cl}(W))$ is the exterior algebra $\wedge(W)$.

The Clifford algebra has a \mathbb{Z}_2 -grading compatible with the algebra structure, in such a way that the generators $w \in W$ are odd. With the usual sign conventions for \mathbb{Z}_2 -graded ('super') algebras, the defining relations may be written $[w_1, w_2] = \langle w_1, w_2 \rangle 1$ where $[\cdot, \cdot]$ denotes the super-commutator. A *module over the Clifford algebra* $\text{Cl}(W)$ is a vector space \mathcal{S} together with an algebra homomorphism

$$\varrho: \text{Cl}(W) \rightarrow \text{End}(\mathcal{S}).$$

Equivalently, the module structure is described by a linear map $\varrho: W \rightarrow \text{End}(\mathcal{S})$ such that

$$\varrho(w)\varrho(w') + \varrho(w')\varrho(w) = \langle w, w' \rangle 1$$

for all $w, w' \in W$. A Clifford module \mathcal{S} is called a *spinor module* if it is irreducible, i.e. if there are no non-trivial sub-modules.

3.2. The Pin group. Let $\Pi: \text{Cl}(W) \rightarrow \text{Cl}(W)$ be the parity automorphism of $\text{Cl}(W)$, equal to $+1$ on the even part and to -1 on the odd part. The *Clifford group* $\Gamma(W)$ is the subgroup of the group $\text{Cl}(W)^\times$ of invertible elements, consisting of all x such that the transformation

$$(5) \quad y \mapsto \Pi(x)yx^{-1}$$

of $\text{Cl}(W)$ preserves the subspace $W \subset \text{Cl}(W)$. Let $A_x \in \text{GL}(W)$ denote the induced transformation of W .

Proposition 3.1. *The homomorphism $A: \Gamma(W) \rightarrow \text{GL}(W)$ has kernel \mathbb{R}^\times and range $\text{O}(W)$. Thus, the Clifford group fits into an exact sequence,*

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \Gamma(W) \longrightarrow \text{O}(W) \longrightarrow 1.$$

Exercise 3.2. Show that any $w \in W$ with $B(w, w) \neq 0$ lies in $\Gamma(W)$, with A_w the reflection defined by w . Since any element of $\text{O}(W)$ may be written as a product of reflections (E.Cartan-Dieudonné theorem), conclude that any element in $\Gamma(W)$ is a product $g = w_1 \cdots w_k$ with $B(w_i, w_i) \neq 0$. Use this to prove the above Proposition.

Let $x \mapsto x^\top$ denote the canonical anti-homomorphism of $\text{Cl}(W)$, i.e. $(w_1 \cdots w_k)^\top = w_k \cdots w_1$ for $w_i \in W$. Then $g^\top g \in \mathbb{R}^\times$ for all $g \in \Gamma(W)$. Letting

$$\text{Pin}(W) = \{g \in \Gamma(W) \mid g^\top g = \pm 1\}$$

one obtains an exact sequence,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Pin}(W) \longrightarrow \text{O}(W) \longrightarrow 1.$$

Thus $\text{Pin}(W)$ is a double cover of $\text{O}(W)$. Its restriction to $\text{SO}(W)$ is denoted $\text{Spin}(W)$.

3.3. Lagrangian subspaces. For any subspace $E \subset W$, we denote by E^\perp the space of vectors orthogonal to E . The subspace E is called *Lagrangian* if $E = E^\perp$. Let $\text{Lag}(W)$ denote the *Lagrangian Grassmannian*, i.e. the set of Lagrangian subspaces. If $\text{Lag}(W) \neq \emptyset$, the bilinear form $\langle \cdot, \cdot \rangle$ is called *split*. Since we are working over \mathbb{R} , the non-degenerate symmetric bilinear forms are classified by their signature, and $\langle \cdot, \cdot \rangle$ is split if and only if the signature is (n, n) . That is, $(W, \langle \cdot, \cdot \rangle)$ is isometric to $\mathbb{R}^{n, n}$, the vector space \mathbb{R}^{2n} with the bilinear form

$$\langle e_i, e_j \rangle = \pm \delta_{ij}, \quad i, j = 1, \dots, 2n$$

with a $+$ sign for $i = j \leq n$ and a $-$ sign for $i = j > n$.

Exercise 3.3. For any invertible matrix $A \in \text{GL}(n)$ let

$$E_A = \{(Av, v) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^{n,n}.$$

- (a) Show that E_A is Lagrangian if and only if $A \in \text{O}(n)$.
- (b) Show that every Lagrangian subspace E is of the form E_A for a unique $A \in \text{O}(n)$.

We will assume for the rest of this Section that the bilinear form $\langle \cdot, \cdot \rangle$ on W is split, of signature (n, n) . The exercise shows that the Lagrangian Grassmannian $\text{Lag}(W)$ is diffeomorphic to $\text{O}(n)$. In particular, it is a manifold of dimension $n(n-1)/2$, with *two connected components*.

Exercise 3.4. Show that $E, F \in \text{Lag}(W)$ are in the same component of $\text{Lag}(W)$ if and only if $n + \dim(E \cap F)$ is even.

The orthogonal group $\text{O}(W) \cong \text{O}(n, n)$ acts transitively on $\text{Lag}(W)$, as does its maximal compact subgroup $\text{O}(n) \times \text{O}(n)$. (By Exercise 3.3, already the subgroup $\text{O}(n, 0)$ acts transitively.)

Remark 3.5. Compare with the situation in symplectic geometry: If (Z, ω) is a *symplectic* vector space (thus $Z \cong \mathbb{R}^{2n}$ with the standard symplectic form ω), a subspace E is called Lagrangian if it coincides with its ω -orthogonal space E^ω . The symplectic group $\text{Sp}(Z, \omega)$ acts transitively on the set $\text{Lag}(Z)$ of Lagrangian subspaces, as does its maximal compact subgroup $\text{U}(n)$, and $\text{Lag}(Z) = \text{U}(n)/\text{O}(n)$. Thus $\text{Lag}(Z)$ is connected and has dimension $n(n+1)/2$.

For any pair of transverse Lagrangian subspaces E, F , the pairing $\langle \cdot, \cdot \rangle$ defines an isomorphism $F \cong E^*$. Equivalently, one obtains an isometric isomorphism

$$W \cong E \oplus E^*$$

where the bilinear form on the right hand side is defined by extension of the pairing between E and E^* .

Exercise 3.6. Show that for any given $E \in \text{Lag}(W)$, the open subset $\{F \in \text{Lag}(W) \mid E \cap F = 0\}$ is (canonically) an affine space, with $\wedge^2 E$ its space of motions. Show that the closure of this subset is a connected component of $\text{Lag}(W)$. Which of the two components is it?

The sub-algebra of $\text{Cl}(W)$ generated by a Lagrangian subspace $E \in \text{Lag}(W)$ is just the exterior algebra $\wedge E$. Given a Lagrangian subspace F transverse to E , and using the commutator relations to ‘write elements of F to the left’, we see that

$$(6) \quad \text{Cl}(W) = \wedge(F) \otimes \wedge(E),$$

thus $\text{Cl}(W) = \wedge(W)$ as a \mathbb{Z}_2 -graded vector space, and also as a filtered vector space (but not as an algebra). If $w \in W$, the isomorphism intertwines the operator $[w, \cdot]$ (graded commutator) on $\text{Cl}(W)$ with the contraction operators $\iota(w)$ on $\wedge(W)$.

Lemma 3.7. (a) *The Clifford algebra $\text{Cl}(W)$ has no non-trivial two-sided ideals.*
 (b) *For $E \in \text{Lag}(W)$, the left-ideal $\text{Cl}(W)E$ is maximal.*

Proof. We use the following simple fact: If a non-zero subspace of an exterior algebra $\wedge(S)$ is stable under all contraction operators $\iota(u)$, $u \in S^*$, then the subspace contains the scalars.

a) Suppose \mathcal{I} is a proper 2-sided ideal in $\text{Cl}(W)$. Then \mathcal{I} is invariant under all $[w, \cdot]$ with $w \in W$. The above isomorphism (6) takes scalars to scalars, and intertwines $[w, \cdot]$ with contractions. Hence $\mathcal{I} = 0$.

b) Let \mathcal{I} be a proper left-ideal containing $\text{Cl}(W)E$. By the isomorphism (6), we have a direct sum decomposition

$$\text{Cl}(W) = \wedge(F) \oplus \text{Cl}(W)E.$$

On $\wedge(F)$, the operators $[w, \cdot]$ for $w \in E \cong F^*$ coincide with the contractions $\iota(w)$. Since $\mathcal{I} \cap \wedge(F)$ is stable under these operators, it must be zero (or else it would contain the scalars). Thus $\mathcal{I} = \text{Cl}(W)E$. \square

Corollary 3.8. *For any non-zero Clifford module \mathcal{S} over $\text{Cl}(W)$, the action map $\varrho: \text{Cl}(W) \rightarrow \text{End}(\mathcal{S})$ is injective.*

Proof. The kernel of the map ϱ is a 2-sided ideal in $\text{Cl}(W)$, hence it must be zero. \square

3.4. The spinor module. A Clifford module \mathcal{S} over $\text{Cl}(W)$ is called a *spinor module* if it is irreducible, i.e. if there are no non-trivial sub-modules.

Example 3.9. If $E \in \text{Lag}(W)$ is Lagrangian, the quotient $\mathcal{S} := \text{Cl}(W)/\text{Cl}(W)E$ is a spinor module. The irreducibility is immediate from the fact that $\text{Cl}(W)E$ is a maximal left-ideal.

We will see below that all spinor modules over $\text{Cl}(W)$ are isomorphic.

Proposition 3.10. *Let \mathcal{S} be a spinor module, and $E \in \text{Lag}(W)$. Then the subspace*

$$\mathcal{S}^E = \{\phi \in \mathcal{S} \mid \varrho(w)\phi = 0 \ \forall w \in E\}$$

of elements fixed by E is 1-dimensional.

Proof. Let F be a complementary Lagrangian subspace, and choose bases e_1, \dots, e_n of E and f^1, \dots, f^n of F with $B(e_i, f^j) = \delta_i^j$. Define $p \in \text{Cl}(W)$ as a product

$$p = \prod_{i=1}^n e_i f^i = \prod_{i=1}^n (1 - f^i e_i)$$

One easily verifies that p has the following properties:

$$(7) \quad p^2 = p, \quad Ep = 0, \quad pF = 0, \quad p - 1 \in \text{Cl}(W)E.$$

Since $p^2 = p$ the operator $\varrho(p) \in \text{End}(\mathcal{S})$ is a projection operator. By $Ep = 0$ its range lies in \mathcal{S}^E , and by $p - 1 \in \text{Cl}(W)E$ it acts as the identity on \mathcal{S}^E . Hence

$$\mathcal{S}^E = \varrho(p) \mathcal{S}.$$

Since ϱ is injective, we have $\varrho(p) \neq 0$, hence $\mathcal{S}^E \neq 0$. Pick a non-zero element $\phi \in \mathcal{S}^E$. Then $\mathcal{S} = \varrho(\text{Cl}(W))\phi = \varrho(\wedge(F))\phi$ by irreducibility, and since the left ideal $\text{Cl}(W)E$ acts trivially on ϕ . Since $pF = 0$, and hence $p \wedge(F) = \mathbb{R}p$ we obtain

$$(8) \quad \mathcal{S}^E = \varrho(p) \mathcal{S} = \varrho(p) \varrho(\wedge(F)) \phi = \mathbb{R}\varrho(p)\phi = \mathbb{R}\phi.$$

Equation (8) proves that \mathcal{S}^E is 1-dimensional. \square

The kernel and range of any homomorphism of Clifford modules are sub-modules. Hence, any non-zero homomorphism of spinor modules is an isomorphism. In particular, this applies to the action map

$$(9) \quad \text{Cl}(W)/\text{Cl}(W)E \otimes \mathcal{S}^E \rightarrow \mathcal{S}, \quad x \otimes \phi \mapsto \varrho(x)\phi.$$

for any spinor module \mathcal{S} . Thus:

Corollary 3.11. *Any two spinor modules $\mathcal{S}_1, \mathcal{S}_2$ over $\text{Cl}(W)$ are isomorphic. Furthermore, the isomorphism is unique up to non-zero scalar, i.e. the space $\text{Hom}_{\text{Cl}(W)}(\mathcal{S}_1, \mathcal{S}_2)$ is 1-dimensional.*

As a consequence, the projectivization $\mathbb{P}(\mathcal{S})$ of the spinor module is canonically defined (i.e. up to a *unique* isomorphism). In other words, the Clifford algebra $\text{Cl}(W)$ has a unique irreducible *projective* module. The map taking E to \mathcal{S}^E defines a canonical equivariant embedding

$$\text{Lag}(W) \rightarrow \mathbb{P}(\mathcal{S})$$

as an orbit for the action of $\text{O}(W)$. The image can be characterized as follows. Given $\phi \in \mathcal{S}$, let $N_\phi \subset W$ be its ‘null space’,

$$N_\phi = \{w \in W \mid \varrho(w)\phi = 0\}.$$

If $w_1, w_2 \in N_\phi$ then

$$0 = \varrho(w_1)\varrho(w_2)\phi + \varrho(w_2)\varrho(w_1)\phi = \varrho([w_1, w_2])\phi = B(w_1, w_2)\phi.$$

Hence, if $\phi \neq 0$ the subspace N_ϕ is isotropic.

Definition 3.12. [14, 13] A non-zero spinor $\phi \in \mathcal{S}$ is called a *pure spinor* if N_ϕ is Lagrangian. Let $\text{Pure}(\mathcal{S})$ denote the set of pure spinors of \mathcal{S} .

Note that the pure spinors defining a given Lagrangian subspace E are exactly the non-zero elements of the line \mathcal{S}^E . We can summarize the discussion in the following commutative diagram, equivariant for the action of $\text{Pin}(W)$:

$$\begin{array}{ccc} \text{Pure}(\mathcal{S}) & \longrightarrow & \mathcal{S}^\times \\ \downarrow & & \downarrow \\ \text{Lag}(W) & \longrightarrow & \mathbb{P}(\mathcal{S}) \end{array}$$

Exercise 3.13. Show that for any spinor module, the map $\varrho: \text{Cl}(W) \rightarrow \text{End}(\mathcal{S})$ is an isomorphism.

Exercise 3.14. Any maximal left ideal $\mathcal{I} \subset \text{Cl}(W)$ defines a spinor module $\text{Cl}(W)/\mathcal{I}$. Prove that the set of maximal left ideals is canonically isomorphic to $\mathbb{P}(\mathcal{S})$, and that the inclusion of $\text{Lag}(W)$ is just the map $E \rightarrow \mathcal{I} = \text{Cl}(W)E$.

3.5. The bilinear pairing on spinors. Let \mathcal{S} be a spinor module over $\text{Cl}(W)$. Then the dual space \mathcal{S}^* is again a spinor module, with Clifford action given as

$$\varrho_{\mathcal{S}^*}(x) = \varrho_{\mathcal{S}}(x^\top)^*.$$

We obtain a 1-dimensional line $K_{\mathcal{S}} = \text{Hom}_{\text{Cl}(W)}(\mathcal{S}^*, \mathcal{S})$. The evaluation map defines an isomorphism of Clifford modules,

$$\mathcal{S} \cong \mathcal{S}^* \otimes K_{\mathcal{S}}.$$

Tensoring with \mathcal{S} , and composing with the duality pairing $\mathcal{S} \otimes \mathcal{S}^* \rightarrow \mathbb{R}$, we obtain a pairing

$$\mathcal{S} \otimes \mathcal{S} \rightarrow K_{\mathcal{S}}, \quad \phi \otimes \psi \mapsto (\phi, \psi).$$

This pairing was introduced by E. Cartan [14, 13], and more transparently by Chevalley [15]. By construction, the pairing satisfies

$$(\phi, \varrho(x)\psi) = (\varrho(x^\top)\phi, \psi)$$

for all $x \in \text{Cl}(W)$. In particular,

$$(10) \quad (g \cdot \phi, g \cdot \psi) = \pm(\phi, \psi)$$

for all $g \in \text{Pin}(W)$, where the dot indicates the Clifford action. This just follows since $g^\top g = \pm 1$ by definition of the Pin group.

Exercise 3.15. Let $\mathcal{S} = \text{Cl}(W)/\text{Cl}(W)E$. Show that there is a canonical isomorphism $K_{\mathcal{S}} = \wedge^n(E^*)$, where $n = \frac{1}{2} \dim W$. (Hint: \mathcal{S}^* is identified with the submodule of $\text{Cl}(W)$ generated by the line $\wedge^n(E) \subset \text{Cl}(W)$.) Choose a Lagrangian subspace complementary to E to identify $\mathcal{S} = \wedge E^*$. Show that that $(\phi, \psi) = (\phi^\top \wedge \psi)_{[n]}$ using the wedge product in $\wedge E^*$.

Proposition 3.16 (E. Cartan). [13, Section 111] and [15, Theorem III.2.4]. *If ϕ, ψ are pure spinors, then the Lagrangian subspaces N_ϕ, N_ψ are transverse if and only if $(\phi, \psi) \neq 0$.*

Proof. Let $E = N_\phi$. For any Lagrangian complement $F \cong E^*$ to E , there is a unique isomorphism of spinor modules $\mathcal{S} \cong \wedge E^*$ taking ϕ to the pure spinor $1 \in \wedge E^*$. In this model, $K_S = \wedge^n E^*$, and $(\phi, \psi) = \psi_{[n]}$. Suppose that $(\phi, \psi) \neq 0$, thus $\psi_{[n]} \neq 0$. For $w \in E - \{0\}$ we have

$$(\varrho(w)\psi)_{[n-1]} = \iota_w(\psi_{[n]}) \neq 0.$$

It follows that $N_\psi \cap E = \{0\}$. Conversely, if N_ψ is transverse to N_ϕ , we may take $F = N_\psi$. Then $\psi \in \wedge^n F^* - \{0\}$, and in particular $(\phi, \psi) = \psi_{[n]} \neq 0$. \square

In particular, pure spinors in $\mathcal{S} = \text{Cl}(W)/\text{Cl}(W)E$, for any pair of transverse Lagrangian subspaces define a non-zero element of $\wedge^n E^*$, i.e. a volume form on E .

4. LINEAR DIRAC GEOMETRY

The field of Dirac geometry was initiated by T. Courant in [16]. One of the original motivations of this theory was to describe manifolds with ‘pre-symplectic foliations’, arising for instance as submanifolds of Poisson manifolds. The term ‘Dirac geometry’ stems from its relation with the Dirac brackets arising in this context. One of the key features of Dirac geometry is that it treats Poisson geometry and pre-symplectic geometry on an equal footing. More recently, it was observed by Hitchin [22] that complex geometry can be understood in this framework as well, leading to the new field of generalized complex geometry [22, 20].

As in Courant’s original paper, we will first discuss the linear case.

4.1. Linear Dirac structures. Let V be any vector space, and $\mathbb{V} = V \oplus V^*$ equipped with the bilinear form $\langle \cdot, \cdot \rangle$ given by the pairing between V and V^* :

$$\langle v_1 \oplus \alpha_1, v_2 \oplus \alpha_2 \rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle$$

for $v_i \in V$ and $\alpha_i \in V^*$. Specializing the constructions from the last section to the case $W = \mathbb{V}$, we note that \mathbb{V} has two distinguished Lagrangian subspaces, V and V^* . We will call the corresponding spinor modules over $\text{Cl}(\mathbb{V})$,

$$\text{Cl}(\mathbb{V})/\text{Cl}(\mathbb{V})V \cong \wedge V^*, \quad \text{Cl}(\mathbb{V})/\text{Cl}(\mathbb{V})V^* \cong \wedge V$$

the contravariant and covariant spinor modules, respectively. The star operator for any volume form on V defines an isomorphism between these two spinor modules.

Definition 4.1. A linear Dirac structure on a vector space V is a Lagrangian subspace $E \in \text{Lag}(\mathbb{V})$.

As we have seen, a linear Dirac structure E may be described by a line \mathcal{S}^E of pure spinors, using e.g. the covariant or contravariant spinor module.

Examples 4.2. Consider the contravariant spinor representation $\wedge V^*$. Here are some examples of pure spinors ϕ and associated Lagrangian subspaces N_ϕ :

- (a) $\phi = 1$ corresponds to $N_\phi = V$.
- (b) For any 2-form $\omega \in \wedge^2 V^*$, the exponential $\phi = e^{-\omega}$ is a pure spinor, with N_ϕ the graph

$$\text{Gr}_\omega = \{v \oplus \alpha \mid \alpha = \omega(v, \cdot)\}.$$

- (c) Any volume form $\mu \in \wedge^{\text{top}} V^* \setminus \{0\}$ defines a pure spinor $\phi = \mu$, with $N_\phi = V^*$.
- (d) If μ is a volume form and $\pi \in \wedge^2 V$, the element $\phi = e^{-\iota(\pi)\mu}$ (where $\iota: \wedge V \rightarrow \text{End}(\wedge V^*)$ is the algebra homomorphism extending the contraction operators $v \mapsto \iota(v)$) is a pure spinor, with N_ϕ the graph

$$\text{Gr}_\pi = \{v \oplus \alpha \mid v = \pi(\alpha, \cdot)\}.$$

For any Lagrangian subspace $E \subset \mathbb{V} = V \oplus V^*$, define its *range* $\text{ran}(E)$ to be the projection onto V . One observes that $S = \text{ran}(E)$ carries a well-defined 2-form,

$$(11) \quad \omega_S(v_1, v_2) = \langle \alpha_1, v_2 \rangle = -\langle \alpha_2, v_1 \rangle$$

where $v_i \oplus \alpha_i \in E$ are lifts of $v_i \in \text{ran}(E)$. The kernel of this 2-form is $\ker \omega_S = \{v \in V \mid (v, 0) \in E\}$. Conversely, E may be recovered from S together with the 2-form $\omega_S \in \wedge^2 S^*$, as $E = \{(v, \alpha) \mid v \in S, \alpha|_S = \omega_S(v, \cdot)\}$.

Let $\text{ann}(S) \subset V^*$ be the annihilator of S , and choose a non-zero element $\mu \in \wedge^{\text{top}} \text{ann}(S) \subset \wedge V^*$.

Exercise 4.3. Show that

$$\phi = e^{-\omega_S} \mu \in \wedge V^*$$

is a pure spinor with $N_\phi = E$. (Here we have chosen an arbitrary extension of ω_S to a 2-form on V . Note that the element ϕ does not depend on this choice.) Conversely, show that every contravariant pure spinor has the form $\phi = e^{-\omega} \mu$, for uniquely given $S, \mu \in \wedge^{\text{top}} \text{ann}(S), \omega \in \wedge^2 S^*$.

Put differently, a contravariant pure spinor is equivalent to a Lagrangian subspace E together with a volume form on $V/\text{ran}(E)$.

Exercise 4.4. Work out a similar description for covariant pure spinors.

4.2. Dirac maps. Let $A: V \rightarrow V'$ be a linear map. We say that two elements $w = v \oplus \alpha \in \mathbb{V}$ and $w' = v' \oplus \alpha' \in \mathbb{V}'$ are *A-related*, and write

$$w \sim_A w'$$

if $v' = A(v)$ and $\alpha = A^*(\alpha')$. Then the pull-back map of contravariant spinors has the property,

$$\varrho(w)(A^* \phi') = A^*(\varrho(w') \phi')$$

for $w \sim_A w'$ and $\phi' \in \wedge(V')^*$. From this, we see that the pull-back of a contravariant pure spinor ϕ' is again a pure spinor *unless* $A^* \phi' = 0$. Hence, if $F' \subset \mathbb{V}'$ is a Lagrangian subspace, and $\mathcal{S}^{F'} \subset \wedge(V')^*$ the pure spinor line in the contravariant spinor module, then $A^* \mathcal{S}^{F'}$ is either zero, or is a pure spinor line corresponding to some Lagrangian subspace $F \subset \mathbb{V}$.

Exercise 4.5. Suppose $\mathcal{S}^F = A^* \mathcal{S}^{F'}$. Show that

$$(12) \quad F = \{w \in \mathbb{V} \mid \exists w' \in F': w \sim_A w'\}.$$

Similarly, in the covariant spinor representation we have

$$\varrho(w')(A_* \chi) = A_*(\varrho(w) \chi)$$

for $w \sim_A w'$ and $\chi \in \wedge V$. Hence, if $E \subset \mathbb{V}$ is a Lagrangian subspace, and \mathcal{S}^E is the pure spinor line in the covariant spinor representation, then $A_*(\mathcal{S}^E)$ is either zero, or is the pure spinor line for a Lagrangian subspace $E' \subset \mathbb{V}'$.

Exercise 4.6. Suppose $\mathcal{S}^{E'} = A_*(\mathcal{S}^E)$. Show that

$$(13) \quad E' = \{w' \in \mathbb{V}' \mid \exists w \in E: w \sim_A w'\}.$$

Definition 4.7. Let V, V' be vector spaces with linear Dirac structures E, E' . A linear map $A: V \rightarrow V'$ is called a *Dirac map* if the spaces E, E' are related by (13). It is a *strong Dirac map* if the induced map $A_*: \mathcal{S} = \wedge(V) \rightarrow \mathcal{S}' = \wedge(V')$ satisfies $A(\mathcal{S}^E) = (\mathcal{S}')^{E'}$.

Strong Dirac maps are also called *Dirac realizations* in the literature.

Exercise 4.8. Show that a Dirac map A is a strong Dirac map if and only if $E \cap (\ker(A) \oplus 0) = 0$.

Example 4.9. Let $\pi \in \wedge^2 V, \pi' \in \wedge^2 V'$ be 2-forms, with $\pi' = A_* \pi$. Then the map A is a strong Dirac map relative to $E' = \text{Gr}_{\pi'}$ and $E = \text{Gr}_\pi$.

Example 4.10. Let E be a linear Dirac structure on V . Recall that $S = \text{ran}(E)$ carries a unique 2-form ω , with

$$E = \{v \oplus \alpha \mid v \in S, \alpha|_S = \omega(v, \cdot)\}$$

View (S, ω) as a Dirac space, with Dirac structure defined by the graph $\text{Gr}_\omega \subset S \oplus S^*$. Then the inclusion map $\iota_S: S \rightarrow V$ is a strong Dirac map.

Exercise 4.11. Let V carry the Dirac structure E . Then the collapsing map $V \rightarrow \{0\}$ is (trivially) a Dirac map. Show that it is a strong Dirac map if and only if the 2-form ω on $S = \text{ran}(E)$ is non-degenerate, if and only if $E = \text{Gr}_\pi$ for a bi-vector π .

Suppose $E, F \in \text{Lag}(\mathbb{V})$ are Lagrangian subspaces. Then E, F are transverse if and only if $\langle \psi, \chi \rangle \neq 0$, where $\psi \in \wedge V^*$ is a contravariant pure spinor defining F and $\chi \in \wedge(V)$ is a covariant pure spinor defining E . (This is equivalent to Proposition 3.16.) The following result says that Lagrangian complements may be ‘pulled back’ under strong Dirac maps.

Proposition 4.12. *Suppose $A: V \rightarrow V'$ is a strong Dirac map relative to Lagrangian subspaces $E \subset \mathbb{V}$, $E' \subset \mathbb{V}'$. Let $\psi' \in \wedge(V')^*$ be a covariant pure spinor, with $N_{\psi'} = F'$ transverse to E' . Then $\psi = A^*\psi'$ is non-zero, and $N_\psi = F$ is transverse to E . Equivalently, if ϕ is a contravariant pure spinor with $N_\phi = E$ we have*

$$\langle \phi, A^*\psi' \rangle \neq 0.$$

Proof. Let $\chi \in \wedge V$ be a covariant pure spinor defining E . Then $A_*(\chi) \in \wedge(V')$ is a pure spinor defining E' . Since E', F' are transverse,

$$0 \neq \langle \psi', A_*\chi \rangle = \langle A^*\psi', \chi \rangle.$$

This shows that $\psi = A^*\psi'$ is a pure spinor, with $N_\psi = F$ transverse to E . \square

4.3. The map $\text{O}(V) \rightarrow \text{Lag}(\mathbb{V})$. Suppose now that V itself carries a non-degenerate symmetric bilinear form B . Let \bar{V} denote the same vector space, but with the bilinear form $-B$. There is an isometric isomorphism

$$\kappa: V \oplus \bar{V} \rightarrow \mathbb{V} = V \oplus V^*, \quad v \oplus w \mapsto (v - w) \oplus \frac{1}{2}B(v + w, \cdot).$$

This identifies $\text{O}(V \oplus \bar{V}) \cong \text{O}(\mathbb{V})$, and in particular yields an inclusion of the subgroup $\text{O}(V)$:

$$\text{O}(V) \hookrightarrow \text{O}(\mathbb{V}), \quad A \mapsto A^\kappa = \kappa \circ \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1}.$$

If we use B to identify V and V^* , the matrix on the right is easily computed to be,

$$(14) \quad A^\kappa = \begin{pmatrix} (A + I)/2 & (A - I) \\ (A - I)/4 & (A + I)/2 \end{pmatrix}.$$

Its action on V describes a new Lagrangian subspace,

$$F = A^\kappa(V).$$

Let $\Gamma(V) \rightarrow \Gamma(\mathbb{V})$ be the inclusion of Clifford groups defined by the homomorphism $\text{Cl}(V) \subset \text{Cl}(V \oplus \bar{V}) \cong \text{Cl}(\mathbb{V})$. This lifts the map $\text{O}(V) \rightarrow \text{O}(\mathbb{V})$, and restricts to a homomorphism of Pin groups. For any lift $\tilde{A} \in \Gamma(V)$ of A , we obtain a lift $\tilde{A}^\kappa \in \Gamma(\mathbb{V})$ of A^κ . The pure spinor $1 \in \wedge(V^*)$ defines the Lagrangian subspace $V \subset \mathbb{V}$, hence

$$\psi = \varrho(\tilde{A}^\kappa) 1$$

represents $F = A^\kappa(V)$. The situation is described in the following diagram:

$$\begin{array}{ccc} \Gamma(V) & \longrightarrow & \text{Pure}(\mathbb{V}) \\ \downarrow & & \downarrow \\ \text{O}(V) & \longrightarrow & \text{Lag}(\mathbb{V}) \end{array}$$

where the lower map is $A \mapsto A^\kappa(v)$ and the upper map is $\tilde{A} \mapsto \varrho(\tilde{A}^\kappa)(1)$. Since $\text{Pin}(V) \subset \Gamma(V)$ is a double cover of $\text{O}(V)$, there is a lift $\tilde{A} \in \text{Pin}(V)$ that is unique up to sign. One has an explicit formula for the resulting ψ , valid for $\det(A + I) \neq 0$:

$$(15) \quad \psi = \det^{1/2}\left(\frac{A+I}{2}\right) \exp\left(\frac{1}{4} \sum_i \left(\frac{I-A}{I+A} v_i\right) \wedge v^i\right).$$

See [5] for a proof. Here we have used B to identify $V^* \cong V$, and v_i, v^i are bases with $B(v_i, v^j) = \delta_i^j$. The sign of the square root depends on the choice of lift \tilde{A} .

Similarly, the action of A^κ on V^* defines a Lagrangian subspace transverse to F ,

$$E = A^\kappa(V^*).$$

Given an orientation on V , the associated Riemannian volume form μ is a pure spinor defining V^* , hence

$$\phi = \varrho(\tilde{A}^\kappa) \mu$$

is a pure spinor defining E .

Remark 4.13. If the scalar product B on V is definite, the inclusion $\text{O}(V) \rightarrow \text{Lag}(V)$ is a bijection: This is exactly the diffeomorphism $\text{Lag}(V) \cong \text{O}(n)$ mentioned earlier. (This isomorphism is described in the paper [16] under the name ‘generalized Cayley transform’.) Similarly, the map $\Gamma(V) \rightarrow \text{Pure}(\mathbb{V})$, $g \mapsto \varrho(g)1$ defines a bijection of the set of pure spinors with the Clifford group:

$$\text{Pure}(\mathbb{V}) \cong \Gamma(n) := \Gamma(\mathbb{R}^n).$$

5. THE CARTAN-DIRAC STRUCTURE

5.1. Almost Dirac structures. It is straightforward to generalize the above considerations from vector spaces to vector bundles, and in particular to the tangent bundle of a manifold. Thus, let

$$\mathbb{T}M = TM \oplus T^*M$$

be the generalized tangent bundle, with fiberwise inner product $\langle \cdot, \cdot \rangle$ given by the pairing of 1-forms with vector fields, and $\text{Cl}(\mathbb{T}M)$ the corresponding bundle of Clifford algebras. Covariant spinors are multi-vector fields, $\chi \in \mathfrak{X}^\bullet(M) = \Gamma(M, \wedge TM)$, while contravariant spinors are differential forms, $\phi \in \Omega(M) = \Gamma(M, \wedge(T^*M))$.

An *almost Dirac structure* on M is a Lagrangian sub-bundle $E \subset \mathbb{T}M$. (In Section 6, we will discuss the integrability condition turning an almost Dirac structure to a Dirac structure.) A smooth map between almost Dirac manifolds $f: M \rightarrow M'$ is called a (strong) Dirac map if each tangent map $(df)_x$ is a (strong) Dirac map.

Any almost Dirac structure may be described (at least locally) by a contravariant pure spinor $\phi \in \Omega(M)$, or by a covariant pure spinor $\chi \in \mathfrak{X}(M)$. Our basic examples for vector spaces carry over to manifolds: Any 2-form on a manifold defines an almost Dirac structure, as does any bi-vector field. If E is an almost Dirac structure, described (locally) by a pure spinor ϕ , and $\tau \in \Omega^2(M)$ any 2-form, one may define a new almost Dirac structure E^τ described locally by pure spinor $e^{-\tau}\phi$. One calls E^τ the *gauge transformation* of E by the 2-form τ . For instance, taking $E = TM$, one obtains the graph of τ :

$$(TM)^\tau = \text{Gr}_\tau.$$

Exercise 5.1. In general, show that E^τ is the image of E under the automorphism $v \oplus \alpha \mapsto v \oplus (\alpha + \iota_v \tau)$ of $\mathbb{T}M$.

Given a pseudo-Riemannian metric on M , any section A of the group bundle $O(TM)$ defines a pair of transverse Lagrangian sub-bundles

$$E = A^\kappa(T^*M), \quad F = A^\kappa(TM)$$

of TM . A lift \tilde{A} to a section of $\text{Pin}(TM)$ defines pure spinors ϕ, ψ corresponding to E, F , where ϕ depends on the choice of an orientation on M .

5.2. The case $M = G$. Let G be a Lie group. For any $\xi \in \mathfrak{g}$, let $\xi^L, \xi^R \in \mathfrak{X}^1(G)$ the corresponding left-,right-invariant vector fields. The bundle $\text{GL}(TG)$ has a unique section A with the property

$$A(\xi^L) = \xi^R.$$

Suppose G carries a bi-invariant pseudo-Riemannian metric, and let B the corresponding inner product on the Lie algebra \mathfrak{g} . Then A is an $\text{Ad}(G)$ -invariant section of $O(TG) \subset \text{GL}(TG)$. Hence, it determines transverse $\text{Ad}(G)$ -invariant Lagrangian sub-bundles $E, F \subset TG$. Recall that $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ denote the Maurer-Cartan forms.

Proposition 5.2. *Define bundle maps $e, f: \mathfrak{g} \rightarrow TG$ by*

$$\begin{aligned} e(\xi) &= (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right) \\ f(\xi) &= \left(\frac{\xi^L + \xi^R}{2}\right) \oplus B\left(\frac{\theta^L - \theta^R}{4}, \xi\right) \end{aligned}$$

The maps e, f are injective, and have range E, F .

Proof. Under left-trivialization of the tangent bundle $TG = G \times \mathfrak{g}$, the section A is just the adjoint action, $g \rightarrow \text{Ad}_g$. Hence, the section A^κ is given by (14), with A replaced by Ad_g . Writing elements $\xi_1 \oplus \xi_2 \in \mathfrak{g} \oplus \mathfrak{g} \cong T_g G \oplus T_g^* G$ as column vectors, we see that

$$A^\kappa(\xi_1 \oplus \xi_2) = f(\xi_1) + e(\xi_2).$$

□

The sub-bundle E is called the *Cartan-Dirac structure*. (It satisfies the integrability condition discussed below.) Since $\xi^\sharp = \xi^L - \xi^R$ are the generating vector fields for the conjugation action, the generalized distribution $\text{ran}(E) = \text{pr}_{TM}(E)$ is just the distribution tangent to the conjugacy classes \mathcal{C} of G . Hence, by (11) the conjugacy classes $\mathcal{C} \subset G$ acquire $\text{Ad}(G)$ -invariant 2-forms ω .

Proposition 5.3. *The 2-forms ω on conjugacy classes \mathcal{C} are exactly the GHJW 2-forms.*

We leave the proof as an exercise. Equivalently, the inclusion maps

$$\iota_{\mathcal{C}}: \mathcal{C} \hookrightarrow G$$

are strong Dirac maps, relative to the (almost) Dirac structures given by the GHJW 2-form ω on \mathcal{C} and the Lagrangian sub-bundle $E \subset TG$.

Let us now *assume* that the adjoint action $\text{Ad}: G \rightarrow O(\mathfrak{g})$ lifts to a group homomorphism $\widetilde{\text{Ad}}: G \rightarrow \text{Pin}(\mathfrak{g})$ into the Pin group. (This is automatic if G is simply connected.) The lift $\widetilde{\text{Ad}}$ determines lifts $\tilde{A}, \tilde{A}^\kappa$. Hence it defines an invariant pure spinor $\tilde{\psi} = \varrho(\tilde{A}^\kappa)1$ with $N_{\tilde{\psi}} = F$, and given an invariant volume form μ it also defines a pure spinor $\phi = \varrho(\tilde{A}^\kappa)\mu$ with $N_\phi = E$.

Consider now a conjugacy class \mathcal{C} , with GHJW 2-form ω . By Lemma 4.12, the pull-back $\iota_{\mathcal{C}}^*\tilde{\psi}$ is a pure spinor defining a Lagrangian sub-bundle transverse to Gr_ω . Equivalently, the pairing between the two pure spinors $e^{-\omega}, \iota_{\mathcal{C}}^*\tilde{\psi}$ is non-vanishing, that is

$$0 \neq (e^{-\omega}, \iota_{\mathcal{C}}^*\tilde{\psi}) = (e^\omega \iota_{\mathcal{C}}^*\tilde{\psi})_{[\text{top}]}$$

is a volume form on \mathcal{C} . We have shown the following more precise version of the **FACT**:

Theorem 5.4. *Suppose that the adjoint action $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ lifts to a homomorphism $\widetilde{\text{Ad}}: G \rightarrow \text{Pin}(\mathfrak{g})$, and let $\psi \in \Omega(G)$ be the pure spinor defined by such a lift. Then, for any conjugacy class \mathcal{C} in G the top degree part of*

$$e^\omega \iota_{\mathcal{C}}^* \psi$$

defines an invariant volume form on \mathcal{C} .

The explicit formula (3) for $\psi = \tilde{A}^\kappa \cdot 1$ is obtained as a special case from (15):

Proposition 5.5. *If G is connected and the adjoint action $G \rightarrow \text{SO}(\mathfrak{g})$ lifts to a group homomorphism $G \rightarrow \text{Spin}(\mathfrak{g})$, the formula (3) defines a pure spinor ψ with $N_\psi = F$.*

Up to a scalar function the expression (3) for ψ can be directly obtained, as follows:

Exercise 5.6. Over the set where $\text{Ad}_g + 1$ is invertible, the vector fields $\frac{\xi^L + \xi^R}{2}$ span the tangent space. Hence, there is a unique 2-form ς on this set, with

$$\iota\left(\frac{\xi^L + \xi^R}{2}\right)\varsigma + B\left(\frac{\theta^L - \theta^R}{4}, \xi\right) = 0.$$

Deduce that e^ς is a pure spinor defining F , hence coincides with ψ up to a scalar function. Next, check that

$$\varsigma = \frac{1}{4}B\left(\frac{1 - \text{Ad}_g}{1 + \text{Ad}_g}\theta^L, \theta^L\right)$$

is the unique solution of the defining equation for ς .

6. DIRAC STRUCTURES

6.1. Courant's integrability condition. One of Courant's main discoveries in [16] was the existence of a natural integrability condition for almost Dirac structures $E \subset \mathbb{T}M$. Following Alekseev-Xu [7] and Gualtieri [20], we will express the Courant integrability condition in terms of the spinor representation. The Lagrangian sub-bundle E defines a filtration on the spinor module $\Omega(M)$:

$$\Omega(M) = \Omega^{(n)}(M) \supset \dots \supset \Omega^{(1)}(M) \supset \Omega^{(0)}(M).$$

Here $\Omega^{(k)}(M)$ consists of differential forms γ with $\rho(w_0) \cdots \rho(w_k)\gamma = 0$ for all $w_i \in \Gamma(E)$.

Let us fix a closed 3-form $\eta \in \Omega^3(M)$ (possibly zero). Note that $d + \eta$ is again a differential.

Lemma 6.1. *Let $\phi \in \Omega(M)$ be a (locally defined) pure spinor with $N_\phi = E$. Then*

$$(d + \eta)\phi \in \Omega^{(3)}(M).$$

Proof. Let $w_i \in \Gamma(E)$. Since $\rho(w_i)$ annihilates ϕ , we have

$$(16) \quad \rho(w_1)\rho(w_2)\rho(w_3)(d + \eta)\phi = [\rho(w_1), [\rho(w_2), [\rho(w_3), d + \eta]]]\phi,$$

using graded commutators of operators on $\Omega(M)$. A calculation (cf. Exercise 6.8 below) shows that the triple commutator of operators is multiplication by a smooth function. Thus $\rho(w_1)\rho(w_2)\rho(w_3)(d + \eta)\phi$ is a function times ϕ , and hence is annihilated by $\rho(w_0)$. \square

We may now state the Courant integrability condition.

Definition 6.2. An almost Dirac structure $E \subset \mathbb{T}M$ is called integrable relative to the closed 3-form η if, for any (locally defined) pure spinor ϕ with $N_\phi = E$,

$$(17) \quad \text{gr}^3((d + \eta)\phi) = 0.$$

Note that this condition does not depend on the choice of ϕ , since

$$\mathrm{gr}^3((d + \eta)(f\phi)) = f \mathrm{gr}^3((d + \eta)(\phi)).$$

By (17), E is integrable if and only if $(d + \eta)\phi \in \Omega^{(2)}(M)$. Since ϕ and $(d + \eta)\phi$ have opposite parity, this is in fact equivalent to the condition

$$(18) \quad (d + \eta)\phi \in \Omega^{(1)}(M).$$

Definition 6.3. A *Dirac manifold* is a triple (M, E_M, η_M) , consisting of a manifold M , an almost Dirac structure E_M , and a closed 3-form η_M such that E_M is integrable relative to η_M . A smooth map $\Phi: M \rightarrow M'$ between two Dirac manifolds is called a (*strong*) *Dirac map* if each $d_x\Phi: T_xM \rightarrow T_{\Phi(x)}M'$ is a linear (strong) Dirac map, and in addition

$$\Phi^*\eta_{M'} = \eta_M.$$

Remark 6.4. The integrability condition may be rephrased as $(d + \eta)\phi = \rho(w)\phi$ for *some* section $w \in \Gamma(\mathbb{T}M)$. It is not always possible to choose ϕ in such a way that $(d + \eta)\phi = 0$. As shown by Alekseev-Xu [7], the obstruction is the ‘modular class’ of E .

6.2. Examples.

Examples 6.5. (a) Let ω be a 2-form and $\phi = e^{-\omega}$. Then $(d + \eta)\phi = (-d\omega + \eta) \wedge \phi$ lies in $\Omega^{(1)}(M)$ if and only if $d\omega = \eta$. From now on, we will view any manifold M with 2-form ω as a Dirac manifold, taking $E_M = \mathrm{Gr}_\omega$. Observe that $\Phi: M \rightarrow \mathrm{pt}$ is a strong Dirac map if and only if ω is symplectic (closed and non-degenerate).

- (b) More generally, if E is integrable with respect to η , and τ is any 2-form, then E^τ is integrable with respect to $\eta + d\tau$.
- (c) Let π be a bi-vector field and μ_M a volume form on M . Then $\phi = e^{-\iota(\pi)}\mu_M$ satisfies

$$(d + \eta)\phi = \iota\left(-\frac{1}{2}[\pi, \pi]_{\mathrm{Sch}} - \pi^\sharp(\eta) + X_\pi + Y_{\pi, \eta}\right)\phi.$$

Here $[\cdot, \cdot]_{\mathrm{Sch}}$ is the Schouten bracket on multi-vector fields, $X_\pi \in \mathcal{X}^1(M)$ is the vector field defined by $d\iota(\pi)\mu_M = -\iota(X_\pi)\mu_M$, π^\sharp is the bundle map from $\wedge T^*M$ to $\wedge TM$, and $Y_{\pi, \eta}$ is the vector field $Y_{\pi, \eta} = \pi^\sharp(\iota(\pi)\eta)$. The Courant integrability condition reduces to the condition

$$\frac{1}{2}[\pi, \pi]_{\mathrm{Sch}} + \pi^\sharp(\eta) = 0,$$

defining a *twisted Poisson structure*. These structures were introduced by Klimcik-Strobl [24] and further studied by Ševera-Weinstein [30]. It was argued by Kosmann-Schwarzbach-Laurent-Gengoux [26, Theorem 6.1] (see also [7, Example 6.2]) that the sum $X_\pi + Y_{\pi, \eta}$ plays the role of the *modular vector field* for a twisted Poisson structure.

- (d) Take $\eta = 0$, and let $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$ be a collection of pointwise linearly independent 1-forms, and $K \subset TM$ be the codimension k distribution given as the intersection of their kernels. Then $\phi = \alpha_1 \wedge \dots \wedge \alpha_k$ is pure spinor defining $E = K \oplus \mathrm{ann}(K)$. The integrability condition $d\phi \in \Omega^{(1)}(M)$ holds if and only if $d\phi = \beta \wedge \phi$ for a 1-form β . This is one version of the standard (Frobenius) integrability condition for distributions.
- (e) The Courant integrability condition has an obvious generalization to complex almost Dirac structures $E \subset TM \otimes \mathbb{C}$. Given an almost complex structure on M , i.e. a linear complex structure on the tangent bundle, $J \in \Gamma(\mathrm{End}(TM))$, $J^2 = -\mathrm{Id}_{TM}$, one obtains a linear complex structure $\mathbb{J} = J \oplus (-J^*) \in \Gamma(\mathrm{End}(TM))$, $\mathbb{J}^2 = -\mathrm{Id}_{TM}$. Let $E \subset TM \otimes \mathbb{C}$ be the $+i$ eigenbundle of \mathbb{J} . It turns out that E is Courant integrable if and only if the almost complex structure J is integrable, i.e. comes from complex coordinate charts with holomorphic transition functions. This is the motivating example for the *generalized complex geometry*, developed by Hitchin [22] and Gualtieri [20].

Exercise 6.6. In Example 6.5(e), give a pure spinor $\phi \in \Omega(M) \otimes \mathbb{C}$ defining E .

Exercise 6.7. (See [18].) Work out a formula for

$$e^{\iota(\pi)} \circ d \circ e^{-\iota(\pi)} = d + [\iota(\pi), d] + \frac{1}{2}[\iota(\pi), [\iota(\pi), d]] + \dots$$

(The resulting expression contains terms at most quadratic in π .) Use this to show

$$d(e^{-\iota(\pi)} \mu_M) = \iota(-\frac{1}{2}[\pi, \pi]_{\text{Sch}} + X_\pi) \mu_M$$

for any volume form μ_M . Similarly show that

$$\eta \wedge (e^{-\iota(\pi)} \mu_M) = \iota(-\pi^\sharp(\eta) + Y_{\pi, \eta})(e^{-\iota(\pi)} \mu_M).$$

Exercise 6.8. (a) Verify that the following formula defines a bilinear map $[\cdot, \cdot]: \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \rightarrow \Gamma(\mathbb{T}M)$:

$$\rho([\![w_1, w_2]\!]) = [\rho(w_1), [\rho(w_2), d + \eta]].$$

This is the definition of the (non skew-symmetric) *Courant bracket* $[\![\cdot, \cdot]\!]$ on $\Gamma(\mathbb{T}M)$ as a *derived bracket*. See Roytenberg [29], Alekseev-Xu [7] and Kosmann-Schwarzbach [25].

(b) Conclude that for any $w_1, w_2, w_3 \in \Gamma(\mathbb{T}M)$, the operator

$$[\rho(w_1), [\rho(w_2), [\rho(w_3), d + \eta]]]$$

on $\Omega(M)$ is multiplication by the smooth function $Y(w_1, w_2, w_3)$.

(c) Show that for any almost Dirac structure $E \subset \mathbb{T}M$, the restriction of Y to sections of E defines an anti-symmetric tensor $Y_E \in \wedge^3 E^*$.

Proposition 6.9. *The almost Dirac structure E is integrable if and only if $\Gamma(E)$ is closed under Courant bracket $[\![\cdot, \cdot]\!]$. In this case, the restriction $[\![\cdot, \cdot]\!]_E$ of the Courant bracket to $\Gamma(E)$ defines a Lie algebroid structure on E : That is, it is a Lie bracket, the projection map $a: \Gamma(E) \rightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism, and*

$$[w_1, fw_2]_E = f[w_1, w_2]_E + v_1(f)w_2, \quad w_i \in \Gamma(E)$$

where $v_1 = a(w_1)$.

Proof. Since E is Lagrangian, we have $[\![w_2, w_3]\!] \in \Gamma(E)$ for all $w_2, w_3 \in \Gamma(E)$ if and only if

$$Y(w_1, w_2, w_3) = [\rho(w_1), \rho([\![w_2, w_3]\!])] = \langle w_1, [\![w_2, w_3]\!] \rangle = 0$$

for all $w_1, w_2, w_3 \in \Gamma(E)$. The remaining claims are left as an exercise. \square

The theory of Lie algebroids [17, 28] shows that the generalized distribution $\text{ran}(E) = \text{pr}_{TM}(E)$ is *integrable*, i.e. defines a generalized foliation. Moreover, the leaves $S \subset M$ of this foliation carry *2-forms* $\omega_S \in \Omega^2(S)$, defined pointwise by (11).

For $E = \text{Gr}_\pi$ the graph of a Poisson bi-vector field (i.e. $\eta = 0$), this is just the usual foliation by symplectic leaves $S \subset M$, with ω_S the symplectic 2-forms. More generally, in the twisted Poisson case $\frac{1}{2}[\pi, \pi]_{\text{Sch}} + \pi^\sharp \eta = 0$ one still obtains a foliation. The 2-forms on the leaves are again non-degenerate (since $E \cap TM = \{0\}$), but are not closed in general:

Proposition 6.10. *Let $E \subset \mathbb{T}M$ be a Dirac structure (relative to a closed 3-form $\eta \in \Omega^3(M)$). Then the 2-forms ω_S on the leaves $S \subset M$ satisfy $d\omega_S = \iota_S^* \eta$.*

Proof. Given any point $x \in S$, we may pass to a neighbourhood of x to reduce to the case $M = S \times N$, with ι_S the inclusion as $S \times \{y\}$ for some $y \in N$. View ω_S as a form on $S \times N$, and define $\gamma := e^{\omega_S} \phi$. Then $\gamma|_S$ is a nowhere vanishing section of the top exterior power of $T^*N|_S \cong \text{ann}(S) \subset T^*M|_S$. By assumption, there exists a vector field v and a 1-form α such that $(d + \eta)\phi = \iota(v)\phi + \alpha\phi$. This yields:

$$0 = e^{\omega_S} (d + \eta - \iota(v) - \alpha)\phi = (\eta - d\omega_S + \iota(v)\omega - \alpha)\gamma + (d\gamma - \iota(v)\gamma).$$

Restricting to S , and taking the component in $\Gamma(\wedge^3 T^*S \otimes \wedge^{\text{top}} T^*N|_S)$ we find $(\iota_S^* \eta - d\omega_S)\gamma|_S = 0$. Hence $\iota_S^* \eta = d\omega_S$. \square

6.3. Integrability of the Cartan-Dirac structure. Let us now return to the example of a Lie group G with an invariant inner product B on \mathfrak{g} . Suppose that G admits an invariant orientation and that $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ lifts to the Pin group. Let $\phi, \psi \in \Omega(G)$ be the pure spinors defining the almost Dirac structures E, F . By construction, both ϕ and ψ are Ad-invariant differential forms.

Now let $\eta \in \Omega^3(G)$ be the left-invariant 3-form

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]).$$

Since B is invariant, one may replace θ^L with θ^R in this formula, thus η is also right-invariant. In particular, η is closed (since any bi-invariant differential form on a Lie group is closed). Letting $\xi^\# = \xi^L - \xi^R$ be the generating vector fields for the conjugation action, one finds

$$\iota(\xi^\#)\eta = -d B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

As a consequence, we see that the commutator of $d + \eta$ with the generating sections $e(\xi)$ of E are,

$$[\rho(e(\xi)), d + \eta] = [\iota(\xi^\#) + B\left(\frac{\theta^L + \theta^R}{2}, \xi\right), d + \eta] = L(\xi^\#).$$

(Here $L(X) = [\iota(X), d]$ denotes the Lie derivative in the direction of a vector field X .) It hence follows that

$$\rho(e(\xi))(d + \eta)\phi = [\rho(e(\xi)), d + \eta]\phi = L(\xi^\#)\phi = 0.$$

Thus $(d + \eta)\phi \in \Omega^{(0)}(M)$. Since the parity of $(d + \eta)\phi$ is opposite to that of ϕ , we obtain:

Theorem 6.11. *The pure spinor ϕ satisfies*

$$(d + \eta)\phi = 0.$$

In particular, we see that E is a Dirac structure.

Definition 6.12. The Dirac structure E on G is called the *Cartan-Dirac structure*.

The integrability of E explains our earlier observation that the distribution $\text{ran}(E)$ is just the tangent distribution for the generalized foliation by conjugacy classes. Furthermore, Proposition 6.10 tells us that the GHJW 2-form ω_C on the conjugacy classes satisfies,

$$d\omega_C = \iota_C^* \eta.$$

Remark 6.13. The Cartan-Dirac structure was discovered independently by Anton Alekseev, Pavol Ševera and Thomas Strobl, around the end of the last century.

Remark 6.14. By contrast, the almost Dirac structure F is not integrable. Instead, one has [1]

$$(d + \eta)\psi = \rho(e(\Xi))\psi$$

where $\Xi \in \wedge^3 \mathfrak{g}$ is the ‘structure constants tensor’, and $e(\Xi) \in \Gamma(\wedge^3 E)$ is defined using the extension of $e: \mathfrak{g} \rightarrow \Gamma(E)$ to an algebra homomorphism $\wedge \mathfrak{g} \rightarrow \Gamma(\wedge(E))$.

7. GROUP-VALUED MOMENT MAPS

The theory of G -valued moment maps was introduced in the paper [4]. One of its main applications is that it provides a natural framework for the construction of symplectic forms on moduli spaces of flat connections.

7.1. Definition of q-Hamiltonian G -spaces. Let G be a connected Lie group with a bi-invariant pseudo-Riemannian metric, and let B be the corresponding invariant inner product on \mathfrak{g} . Let M be a manifold. A G -action on M is a group homomorphism $\mathcal{A}: G \rightarrow \text{Diff}(M)$ such that the action map $G \times M \rightarrow M$, $(g, x) \mapsto \mathcal{A}(g).x$ is smooth. Similarly, a \mathfrak{g} -action is a Lie algebra homomorphism $\mathcal{A}: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ such that the map $\mathfrak{g} \times M \rightarrow TM$, $(\xi, x) \mapsto \mathcal{A}(\xi)_x$ is smooth. We will write $\xi^\sharp = \mathcal{A}(\xi)$. For any G -action, the generating vector fields (defined with the appropriate sign) give a \mathfrak{g} -action. Conversely, if M is compact and G is simply connected, any \mathfrak{g} -action integrates to a G -action.

Definition 7.1. [4] A Hamiltonian \mathfrak{g} -space with G -valued moment map is a \mathfrak{g} -manifold M , together with a \mathfrak{g} -invariant 2-form $\omega \in \Omega^2(M)$ and a \mathfrak{g} -equivariant map $\Phi \in C^\infty(M, G)$ such that

- (a) $d\omega = \Phi^*\eta$,
- (b) $\iota(\xi^\sharp)\omega = \Phi^*B(\frac{\theta^L + \theta^R}{2}, \xi)$, $\xi \in \mathfrak{g}$ (Moment map condition.)
- (c) $\ker(\omega_x) = \{\xi^\sharp(x) \mid \text{Ad}_{\Phi(x)} \xi = -\xi\}$, $x \in M$ (Minimal degeneracy condition.)

Remark 7.2. As pointed out in [4], (b) is the simplest G -valued analogue to the defining property for \mathfrak{g}^* -valued moment maps $\Phi_0: M \rightarrow \mathfrak{g}^*$, $\iota(\xi^\sharp)\omega_0 = -d\langle \Phi_0, \xi \rangle$. It follows from the work of Bursztyn-Crainic [10] and Xu [32] (see also [1]) that (c) may be replaced by the more elegant condition,

$$\ker(\omega_x) \cap \ker(d_x\Phi) = 0.$$

The theory of G -valued moment maps was developed in [4], and subsequent papers, in full analogy to the familiar theory of \mathfrak{g}^* -valued moment maps. However, the proofs were much more complicated than in the \mathfrak{g}^* -valued theory, and for technical reasons it was necessary to assume that B is positive definite. Unfortunately, this restriction excludes several interesting examples, such as representation varieties for non-compact semi-simple Lie groups. (The Killing form of such groups is indefinite.) In the following approach to group-valued moment maps via Dirac structures these difficulties are no longer present.

Theorem 7.3 (Bursztyn-Crainic). *Definition 7.1 is equivalent to the following Definition 7.4.*

Definition 7.4. A Hamiltonian \mathfrak{g} -space with G -valued moment map is a manifold M with a 2-form ω , together with a strong Dirac map $\Phi: M \rightarrow G$.

Here M is viewed as a Dirac manifold with $E_M = \text{Gr}_\omega$ and 3-form $\eta_M = d\omega$, while G carries the Cartan-Dirac structure. Recall that $\Phi^*\eta = \eta_M$ as part of the definition of a Dirac map from $(M, \text{Gr}_\omega, d\omega)$ to (G, E, η) .

Note that Definition 7.4 no longer mentions the \mathfrak{g} -action on M , the equivariance of ω and Φ , or the minimal degeneracy property: as shown by Bursztyn-Crainic, all of this comes for free!

Remarks 7.5. One is immediately led to consider arbitrary Dirac manifolds (M, E_M, η_M) together with strong Dirac maps $M \rightarrow G$. As shown by Bursztyn-Crainic, one recovers the theory of q-Poisson manifolds [2, 3]. Definition 7.4 is parallel to the definition of Hamiltonian \mathfrak{g} -spaces with \mathfrak{g}^* -valued moment maps: These may be defined as manifolds M with closed 2-forms ω and strong Dirac maps $\Phi: M \rightarrow \mathfrak{g}^*$. Here \mathfrak{g}^* carries the Dirac structure coming from its Kirillov-Poisson structure. Similarly, Lu's notion [27] of moment maps $\Phi: M \rightarrow G^*$ for Poisson G -actions on symplectic manifolds (where G, G^* are dual Poisson Lie groups) can be phrased in this way.

In most cases of interest, the \mathfrak{g} -action on M exponentiates to an action of G :

Definition 7.6. Let M be a G -manifold, together with a G -invariant 2-form ω and a G -equivariant map $\Phi: M \rightarrow G$. Then (M, ω, Φ) is called a Hamiltonian G -space with G -valued moment map, or simply a q -Hamiltonian G -space, if Φ is a Dirac map, and the \mathfrak{g} -action generated by Φ is the infinitesimal G -action.

Example 7.7. Every conjugacy class $\mathcal{C} \subset G$, equipped with the GHJW 2-form, is a q-Hamiltonian G -space, with moment map the inclusion.

7.2. Volume forms. Definition 7.4 greatly simplifies many of the constructions with G -valued moment maps. For instance, generalizing our arguments for the FACT about conjugacy classes, one obtains the following

Theorem 7.8. [1] *Assume that the homomorphism $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ lifts to the group $\text{Pin}(\mathfrak{g})$, and let $\psi \in \Omega(G)$ be the pure spinor with $N_\psi = F$, defined by this lift $\widetilde{\text{Ad}}$. Let (M, ω, Φ) be a q -Hamiltonian G -space. Then*

$$(e^\omega \Phi^* \psi)_{[\text{top}]} \in \Omega(M)$$

is a G -invariant volume form.

If G is connected, we see that $\dim M$ must be even (since ψ is an even form in this case).

7.3. Products. Let us next consider products of q -Hamiltonian G -spaces.

For ordinary Hamiltonian G -spaces (M_i, ω_i) with moment maps $\Phi_i: M_i \rightarrow \mathfrak{g}^*$, the product is simply the direct product $M_1 \times M_2$ with the diagonal action, the sum of the 2-forms $\omega_1 + \omega_2$ and the sum $\Phi_1 + \Phi_2$ of the moment maps. Similarly, if G is a Poisson Lie group with dual Poisson-Lie group G^* , the product operation for Lu's Hamiltonian G -spaces with G^* -valued moment maps takes the sum of the 2-forms and the pointwise product of the moment maps. In this case, the G -action is a certain twist of the diagonal action, see [19, 27].

These two constructions work because the addition map $\text{Add}: \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, respectively the product map $\text{Mult}: G^* \times G^* \rightarrow G^*$, are Poisson. For G -valued moment maps, the situation is slightly different since the multiplication map $\text{Mult}: G \times G \rightarrow G$, *as it stands*, is not a strong Dirac map if one simply takes the direct product Dirac structure on $G \times G$. Instead, the product operation involves a *gauge transformation*.

Definition 7.9. Let E_M be an almost Dirac structure on M , defined (locally) by a pure spinor ϕ_M , and $\tau \in \Omega^2(M)$ a 2-form. Then the *gauge transformation* E_M^τ is the almost Dirac structure defined (locally) by the pure spinor $e^{-\tau} \phi_M$.

Note that if E_M is integrable with respect to a closed 3-form η_M , then E_M^τ is integrable with respect to $\eta_M + d\tau$. In our case, we need a suitable gauge transformation of $E_{G \times G} := E_G \times E_G$. Let

$$\tau := \frac{1}{2} B(\text{pr}_1^* \theta^L, \text{pr}_2^* \theta^R) \in \Omega^2(G \times G).$$

This 2-form has the property [31],

$$\text{Mult}^* \eta = \text{pr}_1^* \eta + \text{pr}_2^* \eta + d\tau.$$

Theorem 7.10. [1] *The multiplication map $\text{Mult}: G \times G \rightarrow G$ is a strong Dirac map from $(G \times G, E_{G \times G}^\tau)$ to (G, E_G) .*

One may use this result to define the *fusion product* of two q -Hamiltonian G -spaces M_1, M_2 , or more generally to pass to the diagonal action in a q -Hamiltonian $G \times G$ -space M (e.g. $M = M_1 \times M_2$). Indeed let (M, ω, Φ) be a q -Hamiltonian $G \times G$ -space with moment map $\Phi = \Phi_1 \times \Phi_2$. Put

$$\Phi^{\text{fus}} = \Phi_1 \Phi_2, \quad \omega^{\text{fus}} = \omega + (\Phi_1, \Phi_2)^* \tau.$$

Then $(M, \omega^{\text{fus}}, \Phi^{\text{fus}})$, with diagonal G -action, is a q -Hamiltonian G -space. This follows rather easily from Theorem 7.10, since the composition of two strong Dirac maps is again a strong Dirac map.

Remark 7.11. For the case of compact Lie groups, and working with the Definition 7.1, this result was obtained in [4] by a fairly complicated argument. The main difficulty in this approach was to show that ω^{fus} is again minimally degenerate: It is not easy to compute the kernel of ω^{fus} by ‘direct calculation’!

7.4. Exponentials. Let the dual of the Lie algebra \mathfrak{g}^* be equipped with the Kirillov-Poisson structure π . Its graph Gr_π defines a Dirac structure. Use the inner product B to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let $\varpi \in \Omega^2(\mathfrak{g})$ be the 2-form, obtained by applying the de Rham homotopy operator to $\exp^* \eta$. Thus $(\text{Gr}_\pi)^\varpi$ is a Dirac structure relative to the closed 3-form $d\varpi = \exp^* \eta$. Now let $\mathfrak{g}_\natural \subset \mathfrak{g}$ be the open subset where \exp is a local diffeomorphism.

Theorem 7.12. [1] *The restriction of \exp to the subset \mathfrak{g}_\natural is a strong Dirac map, relative to the Dirac structures $(\text{Gr}_\pi)^\varpi$ on \mathfrak{g} and the Cartan-Dirac structure on G .*

Suppose now that (M, ω_0, Φ_0) is an ordinary Hamiltonian G -space (thus ω_0 is a symplectic form, and $\Phi_0: M \rightarrow \mathfrak{g}^* \cong \mathfrak{g}$ a moment map in the usual sense). Let $\omega = \omega_0 + \Phi_0^* \varpi$ and $\Phi = \exp \circ \Phi_0$. Then (M, ω, Φ) is a q -Hamiltonian G -space *provided that* $\Phi_0(M) \subset \mathfrak{g}_\natural$. Again, this just follows from the fact that the composition of two strong Dirac maps is again a strong Dirac map. Conversely, suppose $U \subset \mathfrak{g}_\natural$ is an open subset where \exp is a diffeomorphism (with inverse denoted \log), and (M, ω, Φ) is a q -Hamiltonian G -space. Put $\Phi_0 = \log(\Phi)$ and $\omega_0 = \omega - \Phi_0^* \varpi$. Then ω_0 is symplectic, and (M, ω_0, Φ_0) is a Hamiltonian G -space in the usual sense. For (M, ω_0, Φ_0) one has all the standard results from symplectic geometry, which one may then translate back to the q -Hamiltonian setting. For instance, the Meyer-Marsden-Weinstein reduction theorem for Hamiltonian manifolds (see Bates-Lerman [9] for a very general version) yields:

Proposition 7.13 (Symplectic reduction of q -Hamiltonian manifolds). *Let (M, ω, Φ) be a q -Hamiltonian G -space, with proper moment map. Suppose the action of G is proper, and that e is a regular value of the moment map. Then G acts locally free on $\Phi^{-1}(e)$, and the reduced space*

$$M//G = \Phi^{-1}(e)/G$$

is a symplectic orbifold. (If e is not a regular value, $M//G$ is a stratified symplectic space.)

7.5. Examples.

7.5.1. Homogeneous spaces. Let G be a Lie group with involution $\sigma \in \text{Aut}(G)$, $\sigma^2 = 1$, and consider the symmetric space $M = G/G^\sigma$. Let $\widehat{G} = \mathbb{Z}_2 \times G$ be the semi-direct product, defined using the action of $\mathbb{Z}_2 = \{1, \sigma\}$ on G . Then M may be viewed as the conjugacy class of the element $(\sigma, e) \in \widehat{G}$. Hence, if \mathfrak{g} carries an invariant scalar product which is preserved under the involution, the space M becomes a q -Hamiltonian \widehat{G} -space, with moment map

$$\Phi: M \rightarrow \widehat{G}, \quad gG^\sigma \mapsto (\sigma, g\sigma(g)^{-1}).$$

Since (σ, e) squares to the group unit, Exercise 2.3 shows that the 2-form ω on M is identically zero. Note also that the action of $\mathbb{Z}_2 \subset \widehat{G}$ is trivial, so that the action of \widehat{G} descends to G .

Consider now the fusion product of M with itself. Letting $\Phi_i = \Phi \circ \text{pr}_i: M \times M \rightarrow \widehat{G}$, the map $\Phi = \Phi_1 \Phi_2$ takes values in the subgroup $G \subset \widehat{G}$:

$$\Phi(g_1 G^\sigma, g_2 G^\sigma) = \sigma(g_1) g_1^{-1} g_2 \sigma(g_2^{-1}).$$

Hence $M \times M$ is a q -Hamiltonian G -space.

7.5.2. The double. Any Lie group G may be viewed as a symmetric space for the group $G \times G$, with action $(g_1, g_2).a = g_1 a g_2^{-1}$. Here $\sigma \in \text{Aut}(G \times G)$ is the involutive automorphism $\sigma(g_1, g_2) = (g_2, g_1)$, fixing the diagonal subgroup, and the inclusion of the first factor identifies the quotient $(G \times G)/G$ with G . Hence, given an invariant scalar product on \mathfrak{g} , the example in Section 7.5.1 shows that G is a q -Hamiltonian $\mathbb{Z}_2 \times (G \times G)$ -space. Taking a fusion product of G with itself, we find that $D(G) := G \times G$ is a q -Hamiltonian $G \times G$ -space, with action

$$(g_1, g_2).(a, b) = (g_1 a g_2^{-1}, g_2 b g_1^{-1})$$

and moment map $(a, b) \mapsto (ab, a^{-1}b^{-1})$. The space $D(G)$ is called the *double* of G .

Remark 7.14. The double $D(G)$ is the counterpart, in the q-Hamiltonian category, of the cotangent bundle T^*G in the usual Hamiltonian category. In fact, as observed in Bursztyn-Crainic-Weinstein-Zhu [11] the double $D(G) \rightrightarrows G$ (viewed as a groupoid over G , with source and target maps the two components of the moment map) ‘integrates’ the Dirac manifold G in a similar sense as $T^*G \rightrightarrows \mathfrak{g}^*$ integrates the Poisson manifold \mathfrak{g}^* . Ping Xu [32] presents $D(G) \rightrightarrows G$ as an example of a *quasi-symplectic groupoid*.

Passing to the diagonal action, $G \times G$ becomes a q-Hamiltonian G -space with moment map the group commutator:

$$(a, b) \mapsto aba^{-1}b^{-1}.$$

This is called the *fused double*, denoted $\tilde{D}(G)$. Taking a fusion product of several copies of $\tilde{D}(G)$ with itself, the space G^{2h} becomes a q-Hamiltonian G -space with moment map

$$\Phi: (a_1, b_1, \dots, a_h, b_h) \mapsto \prod_{i=1}^h a_i b_i a_i^{-1} b_i^{-1}.$$

The symplectic quotient $M//G$ is just the representation variety for a closed oriented surface of genus h :

$$M//G = \text{Hom}(\pi_1(\Sigma), G)/G$$

Equivalently, $M//G$ is the moduli space of flat principal G -bundles on Σ . It was shown in [4] that the symplectic structure obtained by this finite-dimensional reduction, coincides with that coming from Atiyah-Bott’s [8] gauge theory construction. More generally, if $\mathcal{C}_1, \dots, \mathcal{C}_r$ are conjugacy classes in G , the symplectic quotient

$$(G^{2s} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r)//G$$

is the moduli space of flat G -bundles over an oriented surface Σ of genus h with r boundary components, with restrictions to the j th boundary component $(\partial\Sigma)_j \cong S^1$ in the given conjugacy classes. (Note $\text{Hom}(\pi_1(S^1), G)/G = G/\text{Ad}(G)$ is the set of conjugacy classes.)

Remark 7.15. We stress that no compactness assumption is needed for these results. In fact, one could even work over the complex numbers, and obtain a complex symplectic structure over the representation variety for a complex Lie group.

7.5.3. Spheres. There are other examples of q-Hamiltonian spaces which are unrelated to moduli spaces, such as various examples of *multiplicity-free* q-Hamiltonian spaces. Let $\text{SU}(n)$ act on \mathbb{C}^n in the standard way, and consider the unit sphere $S^{2n} \subset \mathbb{C}^n \times \mathbb{R}$ with the restricted action. By a result of Hurtubise-Jeffrey-Sjamaar [23], there exists an invariant 2-form ω and an equivariant map $\Phi: S^{2n} \rightarrow \text{SU}(n)$ for which (S^{2n}, ω, Φ) is a q-Hamiltonian $\text{SU}(n)$ -space. (The special case $n = 2$ was discussed in [6].)

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