COURANT ALGEBROIDS AND POISSON GEOMETRY

D. LI-BLAND AND E. MEINRENKEN

ABSTRACT. Given a manifold M with an action of a quadratic Lie algebra $\mathfrak{d}$, such that all stabilizer algebras are co-isotropic in $\mathfrak{d}$, we show that the product $M \times \mathfrak{d}$ becomes a Courant algebroid over M. If the bilinear form on $\mathfrak{d}$ is split, the choice of transverse Lagrangian subspaces $\mathfrak{g}_1, \mathfrak{g}_2$ of $\mathfrak{d}$ defines a bivector field π on M, which is Poisson if $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ is a Manin triple. In this way, we recover the Poisson structures of Lu-Yakimov, and in particular the Evens-Lu Poisson structures on the variety of Lagrangian Grassmannians and on the de Concini-Procesi compactifications. Various Poisson maps between such examples are interpreted in terms of the behaviour of Lagrangian splittings under Courant morphisms.

CONTENTS

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0. Introduction

Let $\mathfrak{d}$ be a (real or complex) Lie algebra, equipped with an invariant symmetric bilinear form of split signature. A pair $\mathfrak{g}_1, \mathfrak{g}_2$ of transverse Lagrangian subalgebras of $\mathfrak{d}$ defines a *Manin triple* $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$. Manin triples were introduced by Drinfeld [10], and are of fundamental importance in his theory of Poisson Lie groups and Poisson homogeneous spaces. In their paper [12], S. Evens and J.-H. Lu found that every Manin triple defines a Poisson structure on the variety X of all Lagrangian subalgebras $\mathfrak{d}$. If $\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ (where $\mathfrak{g}$ carries the Killing form and $\overline{\mathfrak{g}}$ indicates the same Lie algebra with the opposite bilinear form), then one of the irreducible components of X is the de Concini-Procesi 'wonderful compactification' of the adjoint group G integrating $\mathfrak{g}$. More recently, J.-H. Lu and M. Yakimov [19] studied Poisson structures on homogeneous spaces of the form D/Q, where D is a Lie group integrating $\mathfrak{d}$, and Q is a closed subgroup whose Lie algebra is co-isotropic for the inner product on $\mathfrak{d}$. Their results show that if M is a D-manifold such that all stabilizer algebras are co-isotropic, then the Manin triple $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ defines a Poisson structure on M.

In this paper, we put this construction into the framework of Courant algebroids. Suppose M is a manifold with a Lie algebra action $\mathsf{a}\colon M \times \mathfrak{d} \to TM$. We show that the trivial bundle $M \times \mathfrak{d}$ carries the structure of a Courant algebroid, with a as the anchor map, and with Courant bracket extending the Lie bracket on constant sections, *if and only if* all stabilizer algebras are co-isotropic. If this is the case, any Lagrangian subalgebra of $\mathfrak{d}$ defines a Dirac structure, and Manin triples $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ define pairs of transverse Dirac structures. In turn, by a result of Mackenzie-Xu [20], any pair of transverse Dirac structures determines a Poisson bivector π. In this way, we recover the Lu-Yakimov Poisson bivector. Various Poisson maps between examples of this type are interpreted in terms of the behaviour under Courant morphisms.

More generally, we give an explicit formula for the Schouten bracket $[\pi, \pi]$ for the bivector resulting from any pair of transverse Lagrangian subspaces (not necessarily Lie subalgebras). If $\mathfrak{g}$ is a Lie algebra with invariant inner product, then $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ with Lagrangian splitting given by the diagonal $\mathfrak{g}_\Delta = \{(x,x)|\ x \in \mathfrak{g}\}$ and the anti-diagonal $\mathfrak{g}_{-\Delta} = \{(x,-x)|\ x \in \mathfrak{g}\}$ is a typical example. The obvious $\mathfrak{d}$-action on the Lie group G integrating $\mathfrak{g}$ has co-isotropic (in fact Lagrangian) stabilizers, and the bivector defined by $(\mathfrak{d}, \mathfrak{g}_\Delta, \mathfrak{g}_{-\Delta})$ is the quasi-Poisson structure on G described in [2, 3]. In a similar fashion, one obtains a natural quasi-Poisson structure on the variety of Lagrangian subalgebras of $\mathfrak{g} \oplus \overline{\mathfrak{g}}$.

Our theory involves some general constructions with Courant algebroids, which may be of independent interest. Suppose $\mathbb{A} \to N$ is a given Courant algebroid. We introduce the notion of an *action of a Courant algebroid* $\mathbb{A} \to N$ on a manifold M, similar to the Higgins-Mackenzie definition of a Lie algebroid action [15]. A Courant algebroid action is given by two maps $\Phi\colon M \to N$, $\varrho\colon \Phi^*\mathbb{A} \to TM$ with a suitable compatibility condition, and we show that the vector bundle pull-back $\Phi^*\mathbb{A}$ acquires the structure of a Courant algebroid, provided all 'stabilizers' $\ker(\varrho_m)$ are co-isotropic.

The organization of this article is as follows. Section 1 gives a review of Courant algebroids $\mathbb{A} \to M$, Dirac structures, and Courant morphisms. In Section 2 we discuss coisotropic reductions of Courant algebroids, which we then use to define pull-backs $\Phi^!\mathbb{A}$ under smooth maps. Such a pull-back is different from the vector bundle pull-back $\Phi^*\mathbb{A}$, and to define a Courant algebroid structure on the latter we need a Courant algebroid action (Φ, ϱ) with co-isotropic stabilizers. In Section 3 we consider Lagrangian sub-bundles of Courant algebroids. We show that the Courant tensors of Lagrangian sub-bundles behave naturally under 'backward image', and give a simple construction of the bivector π for a Lagrangian splitting as the backward image under a 'diagonal morphism'. We give a formula for the rank of π and compute its Schouten bracket $[\pi, \pi]$. Finally, we present a compatibility condition for Lagrangian splittings relative to Courant morphisms, which guarantees that the underlying map of manifolds is a bivector map. Section 4 specializes the theory to Courant algebroids coming from actions of quadratic Lie algebras, and puts the examples mentioned above into this framework. In Section 5 we consider the Lagrangian splittings of $M \times \mathfrak{d}$ coming from Lagrangian splittings of $\mathfrak{d}$. In the final Section 6 we show how to interpret the basic theory of Poisson Lie groups from the Courant algebroids perspective.

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1. Courant algebroids

1.1. **Definition of a Courant algebroid.** The notion of a *Courant algebroid* was introduced by Liu-Weinstein-Xu [18] to provide an abstract framework for Courant's theory of Dirac structures [8]. The original definition was later simplified by Roytenberg [22] (using work of Dorfman [9]) and Uchino [25] to the following set of axioms. See Kosmann-Schwarzbach [17] for a slightly different version.

Definition 1.1. A *Courant algebroid* over a manifold M is a quadruple $(\mathbb{A}, \langle\cdot,\cdot\rangle, \mathsf{a}, [\![\cdot,\cdot]\!])$, consisting of a vector bundle $\mathbb{A} \to M$, a non-degenerate bilinear form (*inner product*) $\langle\cdot,\cdot\rangle$ on the fibers of $\mathbb{A}$, a bundle map $\mathsf{a}\colon \mathbb{A} \to TM$ called the *anchor*, and a bilinear *Courant bracket* $[\![\cdot,\cdot]\!]$ on the space $\Gamma(\mathbb{A})$ of sections, such that the following three axioms are satisfied:

- c1) $[\![x_1, [\![x_2, x_3]\!]]\!] = [\![[\![x_1, x_2]\!], x_3]\!] + [\![x_2, [\![x_1, x_3]\!]]\!]$,
- c2) $\mathsf{a}(x_1)\langle x_2, x_3\rangle = \langle [\![x_1, x_2]\!],\, x_3\rangle + \langle x_2,\, [\![x_1, x_3]\!]\rangle$,
- c3) $[\![x_1, x_2]\!] + [\![x_2, x_1]\!] = \mathsf{a}^*(\mathrm{d}\langle x_1, x_2\rangle)$,

for all sections $x_1, x_2, x_3 \in \Gamma(\mathbb{A})$. Here a^* is the dual map $\mathsf{a}^*\colon T^*M \to \mathbb{A}^* \cong \mathbb{A}$, using the isomorphism given by the inner product.

Axioms c1) and c2) say that $[\![x, \cdot]\!]$ is a derivation of both the Courant bracket and of the inner product, while c3) relates the symmetric part of the bracket with the inner product. From the axioms, one can derive the following properties of Courant algebroids:

p1) $[\![x_1, fx_2]\!] = f[\![x_1, x_2]\!] + (\mathsf{a}(x_1)f)x_2$,
p2) $\mathsf{a}([\![x_1, x_2]\!]) = [\mathsf{a}(x_1), \mathsf{a}(x_2)]$,
p3) $\mathsf{a} \circ \mathsf{a}^* = 0$,
p4) $[\![x, \mathsf{a}^*(\mu)]\!] = \mathsf{a}^*(\mathcal{L}_{\mathsf{a}(x)}\mu)$,
p5) $[\![\mathsf{a}^*(\mu), x]\!] = -\mathsf{a}^*(\iota_{\mathsf{a}(x)}\mathrm{d}\mu)$

for $f \in C^\infty(M),\ \mu \in \Omega^1(M)$, and with $\mathcal{L}_v, \iota_v$ denoting Lie derivative, contraction relative to a vector field v. The properties p1) and p2) are sometimes included as part of the axioms; their redundancy was first observed by Uchino [25]. The basic examples are as follows:

Examples 1.2. (a) A Courant algebroid over $M = \mathrm{pt}$ is just a *quadratic Lie algebra*, i.e. a Lie algebra $\mathfrak{g}$ with an invariant non-degenerate symmetric bilinear form.

(b) The *standard Courant algebroid* $\mathbb{T}M$ over a manifold M is the vector bundle $\mathbb{T}M = TM \oplus T^*M$, with $\mathsf{a}\colon \mathbb{T}M \to TM$ the projection along T^*M, and with inner product and Courant bracket

$$\langle (v_1, \mu_1), (v_2, \mu_2)\rangle = \iota_{v_2}\mu_1 + \iota_{v_1}\mu_2,$$
$$[\![(v_1, \mu_1), (v_2, \mu_2)]\!] = ([v_1, v_2],\ \mathcal{L}_{v_1}\mu_2 - \iota_{v_2}\mathrm{d}\mu_1),$$

for vector fields $v_i \in \mathfrak{X}(M)$ and 1-forms $\mu_i \in \Omega^1(M)$. More generally, given a closed 3-form $\eta \in \Omega^3(M)$ one defines a Courant algebroid $\mathbb{T}M^{(\eta)}$, by adding an extra term $(0, \iota_{v_2}\iota_{v_1}\eta)$ to the Courant bracket. By a result of Ševera, a Courant algebroid $\mathbb{A} \to M$ is isomorphic to $TM^{(\eta)}$ if and only if it is *exact*, in the sense that the sequence

$$0 \to T^*M \xrightarrow{\mathsf{a}^*} \mathbb{A} \xrightarrow{\mathsf{a}} TM \to 0$$

is exact.

1.2. **Dirac structures.** A subbundle $E \subset \mathbb{A}$ of a Courant algebroid is called a *Dirac structure* if it is *Lagrangian* (i.e. $E^\perp = E$) and if its space $\Gamma(E)$ of sections is closed under the Courant bracket. This then implies that E, with the restriction of the Courant bracket and the anchor map, is a Lie algebroid. For any Dirac structure, the generalized distribution $\mathsf{a}(E) \subset TM$ is integrable in the sense of Stefan-Sussmann, hence it defines a generalized foliation of M.

The lack of integrability of a given Lagrangian subbundle $E \subset \mathbb{A}$ is measured by the *Courant tensor* $\Upsilon^E \in \Gamma(\wedge^3 E^*)$,

$$\Upsilon^E(x_1, x_2, x_3) = \langle x_1, [\![x_2, x_3]\!]\rangle, \quad x_i \in \Gamma(E).$$

(Using the Courant axioms, one checks that the right hand side is tensorial and anti-symmetric in $x_1, x_2, x_3 \in \Gamma(E)$.)

Example 1.3. Let $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ be a bivector on M. Then the graph of π,

$$\mathrm{Gr}_\pi = \{(\iota_\mu\pi, \mu)|\ \mu \in T^*M\} \subset \mathbb{T}M$$

is a Lagrangian subbundle. Conversely, a Lagrangian subbundle of $\mathbb{T}M$ is of the form Gr_π if and only if its intersection with TM is trivial. Using the pairing of TM and Gr_π given by the inner product, we may identify $\mathrm{Gr}_\pi^* \cong TM$ and hence view the Courant tensor of

Gr_π as a section of $\wedge^3 TM$. A direct calculation using the definition of the Courant bracket on $\mathbb{T}M$ shows [8]

$$\Upsilon^{\mathrm{Gr}_\pi} = \tfrac{1}{2}[\pi,\pi] \tag{1}$$

where $[\pi,\pi]$ is the Schouten bracket of multi-vector fields. Thus Gr_π is a Dirac structure if and only if π is a Poisson bivector.

1.3. **Courant morphisms.** Suppose $\mathbb{A},\mathbb{A}'$ are Courant algebroids over M, M'. We denote by $\overline{\mathbb{A}}$ the Courant algebroid $\mathbb{A}$ with the opposite inner product. Given a smooth map $\Phi\colon M\to M'$ let

$$\mathrm{Graph}(\Phi) = \{(\Phi(m),m)|m\in M\}\subset M'\times M$$

be its graph. The notion of a Courant morphism is due to Alekseev-Xu [5], see [7, 21].

Definition 1.4. Let $\mathbb{A},\mathbb{A}'$ be Courant algebroids over M, M'. A *Courant morphism* $R_\Phi\colon \mathbb{A}\dashrightarrow \mathbb{A}'$ is a smooth map $\Phi\colon M\to M'$ together with a subbundle

$$R_\Phi\subset (\mathbb{A}'\times\bar{\mathbb{A}})|_{\mathrm{Graph}(\Phi)}$$

with the following properties:

- m1) R_Φ is Lagrangian (i.e. $R_\Phi^\perp = R_\Phi$).
- m2) The image $(\mathsf{a}'\times\mathsf{a})(R_\Phi)$ is tangent to the graph of Φ.
- m3) If $x_1,x_2\in\Gamma(\mathbb{A}'\times\overline{\mathbb{A}})$ restrict to sections of R_Φ, then so does their Courant bracket $[\![x_1,x_2]\!]$.

The composition of Courant morphisms is given as the fiberwise composition of relations: $R_{\Psi\circ\Phi} = R_\Psi\circ R_\Phi$. (One imposes the usual transversality conditions to ensure that this composition is a smooth subbundle.) For a Courant morphism $R_\Phi\colon\mathbb{A}\dashrightarrow\mathbb{A}'$, and elements $x\in\mathbb{A},\ x'\in\mathbb{A}'$, we write $x\sim_{R_\Phi} x'$ if $(x',x)\in R_\phi$. Similarly, for $x\in\Gamma(\mathbb{A}), x'\in\Gamma(\mathbb{A}')$ we write $x\sim_{R_\Phi} x'$ if $x'\times x$ restricts to a section of R_Φ. The definition of a Courant morphism shows that for all sections $x_i\in\Gamma(\mathbb{A}),\ x_i'\in\Gamma(\mathbb{A}')$

$$x_1\sim_{R_\Phi} x_1',\ x_2\sim_{R_\Phi} x_2' \ \ \Rightarrow\ \ [\![x_1,x_2]\!]\sim_{R_\Phi}[\![x_1',x_2']\!],\ \ \langle x_1,x_2\rangle = \Phi^*\langle x_1',x_2'\rangle.$$

From now on, we will often describe the subbundle R_Φ in terms of the relation $\sim_{R_\Phi}$.

Examples 1.5.

- (a) An automorphism of a Courant algebroid is an invertible morphism $R_\Phi\colon\mathbb{A}\dashrightarrow\mathbb{A}$, with $\Phi=\mathrm{id}_M$ the identity map on M.
- (b) A morphism $\mathbb{A}\dashrightarrow 0$ to the zero Courant algebroid over a point pt is the same thing as a *Dirac structure* in $\mathbb{A}$.
- (c) Suppose $\Phi\colon M\to M'$ is a given map, $\mathbb{A},\mathbb{A}'$ are Courant algebroids over M,M', and $R\subset\mathbb{A}'\times\bar{\mathbb{A}}$ is a Dirac structure whose foliation is tangent to the graph of Φ. Then the restriction $R_\Phi = R|_{\mathrm{Graph}(\Phi)}$ is a Courant morphism. However, not every Courant morphism arises in this way: For instance, the bracket on $\Gamma(R_\Phi)$ need not be skew-symmetric.
- (d) Associated to any smooth map $\Phi\colon M\to M'$ is a *standard morphism* $R_\Phi\colon\mathbb{T}M\dashrightarrow\mathbb{T}M'$, where
$$(v,\mu)\sim_{R_\Phi}(v',\mu')\ \Leftrightarrow\ v'=\Phi_* v,\ \mu=\Phi^*\mu'.$$
- (e) If $\mathfrak{g},\mathfrak{g}'$ are quadratic Lie algebras (viewed as Courant algebroids over a point), a Courant morphism $R\colon\mathfrak{g}\dashrightarrow\mathfrak{g}'$ is a Lagrangian subalgebra of $\mathfrak{g}'\oplus\overline{\mathfrak{g}}$.

Another example is described in the following Proposition.

Proposition 1.6 (Diagonal morphism)**.** *Let* $\mathbb{A} \to M$ *be any Courant algebroid, and* $\mathrm{diag}\colon M \to M \times M$ *the diagonal embedding. There is a Courant morphism*

$$R_{\mathrm{diag}}\colon \mathbb{T}M \dashrightarrow \mathbb{A} \times \overline{\mathbb{A}}$$

given by

$$(v, \mu) \sim_{R_{\mathrm{diag}}} (x, y) \ \Leftrightarrow\ v = \mathsf{a}(x),\ x - y = \mathsf{a}^*\mu.$$

Proof. Write $f(x, \mu) = (x, x - \mathsf{a}^*(\mu), \mathsf{a}(x), \mu)$, so that R_{diag} is spanned by the elements $f(x, \mu)$ with $(x, \mu) \in \mathbb{A} \oplus T^*M$. The subbundle R_{diag} is Lagrangian since

$$\langle f(x, \mu),\ f(x, \mu)\rangle = \langle x, x\rangle - \langle x - \mathsf{a}^*(\mu), x - \mathsf{a}^*(\mu)\rangle - 2\iota_{\mathsf{a}(x)}\mu = 0.$$

Using p3), we see that the image of $f(x, \mu)$ under the anchor map for $\mathbb{A} \times \overline{\mathbb{A} \times \mathbb{T}M}$ is $(\mathsf{a}(x), \mathsf{a}(x), \mathsf{a}(x))$ which is tangent to the graph of diag. Now let $x_1, x_2 \in \Gamma(\mathbb{A})$ and $\mu_1, \mu_2 \in \Gamma(T^*M)$. Then $f(x_i, \mu_i)$ are sections of $\mathbb{A} \times \overline{\mathbb{A} \times \mathbb{T}M}$ that restrict to sections of R_{diag}. Using p4), p5) we obtain

$$[\![f(x_1, \mu_1),\ f(x_2, \mu_2)]\!] = f(x, \mu)$$

with $x = [\![x_1, x_2]\!]$, $\ \mu = \mathcal{L}_{\mathsf{a}(x_1)}\mu_2 - \iota_{\mathsf{a}(x_2)}\mathrm{d}\mu_1$. Hence, the Courant bracket again restricts to a section of R_{diag}. This completes the proof (cf. [7, Remark 2.5]). □

2. Constructions with Courant algebroids

In this section we describe three constructions involving Courant algebroids. We begin by discussing a reduction procedure relative to co-isotropic subbundles. We then use reduction to define restrictions and pull-backs $\Phi^!\mathbb{A}$ of Courant algebroids. Finally, we introduce the notion of a Courant algebroid action, and describe conditions under which the usual pull-back as a vector bundle $\Phi^*\mathbb{A}$ acquires the structure of a Courant algebroid.

2.1. **Coisotropic reduction.** The reduction procedure is frequently used to obtain new Courant algebroids out of old ones. See [6] and [26] for similar constructions for the case of exact Courant algebroids.

Proposition 2.1. *Let* $S \subset M$ *be a submanifold, and* $C \subset \mathbb{A}|_S$ *a subbundle such that*

r1) *C is co-isotropic (i.e. $C^\perp \subset C$),*
r2) $\mathsf{a}(C) \subset TS$, $\ \mathsf{a}(C^\perp) = 0$,
r3) *if $x, y \in \Gamma(\mathbb{A})$ restrict to sections of C, then so does their Courant bracket.*

Then the anchor map, bracket and inner product on C descend to $\mathbb{A}_C = C/C^\perp$, and make $\mathbb{A}_C$ into a Courant algebroid over S. The inclusion $\Phi\colon S \hookrightarrow M$ lifts to a Courant morphism

$$R_\Phi\colon \mathbb{A}_C \dashrightarrow \mathbb{A},\quad y \sim_{R_\Phi} x \ \Leftrightarrow\ x \in C,\ y = q(x)$$

where $q\colon C \to \mathbb{A}_C$ is the quotient map.

Proof. By r1),r2) the inner product and the anchor map descend to $\mathbb{A}_C$. We want to define the bracket on $\mathbb{A}_C$ by

$$[\![q(x|_S), q(y|_S)]\!] = q([\![x, y]\!]|_S),$$

for any sections $x, y \in \Gamma(\mathbb{A})$ that restrict to sections of C. To see that this is well-defined, we must show that the right hand side vanishes if $q(x|_S) = 0$ or if $q(y|_S) = 0$. Equivalently, letting $x, y, z \in \Gamma(\mathbb{A})$ with $x|_S,\ y|_S,\ z|_S \in \Gamma(C)$, we must show that $\langle [\![x, y]\!], z\rangle$ vanishes on S if one of the two sections $x|_S,\ y|_S$ takes values in $C^\perp$. (i) Suppose $y|_S \in \Gamma(C^\perp)$. Then

$$\langle [\![x, y]\!], z\rangle = \mathsf{a}(x)\langle y, z\rangle - \langle y, [\![x, z]\!]\rangle$$

vanishes on S, since $\langle y, z\rangle|_S = 0$ and $\mathsf{a}(x)$ is tangent to S. (ii) Suppose $x|_S \in \Gamma(C^\perp)$. Then

$$\langle [\![x, y]\!], z\rangle = -\langle [\![y, x]\!], z\rangle + \langle \mathrm{d}\langle x, y\rangle, \mathsf{a}(z)\rangle$$

vanishes on S: The first term vanishes by (i), and the second term vanishes since $\langle x, y\rangle|_S = 0$ and consequently $\mathrm{d}\langle x, y\rangle|_S \in (\Gamma(\mathrm{ann}(TS))$. The Courant axioms for $\mathbb{A}_C$ follow from the Courant axioms for $\mathbb{A}$. □

Example 2.2. Suppose $\ker(\mathsf{a}) = C$ is a smooth subbundle of $\mathbb{A}$, as happens for example if the anchor map a is surjective. Then $C^\perp = \mathrm{ran}(\mathsf{a}^*)$, and $\mathbb{A}_C$ is a Courant algebroid with trivial anchor map. That is, $\mathbb{A}_C$ is simply a bundle of quadratic Lie algebras.

Remark 2.3. Assume $S = M$, and let $C \subset \mathbb{A}$ be as in the Proposition. Write $R = R_\Phi$ where $\Phi = \mathrm{id}_M$. Let $R^t\colon \mathbb{A} \dashrightarrow \mathbb{A}_C$ be the transpose of R, i.e. $(y, x) \in R^t \Leftrightarrow (x, y) \in R$. Then $R^t \circ R$ is the identity morphism of $\mathbb{A}_C$, while the composition $T = R \circ R^t\colon \mathbb{A} \dashrightarrow \mathbb{A}$ satisfies $T \circ T = T$. Explicitly,

$$x \sim_T x' \ \Leftrightarrow\ x, x' \in C,\ q(x) = q(x').$$

This is parallel to a construction in symplectic geometry, see Guillemin-Sternberg [14].

2.2. **Pull-backs of Courant algebroids.** Let $\mathbb{A} \to M$ be a given Courant algebroid, with anchor map $\mathsf{a}\colon \mathbb{A} \to TM$.

Proposition 2.4. *Suppose $S \subset M$ is an embedded submanifold, and assume that a is transverse to TS, in the sense that*

$$TM|_S = TS + \mathrm{ran}(\mathsf{a})|_S. \tag{2}$$

Then

$$\mathbb{A}_S := \mathsf{a}^{-1}(TS)/\mathsf{a}^*(\mathrm{ann}(TS \cap \mathrm{ran}(\mathsf{a}))) \tag{3}$$

is a Courant algebroid $\mathbb{A}_S \to S$ called the restriction of $\mathbb{A}$ to S. *One has* $\mathrm{rk}(\mathbb{A}_S) = \mathrm{rk}(\mathbb{A}) - 2\dim M + 2\dim S$. *The inclusion $\Phi\colon S \hookrightarrow M$ lifts to a Courant morphism $\mathbb{A}_S \dashrightarrow \mathbb{A}$.*

Of course, the restriction $\mathbb{A}_S$ as a Courant algebroid is different from the restriction $\mathbb{A}|_S$ as a vector bundle.

Proof. The transversality condition ensures that $C = \mathsf{a}^{-1}(TS)$ is a smooth subbundle. We verify the conditions from Proposition 2.1: r1) and r2) follow since $\ker(\mathsf{a}) \subset C$, hence

$$C^\perp \subset \ker(\mathsf{a})^\perp = \mathrm{ran}(\mathsf{a}^*) \subset \ker(\mathsf{a}) \subset C.$$

Condition r3) follows from p2), and since the Lie bracket of two vector fields tangent to S is again tangent to S. To describe $C^\perp$, note that $\mathsf{a}^*\mu$ is orthogonal to C if and only if μ annihilates all elements of $\mathsf{a}(C) = TS \cap \mathrm{ran}(\mathsf{a})$. Hence $C^\perp = \mathsf{a}^*(\mathrm{ann}(TS \cap \mathrm{ran}(\mathsf{a})))$. Since C is the kernel of a surjective map $\mathbb{A}|_S \oplus TS \to TM|_S$, one obtains $\mathrm{rk}(C) = \mathrm{rk}(\mathbb{A}) - \dim M + \dim S$. Consequently $\mathrm{rk}(C^\perp) = \dim M - \dim S$, and the formula for $\mathrm{rk}(\mathbb{A}_C)$ follows. □

Remark 2.5. The transversality condition (2) may be replaced by the weaker assumption that C is a smooth subbundle. Note also that the transversality condition is automatic if $\mathrm{ran}(\mathsf{a}) = TM$, e.g. for exact Courant algebroids. For this case, restriction of Courant algebroids is discussed in [6, Lemma 3.7] and in [13, Appendix].

Restriction to submanifolds generalizes to pull-back under maps:

Definition 2.6. Suppose $\Phi\colon S \to M$ is a smooth map whose differential $\mathrm{d}\Phi\colon TS \to TM$ is transverse to $\mathsf{a}\colon \mathbb{A} \to TM$. We define a *pull-back Courant algebroid* $\Phi^!\mathbb{A} \to S$ by restricting the direct product $\mathbb{A} \times \mathbb{T}S \to M \times S$ to the graph $\mathrm{Graph}(\Phi) \cong S$. (The shriek notation is used to distinguish $\Phi^! A$ from the pull-back as a vector bundle.)

Explicitly, we have the following description:

Proposition 2.7. *The pull-back Courant algebroid is a quotient* $\Phi^!\mathbb{A} = C/C^\perp$ *where*

$$C = \{(x; v, \mu) \in \mathbb{A} \times \mathbb{T}S,\ (d\Phi)(v) = \mathsf{a}(x)\}.$$

$$C^\perp = \{(\mathsf{a}^*\lambda; 0, \nu)|\ (d\Phi)^*\lambda + \nu \in \mathrm{ann}((d\Phi)^{-1}\,\mathrm{ran}(\mathsf{a}))\}.$$

One has $\mathrm{rk}(\Phi^!\mathbb{A}) = \mathrm{rk}(\mathbb{A}) - 2(\dim M - \dim S)$.

Proof. This is just a special case of Proposition 2.4. Let us nevertheless give some details on the computation of $C^\perp$. As in the proof of Proposition 2.4, $C^\perp$ is contained the range of the dual of the anchor map. Hence, elements of $C^\perp$ are of the form $(\mathsf{a}^*\lambda, (0, \nu))$ with $\lambda \in T^*M$ and $\nu \in T^*S$. Pairing with $(x; v, \mu) \in C$ we obtain the condition

$$\begin{aligned} 0 &= \langle (\mathsf{a}^*\lambda; 0, \nu),\ (x; v, \mu)\rangle \\ &= \langle \lambda, \mathsf{a}(x)\rangle + \langle \nu, v\rangle \\ &= \langle \lambda, \mathrm{d}\Phi(v)\rangle + \langle \nu, v\rangle \\ &= \langle (\mathrm{d}\Phi)^*\lambda + \nu, v\rangle. \end{aligned}$$

Thus, $(\mathsf{a}^*\lambda; 0, \nu) \in C^\perp$ if and only if $(\mathrm{d}\Phi)^*\lambda + \nu \in T^*S$ annihilates all $v \in TS$ with $\mathrm{d}\Phi(v) \in \mathrm{ran}(\mathsf{a})$. $\square$

Note that for $\mathrm{ran}(\mathsf{a}) = TM$ (e.g. exact Courant algebroids) the description of $C^\perp$ simplifies to the condition $(\mathrm{d}\Phi)^*\lambda + \nu = 0$.

Proposition 2.8. *If* $\Phi\colon S \to M$ *is an embedding, with* a *transverse to* TS*, then the pull-back* $\Phi^!\mathbb{A}$ *is canonically isomorphic to the restriction of* $\mathbb{A}$ *to* S.

Proof. Recall that $\mathbb{A}_S = \mathsf{a}^{-1}(TS)/\mathsf{a}^*(\mathrm{ann}(TS \cap \mathrm{ran}(\mathsf{a})))$. The inclusion

$$\psi\colon \mathsf{a}^{-1}(TS) \to (\mathbb{A} \times \mathbb{T}S)|_{\mathrm{Graph}(\Phi)},\quad x \mapsto (x; \mathsf{a}(x), 0),$$

takes values in C, with $\psi(\mathsf{a}^{-1}(TS)) \cap C^\perp = \psi(\mathsf{a}^*(\mathrm{ann}(TS \cap \mathrm{ran}(\mathsf{a}))))$. The resulting inclusion $\mathbb{A}_S \to \Phi^!\mathbb{A}$ is an isomorphism of vector bundles by dimension count, and it clearly preserves inner products. It is an isomorphism of Courant algebroids since for all sections $x_1, x_2 \in \Gamma(\mathbb{A})$, such that $x_1|_S,\ x_2|_S$ take values in $\mathbb{A}_S$, the Courant bracket $x = [\![x_1, x_2]\!]$ satisfies

$$\begin{aligned} [\![(x_1; \mathsf{a}(x_1), 0), (x_2; \mathsf{a}(x_2), 0)]\!]|_{\mathrm{Graph}(\Phi)} &= ([\![x_1, x_2]\!]; [\mathsf{a}(x_1), \mathsf{a}(x_2)], 0)|_{\mathrm{Graph}(\Phi)} \\ &= ([\![x_1, x_2]\!]; \mathsf{a}([\![x_1, x_2]\!]), 0)|_{\mathrm{Graph}(\Phi)} \\ &= \psi([\![x_1, x_2]\!]|_S). \end{aligned}$$

$\square$

Proposition 2.9. *For any smooth map $\Phi\colon S \to M$, one has a canonical isomorphism*

$$\Phi^!(\mathbb{T}M) = \mathbb{T}S.$$

Proof. If $\mathbb{A} = \mathbb{T}M$ the bundle $C \subset (\mathbb{T}M \times \mathbb{T}S)|_{\mathrm{Graph}(\Phi)}$ has the description,

$$C = \{(\mathrm{d}\Phi(v), \lambda;\, v, \mu)|\ \lambda \in T^*M,\ v \in TS\}.$$

The bundle $\mathbb{T}S$ is embedded in C as the subbundle defined by $\lambda = 0$, and it defines a complement to the subbundle $C^\perp$ given by the conditions $v = 0,\ \mu = -(\mathrm{d}\Phi)^*\lambda$. Since the inclusion $\mathbb{T}S \to C$ preserves Courant brackets, this shows $C/C^\perp = \mathbb{T}S$ as Courant algebroids. $\square$

Proposition 2.10. *There is a canonical Courant morphism*

$$P_\Phi\colon \Phi^!\mathbb{A} \dashrightarrow \mathbb{A} \tag{4}$$

lifting $\Phi\colon S \to M$. Explicitly,

$$y \sim_{P_\Phi} x \ \Leftrightarrow\ \exists v \in TS\colon \mathsf{a}(x) = d\Phi(v),\ (x; v, 0) \in C \textit{ maps to } y.$$

Note that as a space, P_Φ is the fibered product of $\mathbb{A}$ and TS over TM. It is a smooth vector bundle over $S \cong \mathrm{Graph}(\Phi)$ since a is transverse to $\mathrm{d}\Phi$ by assumption.

Proof. We factor $\Phi = \Phi_1 \circ \Phi_2$, where $\Phi_2\colon S \to M \times S$ is the inclusion as $\mathrm{Graph}(\Phi)$, and $\Phi_1\colon M \times S \to M$ is projection to the first factor. Since $\Phi^!\mathbb{A} = (\mathbb{A} \times \mathbb{T}S)|_{\mathrm{Graph}(\Phi)}$, there is a canonical Courant morphism $R_{\Phi_2}\colon \Phi^!\mathbb{A} \dashrightarrow \mathbb{A} \times \mathbb{T}S$ as explained in Proposition 2.4. The Lagrangian subbundle R_{Φ_2} consists of elements of the form $(x; v, \mu; y)$ with $(x; v, \mu) \in C$ mapping to $y \in C/C^\perp$. On the other hand, the projection map $\Phi_1\colon M \times S \to M$ lifts to a Courant morphism $R_{\Phi_1}\colon \mathbb{A} \times \mathbb{T}S \dashrightarrow \mathbb{A}$ (given as the direct product of the identity morphism $\mathbb{A} \dashrightarrow \mathbb{A}$ with the standard morphism $\mathbb{T}S \to \mathbb{T}(\mathrm{pt}) = \mathrm{pt}$). Thus R_{Φ_1} is given by elements of the form $(x; x; v, 0)$ with $x \in \mathbb{A}$ and $v \in TS$. Composition gives the Courant morphism $P_\Phi\colon \Phi^!\mathbb{A} \dashrightarrow \mathbb{A}$ as described above. $\square$

Our construction of a pull-back Courant algebroid is similar to the notion of *pull-back Lie algebroid*, due to Higgins-Mackenzie [15]. Suppose $E \to M$ is a Lie algebroid with anchor map $\mathsf{a}\colon E \to TM$, and $\Phi\colon S \to M$ is a smooth map such that a is transverse to $\mathrm{d}\Phi$. One defines

$$\Phi^!E = \{(x, v)|\ \mathrm{d}\Phi(v) = \mathsf{a}(x)\} \subset (E \times TS)|_{\mathrm{Graph}(\Phi)}. \tag{5}$$

By the transversality assumption, this is a subbundle of rank $\mathrm{rk}(\Phi^!E) = \mathrm{rk}(E) - \dim(M) + \dim(S)$. Given two sections of $E \times TS$ whose restriction to $\mathrm{Graph}(\Phi)$ takes values in $\Phi^!E$, then so does their Lie bracket. Furthermore, if one of the two sections vanishes along $\mathrm{Graph}(\Phi)$, then so does their Lie bracket. This defines a Lie bracket on $\Gamma(\Phi^!E)$, making $\Phi^!E$ into a Lie algebroid. If E is a Lagrangian sub-bundle of $\mathbb{A} \to M$, and such that $\mathrm{d}\Phi$ is transverse to both $\mathsf{a},\ \mathsf{a}|_E$, then $\Phi^!E$ is embedded as a Lagrangian sub-bundle of $\Phi^!\mathbb{A}$ by the map $\Phi^!E \to \Phi^!\mathbb{A}$, taking $(x, v) \in E \times TS$ with $\mathsf{a}(x) = \mathrm{d}\Phi(v)$ to the equivalence class $[(x; v, 0)] \in \Phi^!\mathbb{A}$ of $(x; v, 0) \in C$. Clearly, if E is a Dirac structure, then the Courant bracket on $\Phi^!\mathbb{A}$ restricts to the Lie algebroid bracket on $\Phi^!E$, and in particular $\Phi^!E$ is a Dirac structure in $\Phi^!\mathbb{A}$.

2.3. **Actions of Courant algebroids.** The pull-back $\Phi^!\mathbb{A}$ of a Courant algebroid is different from the pull-back as a vector bundle $\Phi^*\mathbb{A}$. In order to define a Courant algebroid structure on $\Phi^*\mathbb{A}$, one needs the additional structure of an action of $\mathbb{A}$ on the map Φ, such that all stabilizers of the action are co-isotropic. Here actions of Courant algebroids may be defined in analogy with the actions of Lie algebroids [15]:

Definition 2.11. Suppose $\mathbb{A} \to M$ is a Courant algebroid. An *action of* $\mathbb{A}$ *on a manifold* S is a map $\Phi\colon S \to M$ together with an *action map* $\varrho\colon \Gamma(\mathbb{A}) \to \mathfrak{X}(S)$ satisfying

$$\begin{aligned} \mathrm{d}\Phi \circ \varrho(x) &= \mathsf{a}(x), \\ [\varrho(x), \varrho(y)] &= \varrho([\![x, y]\!]), \\ \varrho(fx) &= \Phi^* f\ \varrho(x). \end{aligned}$$

for all $x, y \in \Gamma(\mathbb{A})$ and $f \in C^\infty(M)$.

The last condition shows that ϱ defines a vector bundle map (denoted by the same letter)

$$\varrho\colon \Phi^*\mathbb{A} \to TS. \tag{6}$$

For each $s \in S$, the kernel of the map $\varrho_s\colon \mathbb{A}_{\Phi(s)} \to T_sS$ is called the *stabilizer at* s.

Theorem 2.12. *Suppose the Courant algebroid* $\mathbb{A} \to M$ *acts on* S *with co-isotropic stabilizers. Let* $\Phi\colon S \to M$ *and* $\varrho\colon \mathbb{A} \to TS$ *be the maps defining the action. Then the pull-back vector bundle* $\Phi^*\mathbb{A}$ *carries a unique structure of Courant algebroid with anchor map* ϱ, *such that the pull-back map on sections* $\Phi^*\colon \Gamma(\mathbb{A}) \to \Gamma(\Phi^*\mathbb{A})$ *preserves inner products and Courant brackets. If* $E \subset \mathbb{A}$ *is a Dirac structure, then* $\varrho|_E$ *is an action of the Lie algebroid* E *on* S, *and the resulting Lie algebroid structure on* Φ^*E *coincides with that as a Dirac structure in* $\Phi^*\mathbb{A}$.

Proof. Consider the direct product Courant algebroid $\mathbb{A} \times \mathbb{T}S$ over $M \times S$. For $\mu \in \Omega^1(S)$ and $x \in \Gamma(\mathbb{A})$, let

$$f(x, \mu) = (x; \varrho(x), \mu) \in \Gamma(\mathbb{A} \times \mathbb{T}S).$$

The image of $f(x, \mu)$ under the anchor map is the vector field $(\mathsf{a}(x), \varrho(x))$, which is tangent to the graph of Φ since $\mathrm{d}\Phi(\varrho(x)) = \mathsf{a}(x)$. The Courant bracket between two sections of this form is $[\![f(x_1, \mu_1), f(x_2, \mu_2)]\!] = f(x, \mu)$ with $x = [\![x_1, x_2]\!]$ and $\mu = \mathcal{L}_{\varrho(x_1)}\mu_2 - \iota_{\varrho(x_2)}\mathrm{d}\mu_1$. Let $C \subset (\mathbb{A} \times \mathbb{T}S)|_{\mathrm{Graph}(\Phi)}$ be the subbundle spanned by the restrictions of $f(x, \mu)$ to the graph of Φ. This is a co-isotropic subbundle, since its fiber at $(\Phi(s), s)$ contains the co-isotropic subspace $\ker(\rho_s) \times T_s^*S$. Its orthogonal is given by

$$C^\perp = \{(-\varrho^*\nu; 0, \nu) |\ \nu \in T^*S\}.$$

(Indeed, it is easy to check that the elements on the right hand side lie in $C^\perp$. Equality follows since $\mathrm{rk}(C) = \mathrm{rk}(\mathbb{A}) + \dim S$, hence $\mathrm{rk}(C^\perp) = \mathrm{rk}(\mathbb{A} \times \mathbb{T}S) - \mathrm{rk}(C) = \dim S$.) Since the co-isotropic subbundle satisfies all the conditions from Proposition 2.1, the reduced Courant algebroid $\mathbb{A}_C$ over $\mathrm{Graph}(\Phi) \cong S$ is defined. The sections of the form $f(x, 0)$ span a complement to $C^\perp$ in C, and identify $\mathbb{A}_C = \Phi^*\mathbb{A}$ as a vector bundle. This identification also preserves inner products, since $\langle f(x_1, 0), f(x_2, 0)\rangle = \langle x_1, x_2\rangle$. Furthermore, $[\![f(x_1, 0), f(x_2, 0)]\!] = f([\![x_1, x_2]\!], 0)$ shows that the pull-back map on sections $\Gamma(\mathbb{A}) \to \Gamma(\Phi^*\mathbb{A}) = \Gamma(\mathbb{A}_C)$ preserves Courant brackets. If $E \subset \mathbb{A}$ is a Dirac structure, then it is obvious that the action ϱ restricts to a Lie algebroid action of E (i.e. the map

$\Gamma(E) \to \Gamma(TS)$ defined by ϱ preserves brackets). Also, since the Lie algebroid bracket on $\Gamma(\Phi^*E)$ is determined by the Lie bracket on the subspace $\Phi^*\Gamma(E)$, it is immediate that Φ^*E with this bracket is a Dirac structure in $\Phi^*\mathbb{A}$. $\square$

Example 2.13. Suppose $\mathbb{A} \to M$ is a Courant algebroid, and $\Phi\colon S \hookrightarrow M$ is an embedded submanifold, such that $\mathrm{ran}(\mathsf{a})$ is tangent to S. Then the map $\Gamma(\mathbb{A}) \to \Gamma(TS)$, given by restriction to S followed by a, satisfies the axioms. Hence, $\Phi^*\mathbb{A}$ is a well-defined Courant algebroid.

Example 2.14. Let $\mathfrak{g}$ be a quadratic Lie algebra, viewed as a Courant algebroid over a point. Let M be a manifold with a $\mathfrak{g}$-action whose stabilizer algebras are coisotropic. Then the product $M \times \mathfrak{g}$ (viewed as the vector bundle pull-back of $\mathfrak{g} \to \mathrm{pt}$ by the map $M \to \mathrm{pt}$) acquires the structure of a Courant algebroid. This example will be explored in detail in Section 4.

Since the Courant bracket on $\Phi^*\mathbb{A}$ was defined by reduction, there is a Courant morphism $\Phi^*\mathbb{A} \dashrightarrow \mathbb{A} \times \mathbb{T}S$ lifting the inclusion $S \hookrightarrow M \times S$ as the graph of Φ. Its composition with $\mathbb{A} \times \mathbb{T}S \dashrightarrow \mathbb{A}$ (lifting the projection to the first factor) is a Courant morphism

$$U_\Phi\colon \Phi^*\mathbb{A} \dashrightarrow \mathbb{A} \tag{7}$$

lifting Φ.

3. Manin pairs and Manin triples

A pair $(\mathbb{A}, E)$ of a Courant algebroid $\mathbb{A} \to M$ together with a Dirac structure is called a *Manin pair* (over M). Given a second Dirac structure $F \subset \mathbb{A}$ such that $\mathbb{A} = E \oplus F$, the triple $(\mathbb{A}, E, F)$ is called a *Manin triple* (over M). For $M = \mathrm{pt}$, this reduces to the classical notion of a Manin triple $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ as introduced by Drinfeld [10]: A split quadratic Lie algebra $\mathfrak{d}$ with two transverse Lagrangian subalgebras $\mathfrak{g}_1, \mathfrak{g}_2$.

Remark 3.1. As shown by Liu-Weinstein-Xu [18], Manin triples over M are equivalent to *Lie bialgebroids* $A \to M$. Indeed, for any Lie bialgebroid the direct sum $\mathbb{A} = A \oplus A^*$ carries a unique structure of a Courant algebroid such that $(\mathbb{A}, A, A^*)$ is a Manin triple. The more general concept of a *Manin quasi-triple* requires only that E is integrable. It is equivalent to the notion of quasi-Lie bialgebroid, see Kosmann-Schwarzbach [16, 17], Roytenberg [23], and Ponte-Xu [21].

3.1. Backward images.

The following result was obtained in [1, Proposition 2.10] for the case of exact Courant algebroids, with a very different proof.

Proposition 3.2. *Let $R_\Phi\colon \mathbb{A} \dashrightarrow \mathbb{A}'$ be a Courant morphism. Suppose $E' \subset \mathbb{A}'$ is a Lagrangian subbundle, with the property*

$$x' \in E',\ 0 \sim_{R_\Phi} x' \in E' \ \Rightarrow\ x' = 0. \tag{8}$$

Then the backward image

$$E := E' \circ R_\Phi = \{x \in \mathbb{A} |\ \exists x' \in E'\colon x \sim_{R_\Phi} x'\}$$

is a Lagrangian subbundle of $\mathbb{A}$, and there is a unique bundle homomorphism $\alpha\colon E \to \Phi^*E'$ such that $x \sim_{R_\Phi} \alpha(x)$ for all $x \in E$. The Courant tensors of E, E' are related by

$$\Upsilon^E = \alpha^*(\Phi^*\Upsilon^{E'}), \tag{9}$$

where α^* is the map dual to α, extended to the exterior algebras.

Proof. The fact that the backward image is a Lagrangian subbundle is parallel to Guillemin-Sternberg's construction in symplectic geometry [14], hence we will be brief. We have

$$E = E' \circ R_\Phi \cong ((E' \times R_\Phi) \cap C)\big/((E' \times R_\Phi) \cap C^\perp)$$

where $C \subset (\overline{\mathbb{A}}' \times \mathbb{A}' \times \overline{\mathbb{A}})|_{\mathrm{diag}(M') \times M}$ is the co-isotropic subbundle given as

$$C = \{(x', x', x)|\ x \in \mathbb{A},\ x' \in \mathbb{A}'\}.$$

$C^\perp$ is similarly given as the set of all $(x', x', 0)$. The condition $x' \in E'$, $0 \sim_{R_\Phi} x' \Rightarrow x' = 0$ amounts to the transversality property $(E' \times R_\Phi) \cap C^\perp = 0$, ensuring that E is a smooth subbundle. It is easy to see that E is isotropic, and hence Lagrangian by dimension count. The map α associates to each $x \in E_m$ the unique $x' \in E_{\Phi(m)}$ with $x \sim_{R_\Phi} x'$. To prove (9) we must show that

$$\Upsilon^{E'}(x_1', x_2', x_3') - \Upsilon^E(x_1, x_2, x_3) = 0 \tag{10}$$

for all $x_i \in E_m,\ x_i' = \alpha(x_i) \in E'_{\Phi(m)}$. Think of $E' \times E$ as a Lagrangian subbundle of $\mathbb{A}' \times \overline{\mathbb{A}}$. Its Courant tensor is

$$\Upsilon^{E' \times E} = \Upsilon^{E'} - \Upsilon^E$$

as an element of $(\wedge^3 E')^* + \wedge^3 E^* \subset \wedge^3(E' \times E)^*$. Let $\sigma_1, \sigma_2, \sigma_3$ be three sections of $\mathbb{A}' \times \overline{\mathbb{A}}$ restricting to sections of R_Φ. Then

$$\Upsilon^{E' \times E}(\sigma_1, \sigma_2, \sigma_3) = \langle \sigma_1, [\![\sigma_2, \sigma_3]\!]\rangle$$

vanishes along the graph of Φ: Indeed $[\![\sigma_2, \sigma_3]\!]$ restricts to a section of R_Φ, hence its inner product with σ_1 restricts to 0. Choosing the σ_i such that $\sigma_i|_{(\Phi(m), m)} = (x_i', x_i)$ this gives (10). ☐

As a special case, we see that if $(\mathbb{A}', E')$ is a Manin pair, and E is the backward image of E' under a Courant morphism $R_\Phi \colon \mathbb{A} \to \mathbb{A}'$ satisfying (8), then $(\mathbb{A}, E)$ is a Manin pair.

Example 3.3. Let $\mathbb{A} \to M$ be a Courant algebroid and $\Phi \colon S \to M$ a smooth map, with $\mathrm{d}\Phi$ transverse to the anchor map a. Let $P_\Phi \colon \Phi^! \mathbb{A} \dashrightarrow \mathbb{A}$ be the Courant morphism defined in Section 2.2, and suppose $E \subset \mathbb{A}$ is a Lagrangian subbundle, with the property $x \in E,\ 0 \sim_{P_\Phi} x \Rightarrow x = 0$. Then

$$E \circ P_\Phi = \Phi^! E$$

where $\Phi^! E$ is defined by (5).

Similarly, given a Courant algebroid action (Φ, ϱ) of $\mathbb{A}$ as in Section 2.3, the backward image of a Lagrangian sub-bundle E under the Courant morphism U_Φ is just the pull-back bundle $\Phi^* E$.

3.2. **The bivector associated to a Lagrangian splitting.** Let $\mathbb{A} \to M$ be a Courant algebroid. By a *Lagrangian splitting* of $\mathbb{A}$, we mean a direct sum decomposition

$$\mathbb{A} = E \oplus F$$

where E, F are Lagrangian subbundles. Given a Lagrangian splitting $\mathbb{A} = E \oplus F$, let $\mathrm{pr}_E, \mathrm{pr}_F$ be the projections onto the two summands. Thus $x = \mathrm{pr}_E(x) + \mathrm{pr}_F(x)$ for all $x \in \mathbb{A}$. Note $\langle x, x\rangle = 2\langle \mathrm{pr}_E(x), \mathrm{pr}_F(x)\rangle$, so that $\mathrm{pr}_E(x)$ and $\mathrm{pr}_F(x)$ are orthogonal if x is isotropic. This applies in particular to elements of the form $x = \mathsf{a}^*(\mu)$. Define a bi-vector field $\pi \in \mathfrak{X}^2(M)$ by the identity

$$\iota_\mu \pi = \mathsf{a}(\mathrm{pr}_F(\mathsf{a}^*(\mu)), \quad \mu \in T^*M.$$

This is well-defined since $\iota_\mu \mathsf{a}(\mathrm{pr}_F(\mathsf{a}^*(\mu)) = \langle \mathsf{a}^*(\mu), \mathrm{pr}_F(\mathsf{a}^*(\mu)\rangle = 0$. Since $\mathsf{a}(\mathsf{a}^*(\mu)) = 0$, the formula for π may also be written $\iota_\mu \pi = -\mathsf{a}(\mathrm{pr}_E(\mathsf{a}^*(\mu))$. If e_i is a local frame for E, and f^i the dual frame for F, i.e. $\langle e_i, f^j\rangle = \delta_i^j$, we find,

$$\pi = \tfrac{1}{2} \sum_i \mathsf{a}(e_i) \wedge \mathsf{a}(f^i). \tag{11}$$

Note that π changes sign if the roles of E, F are reversed.

Example 3.4. If π is any bivector on M, with graph $\mathrm{Gr}_\pi \subset \mathbb{T}M$, the splitting $\mathbb{T}M = TM \oplus \mathrm{Gr}_\pi$ is Lagrangian splitting. The bivector associated to this splitting is just π itself.

The following Proposition gives an alternative description of the bivector field π.

Proposition 3.5. *Let $\mathbb{A} = E \oplus F$ be a Lagrangian splitting with associated bivector π. Then $\mathrm{Gr}_{-\pi} \subset \mathbb{T}M$ is the backward image of $E \times F$ under the diagonal morphism R_{diag} from Proposition 1.6. Furthermore, $0 \sim_{R_{\mathrm{diag}}} (x, y) \in E \times F$ implies $x = 0,\ y = 0$.*

Proof. By definition of the diagonal morphism, we have $(v, \mu) \sim_{R_{\mathrm{diag}}} (x, y)$ if and only if $v = \mathsf{a}(x),\ x - y = \mathsf{a}^*(\mu)$. In particular, for $\mu = 0$ and $x \in E,\ y \in F$ we obtain $x = y = 0$. Hence, Proposition 3.2 shows that the backward image of $E \times F$ under R_{diag} is a smooth Lagrangian subbundle of $\mathbb{T}M$, transverse to TM. The backward image is thus of the form $\mathrm{Gr}_{-\pi}$ for *some* bivector π. The relation

$$(-\iota_\mu \pi,\ \mu) \sim_{R_{\mathrm{diag}}} (x, y)$$

with $x \in E$ and $y \in F$ means by definition of R_{diag} that $\mathsf{a}^*\mu = x - y$, thus $y = \mathrm{pr}_F\, \mathsf{a}^*\mu$, and $-\iota_\mu \pi = \mathsf{a}(y)$. Thus $\iota_\mu \pi = \mathsf{a}(\mathrm{pr}_F(\mathsf{a}^*(\mu)))$ proving the Formula (11). □

3.3. **Rank of the bivector.** The rank of the map $\pi^\sharp \colon T^*M \to TM,\ \mu \mapsto \iota_\mu \pi$ at $m \in M$ is called the rank of π at m. If π is integrable (i.e. Poisson), the range of $\pi^\sharp_m$ is the tangent space to the symplectic leaf, and so $\mathrm{rk}(\pi_m)$ is its dimension. By definition of π,

$$\mathrm{ran}\, \pi^\sharp = \mathsf{a}(\mathrm{pr}_F\, \mathrm{ran}\, \mathsf{a}^*) = \mathsf{a}(\mathrm{pr}_E\, \mathrm{ran}\, \mathsf{a}^*). \tag{12}$$

For each m, the subspace $L_m = \mathrm{ran}(\mathsf{a}^*_m) + (\ker(\mathsf{a}_m) \cap F_m)$ is Lagrangian, as one verifies by taking its orthogonal. We have:

Lemma 3.6. *At any point $m \in M$, the rank of the bivector π is given by the formula,*

$$\mathrm{rk}(\pi_m) = \dim(\mathsf{a}_m(F_m)) - \dim(L_m \cap E_m). \tag{13}$$

Proof. Formula (12) shows that π_m has rank

$$\mathrm{rk}(\pi_m) = \dim(\mathrm{pr}_E(\mathrm{ran}(\mathsf{a}^*_m))) - \dim(\ker(\mathsf{a}_m) \cap \mathrm{pr}_E(\mathrm{ran}(\mathsf{a}^*_m))).$$

But $\mathrm{pr}_E(\mathrm{ran}(\mathsf{a}^*_m))^\perp = ((F_m + \mathrm{ran}(\mathsf{a}^*_m)) \cap E_m)^\perp = (F_m \cap \ker(\mathsf{a}_m)) + E_m$ has dimension $\dim E_m + \dim(F_m \cap \ker(\mathsf{a}_m))$. Hence $\dim(\mathrm{pr}_E(\mathrm{ran}(\mathsf{a}^*_m))) = \dim F_m - \dim(F_m \cap \ker(\mathsf{a}_m)) = \dim(\mathsf{a}_m(F_m))$. Similarly, one observes that

$$\ker(\mathsf{a}_m) \cap \mathrm{pr}_E(\mathrm{ran}(\mathsf{a}^*_m)) = (\mathrm{ran}(\mathsf{a}^*_m) + F_m \cap \ker(\mathsf{a}_m)) \cap E_m.$$

□

Equation (12) shows in particular that $\operatorname{ran}(\pi^\sharp)\subset \mathsf{a}(E)\cap\mathsf{a}(F)$. In nice cases, this is an equality:

Proposition 3.7. *Let $\mathbb{A}=E\oplus F$ be a Lagrangian splitting, with associated bivector $\pi\in\mathfrak{X}^2(M)$. Then*

$$\operatorname{ran}(\pi^\sharp)=\mathsf{a}(E)\cap\mathsf{a}(F), \tag{14}$$

if and only if

$$\ker(\mathsf{a})=\operatorname{ran}(\mathsf{a}^*)+(\ker(\mathsf{a})\cap E)+(\ker(\mathsf{a})\cap F). \tag{15}$$

Proof. Suppose Condition (15) is satisfied. Given $v\in\mathsf{a}(E)\cap\mathsf{a}(F)$, let $x\in E,\ y\in F$ with $\mathsf{a}(x)=\mathsf{a}(y)=v$. Then $x-y\in\ker(\mathsf{a})$, and (15) allows us to modify x,y to arrange $x-y\in\operatorname{ran}(\mathsf{a}^*)$. We may thus write $x-y=\mathsf{a}^*(\mu)$. But $v=\mathsf{a}(x),\ x-y=\mathsf{a}^*(\mu)$ means that $(v,\mu)\sim_{R_{\mathrm{diag}}}(x,y)$. Thus $v=-\iota_\mu\pi\in\operatorname{ran}\pi^\sharp$, which proves (14). Conversely, assume (14) and let $w\in\ker(\mathsf{a})$ be given. Write $w=x-y$ with $x\in E,\ y\in F$, and put $v=\mathsf{a}(x)=\mathsf{a}(y)$. By (14) there exists $\mu\in T^*M$ with $v=-\iota_\mu\pi$. By Proposition 3.5, we have $(v,\mu)\sim_{R_{\mathrm{diag}}}(\tilde{x},\tilde{y})$, for some $\tilde{x}\in E,\ \tilde{y}\in F$. Thus $\tilde{x}-\tilde{y}=\mathsf{a}^*(\mu)$ and $\mathsf{a}(\tilde{x})=\mathsf{a}(\tilde{y})=v$. This gives the desired decomposition

$$w=x-y=\mathsf{a}^*(\mu)+(x-\tilde{x})-(y-\tilde{y})$$

with $x-\tilde{x}\in E\cap\ker(\mathsf{a})$ and $y-\tilde{y}\in F\cap\ker(\mathsf{a})$. □

Remark 3.8. For an exact Courant algebroid, $\ker(\mathsf{a})=\operatorname{ran}(\mathsf{a}^*)$ and so Condition (15) is automatic.

3.4. Integrability of the bivector.

Theorem 3.9. *Let $\mathbb{A}=E\oplus F$ be a Lagrangian splitting, with associated bivector $\pi\in\mathfrak{X}^2(M)$, and let*

$$\Upsilon^E\in\Gamma(\wedge^3F),\quad \Upsilon^F\in\Gamma(\wedge^3E)$$

be the Courant tensors, where we are using the isomorphisms $E^*\cong F,\ F^*\cong E$ given by the pairing between E,F. Then

$$\tfrac{1}{2}[\pi,\pi]=\mathsf{a}(\Upsilon^E)+\mathsf{a}(\Upsilon^F). \tag{16}$$

In particular, if $(\mathbb{A},E,F)$ is a Manin triple over M, then π is a Poisson structure. The symplectic leaves of that Poisson structure are contained in the the connected components of the intersections of the leaves of the Dirac structures E and F. Under Condition (15) *they are equal to these components.*

Proof. Since $\mathrm{Gr}_{-\pi}$ is the backward image of $E\times F\subset\mathbb{A}\times\overline{\mathbb{A}}$, Proposition 3.2 shows that its Courant tensor is given by

$$\Upsilon^{\mathrm{Gr}_{-\pi}}=\alpha^*(\mathrm{pr}_1^*\Upsilon^E-\mathrm{pr}_2^*\Upsilon^F). \tag{17}$$

Here $\mathrm{pr}_1^*\colon \wedge F^*\to\wedge(E\times F)^*$ and $\mathrm{pr}_2^*\colon \wedge F^*\to\wedge(E\times F)^*$ are pull-backs under the two projections. (Note that the Courant tensor of F, viewed as a subbundle of $\overline{\mathbb{A}}$, is $-\Upsilon^F$.) The map $\alpha\colon \mathrm{Gr}_{-\pi}\to E\times F$ was computed in the proof of Proposition 3.5:

$$\alpha(-\iota_\mu\pi,\mu)=\big(\mathrm{pr}_E(\mathsf{a}^*(\mu)),-\mathrm{pr}_F(\mathsf{a}^*(\mu))\big).$$

To calculate the dual map

$$\alpha^*\colon F \oplus E \cong E^* \oplus F^* \to \mathrm{Gr}^*_{-\pi} \cong TM,$$

let $x \in E, y \in F$ and $\mu \in T^*M$. We have

$$\begin{aligned}\langle \alpha^*(y,x), (-\iota_\mu \pi, \mu)\rangle &= \langle (y,x),\ \alpha((-\iota_\mu\pi,\mu))\rangle\\ &= \langle y, \mathrm{pr}_E(\mathsf{a}^*(\mu))\rangle - \langle x, \mathrm{pr}_F(\mathsf{a}^*(\mu))\rangle\\ &= \langle \mu, \mathsf{a}(y) - \mathsf{a}(x)\rangle.\end{aligned}$$

Hence, $\alpha^*(y,x) = \mathsf{a}(y) - \mathsf{a}(x), \quad x \in E, y \in F$. It follows that

$$\alpha^*(\mathrm{pr}_1^* \Upsilon^E - \mathrm{pr}_2^* \Upsilon^F) = \mathsf{a}(\Upsilon^E) + \mathsf{a}(\Upsilon^F).$$

On the other hand, as remarked in Section 1.2 we also have $\Upsilon^{\mathrm{Gr}_{-\pi}} = \frac{1}{2}[\pi,\pi]$ using the identification $\mathrm{Gr}^*_{-\pi} \cong TM$. Hence (17) translates into (16). If both E, F are Dirac structures the tensors Υ^E, Υ^F vanish, and (16) shows that $[\pi,\pi] = 0$. The description of the symplectic leaves follows from Proposition 3.7. □

The fact that pairs of transverse Dirac structures define Poisson structures is due to Mackenzie-Xu [20]. Their paper expresses this result in terms of Lie bialgebroids (cf. Remark 3.1).

3.5. **Relations of Lagrangian splittings.** Suppose $\mathbb{A} = E \oplus F$ and $\mathbb{A}' = E' \oplus F'$ are Lagrangian splittings of Courant algebroids over M, M', with associated bivectors π, π'. Assume also that $R_\Phi\colon \mathbb{A} \dashrightarrow \mathbb{A}'$ is a Courant morphism. Our goal in this Section is to formulate a sufficient condition on the two splittings such that the map $\Phi\colon M \to M'$ is a bivector map,

$$\pi \sim_\Phi \pi',$$

i.e. $(\mathrm{d}\Phi)_m(\pi_m) = \pi'_{\Phi(m)}$.

Since the condition will only involve linear algebra we will temporarily just deal with vector spaces W with split bilinear form $\langle\cdot,\cdot\rangle$. For instance, if V is any vector space then $\mathbb{V} = V \oplus V^*$ carries a natural split bilinear form given by the pairing. Suppose $W_1 \subset W$ is a co-isotropic subspace, and $W_0 = W_1^\perp$ its orthogonal. Then $W_{\mathrm{red}} = W_1/W_0$ is again a vector space with split bilinear form. For any Lagrangian subspace $E \subset W$, the quotient $E_{\mathrm{red}} = E_1/E_0$ with $E_i = E \cap W_i$ is a Lagrangian subspace of E_{red}.

A bivector $\Pi \in \wedge^2(W)$ descends to a bivector Π_{red} on W_1/W_0 if and only if it lies in the subspace $\wedge^2(W_1) + W_0 \wedge W$, or equivalently

$$w \in W_1 \Rightarrow \iota(w)\Pi \in W_1. \tag{18}$$

Here $\iota(w)\colon \wedge W \to \wedge W$ is the derivation extension of $w_1 \mapsto \langle w_1, w\rangle$. Note that (18) implies $w \in W_0 \Rightarrow \iota(w)\Pi \in W_0$. Letting $w_{\mathrm{red}} \in W_{\mathrm{red}}$ be the image of $w \in W_1$, the bivector $\Pi_{\mathrm{red}} \in \wedge^2 W_{\mathrm{red}}$ is then given by

$$\iota(w_{\mathrm{red}})\Pi_{\mathrm{red}} = (\iota(w)\Pi)_{\mathrm{red}}.$$

Lemma 3.10. *Suppose that $W = E \oplus F$ is a Lagrangian splitting, and let $\Pi \in \wedge^2 W$ be the bivector defined by $\iota_w \Pi = \frac{1}{2}(\mathrm{pr}_F(w) - \mathrm{pr}_E(w))$, i.e.*

$$\Pi = \tfrac{1}{2}\sum_i e_i \wedge f^i \tag{19}$$

for dual bases e_i, f^i of E, F. Then Π descends to a bivector Π_{red} on W_{red} if and only if $W_0 = E_0 \oplus F_0$, or equivalently $W_1 = E_1 \oplus F_1$. Furthermore, in this case $E_{\mathrm{red}},\ F_{\mathrm{red}}$ are transverse, and Π_{red} is given by a formula similar to (19) with dual bases for $E_{\mathrm{red}},\ F_{\mathrm{red}}$.

Proof. Let $\mathrm{pr}_E,\ \mathrm{pr}_F$ denote the projections from W to E, F. Thus $w = \mathrm{pr}_E(w) + \mathrm{pr}_F(w)$, while on the other hand $\iota(w)\Pi = \frac{1}{2}(\mathrm{pr}_F(w) - \mathrm{pr}_E(w))$. Hence, the condition $w \in W_1 \Rightarrow \iota(w)\Pi \in W_1$ holds if and only if $w \in W_1 \Rightarrow \mathrm{pr}_E(w), \mathrm{pr}_F(w) \in W_1$, that is $W_1 = E_1 \oplus F_1$. Taking orthogonals, this is equivalent to $W_0 = (E + W_0) \cap (F + W_0)$. We claim that this in turn is equivalent to $W_0 = E_0 \oplus F_0$. The implication $\Leftarrow$ is clear. For the opposite implication $\Rightarrow$ let $w \in W_0$, and write $w = e + f$ with $e = \mathrm{pr}_E(w)$ and $f = \mathrm{pr}_F(w)$. Then $f = w - e \in (E + W_0) \cap F = (E + W_0) \cap (F + W_0) \cap F = W_0 \cap F = F_0$, and similarly $e \in E_0$.

Suppose then that $W_0 = E_0 \oplus F_0$. For $w \in W_1$, the decomposition $w = x + y$ with $x = \mathrm{pr}_E(w) \in E_1,\ y = \mathrm{pr}_F(w) \in F_1$ descends to the decomposition $w_{\mathrm{red}} = x_{\mathrm{red}} + y_{\mathrm{red}}$ with $x \in E_{\mathrm{red}}$ and $y \in F_{\mathrm{red}}$. Hence

$$\iota(w_{\mathrm{red}})\Pi_{\mathrm{red}} = (\iota(w)\Pi)_{\mathrm{red}} = \tfrac{1}{2}(y - x)_{\mathrm{red}} = \tfrac{1}{2}(y_{\mathrm{red}} - x_{\mathrm{red}})$$

which shows that Π_{red} is the bivector for the decomposition $W_{\mathrm{red}} = E_{\mathrm{red}} \oplus F_{\mathrm{red}}$. $\square$

Remark 3.11. The property $E_{\mathrm{red}} \cap F_{\mathrm{red}} = 0$ is not automatic. Suppose e.g. that w is an isotropic vector not contained in E, F, and let W_0 be the 1-dimensional subspace spanned by w. Then $x = \mathrm{pr}_E(w) \in E_1$ and $y = -\mathrm{pr}_F(w) \in F_1$ descend to non-zero elements $x_{\mathrm{red}} \in E_{\mathrm{red}},\ y_{\mathrm{red}} \in F_{\mathrm{red}}$, with $x_{\mathrm{red}} - y_{\mathrm{red}} = w_{\mathrm{red}} = 0$. On the other hand $W_0 = E_0 \oplus F_0$ is not a necessary condition to ensure $E_{\mathrm{red}} \cap F_{\mathrm{red}} = 0$, as one can see by taking W_0 to be any Lagrangian subspace transverse to E, F. (In this case the transversality condition is trivial since $W_{\mathrm{red}} = 0$, while $E_0 + F_0 = 0$.)

A morphism of vector spaces W, W' with split bilinear forms is a Lagrangian subspace $R \subset W' \times \overline{W}$, where $\overline{W}$ is the vector space W with opposite bilinear form. As before we write $R\colon W \dashrightarrow W'$, and $w \sim_R w'$ if $(w, w') \in R$. Let

$$\begin{aligned} \ker(R) &= \{w \in W |\ w \sim_R 0\}, \\ \mathrm{ran}(R) &= \{w \in W' |\ \exists\, w \in W\colon\ w \sim_R w'\}. \end{aligned} \tag{20}$$

The *transpose* of the morphism $R\colon W \dashrightarrow W'$ is the morphism $R^t\colon W' \dashrightarrow W$, given by $w' \sim_{R^t} w \ \Leftrightarrow\ w \sim_R w'$.

Lemma 3.12. *We have* $\ker(R) = \mathrm{ran}(R^t)^\perp$ *and* $\mathrm{ran}(R) = \ker(R^t)^\perp$. *Furthermore,*

$$\dim\ker(R) + \dim\mathrm{ran}(R) = \dim R. \tag{21}$$

The morphism $R\colon W \dashrightarrow W'$ gives a well-defined linear isomorphism

$$R_{\mathrm{red}}\colon\ \mathrm{ran}(R^t)/\ker(R) \to \mathrm{ran}(R)/\ker(R^t)$$

with the property

$$w \sim_R w' \ \Rightarrow\ R_{\mathrm{red}}(w_{\mathrm{red}}) = w'_{\mathrm{red}}.$$

Here $w_{\mathrm{red}}, w'_{\mathrm{red}}$ are the images of $w \in \mathrm{ran}(R^t)$ and $w' \in \mathrm{ran}(R)$ in the two quotient spaces.

Proof. We will write

$$W_0 = \ker(R),\ W_1 = \mathrm{ran}(R^t),\ W_0' = \ker(R^t),\ W_1' = \mathrm{ran}(R)$$

and $W_{\mathrm{red}} = W_1/W_0,\ W'_{\mathrm{red}} = W'_1/W'_0$. The dimension formula (21) follows since the spaces W_0 and W_1' are the kernel and range of the projection of R onto W'. Taking the sum and the difference with a similar equation for R^t, we obtain

$$\begin{aligned}(\dim W_0 + \dim W_1) + (\dim W_0' + \dim W_1') &= \dim W + \dim W', \\ \dim W_1 - \dim W_0 &= \dim W_1' - \dim W_0'.\end{aligned} \tag{22}$$

The second equation shows that $\dim W_{\mathrm{red}} = \dim W'_{\mathrm{red}}$.

Recall $w_i \sim_R w_i'$ for $i = 1, 2$ implies $\langle w_1, w_2\rangle = \langle w_1', w_2'\rangle$. If $w_1 \in W_0$ so that $w_1 \sim 0$, and $w_2 \in W_1$ so that $w_2 \sim w_2'$ for some $w_2' \in W'$, this shows $\langle w_1, w_2\rangle = \langle 0, w_2'\rangle = 0$. Thus $W_0 \subset W_1^\perp$, and similarly $W_1' \subset (W_0')^\perp$. To prove equality we observe that

$$\dim W_0 + \dim W_1 = \dim W,\quad \dim W_0' + \dim W_1' = \dim W'. \tag{23}$$

Indeed we already know the inequality $\le$ in these equations, and the equality follows by the first equation in (22). Finally, if $w \sim_R w'$, and $w - \tilde{w} \in W_0,\ w' - \tilde{w}' \in W_0'$, then $\tilde{w} \sim_R \tilde{w}'$. This shows that the map $R_{\mathrm{red}} \colon W_{\mathrm{red}} \to W'_{\mathrm{red}},\ w_{\mathrm{red}} \mapsto w'_{\mathrm{red}}$ is a well-defined isomorphism. □

Definition 3.13. Given a morphism $R\colon W \dashrightarrow W'$ and Lagrangian subspaces $E \subset W,\ E' \subset W'$, we write

$$E \sim_R E' \tag{24}$$

if the isomorphism R_{red} from Lemma 3.12 takes E_{red} to E'_{red}. Given Lagrangian splittings $W = E \oplus F$ and $W' = E' \oplus F'$, we write

$$(E, F) \sim_R (E', F') \tag{25}$$

if $E \sim_R E',\quad F \sim_R F'$, and in addition

(a) $\ker(R)$ is the direct sum of its intersections with E and with F,
(b) $\mathrm{ran}(R)$ is the direct sum of its intersections with E' and with F'.

Remark 3.14. The relation (24) follows if $E = E' \circ R$ (b-Dirac morphism) or $E' = R \circ E$ (f-Dirac morphism). Conversely, if $\ker(R) = 0$ the relation (24) implies $E = E' \circ R$, while for $\mathrm{ran}(R) = W'$ it implies $E' = R \circ E$. In general, (24) implies that R is an f-Dirac morphism followed by a b-Dirac morphism:

$$(W, E) \dashrightarrow (W'_{\mathrm{red}}, E'_{\mathrm{red}}) \dashrightarrow (W', E')$$

with $W'_{\mathrm{red}} = \mathrm{ran}(R)/\mathrm{ran}(R)^\perp$. Note however that $E \sim_R E'$ and $E' \sim_S E''$ do not imply $E \sim_{S\circ R} E''$ in general.

Proposition 3.15. *Let $W = E \oplus F$, $W' = E' \oplus F'$ be Lagrangian splittings, defining bivectors Π, Π', and $R\colon W \dashrightarrow W'$ a morphism with $(E, F) \sim_R (E', F')$. Then the induced map $R_{\mathrm{red}}\colon W_{\mathrm{red}} \to W'_{\mathrm{red}}$ takes Π_{red} to Π'_{red}.*

Proof. This follows since $\Pi_{\mathrm{red}}, \Pi'_{\mathrm{red}}$ are the bivectors defined by the splittings $W_{\mathrm{red}} = E_{\mathrm{red}} \oplus F_{\mathrm{red}}$ and $W'_{\mathrm{red}} = E'_{\mathrm{red}} \oplus F'_{\mathrm{red}}$, and since $R_{\mathrm{red}}(E_{\mathrm{red}}) = E'_{\mathrm{red}},\ R_{\mathrm{red}}(F_{\mathrm{red}}) = F'_{\mathrm{red}}$. □

We now apply these results to Courant morphisms $R_\Phi\colon \mathbb{A} \dashrightarrow \mathbb{A}'$ over manifolds $\Phi\colon M \to M'$. For Lagrangian splittings $\mathbb{A} = E \oplus F$ and $\mathbb{A}' = E' \oplus F'$ we will write

$$(E, F) \sim_{R_\Phi} (E', F')$$

if this relation holds pointwise, at any $m \in M$. The main result of this Section is as follows:

Theorem 3.16. *Let $\mathbb{A} = E \oplus F$ and $\mathbb{A}' = E' \oplus F'$ be Lagrangian splittings of Courant algebroids $\mathbb{A}$, $\mathbb{A}'$, defining bivectors $\pi \in \mathfrak{X}^2(M)$, $\pi' \in \mathfrak{X}^2(M')$. Suppose*

$$R_\Phi\colon \mathbb{A} \dashrightarrow \mathbb{A}'$$

is a Courant morphism. Then

$$(E, F) \sim_{R_\Phi} (E', F') \ \Rightarrow\ \pi \sim_\Phi \pi'.$$

Proof. We have to show $\pi_{\Phi(m)} = (d\Phi)_m(\pi_m)$ for all $m \in M$. To simplify notation, we omit base points in the following discussion. Let Π, Π' be the bivectors defined by the splittings, so that $\pi = \mathsf{a}(\Pi)$ and $\pi' = \mathsf{a}'(\Pi')$. We define

$$\mathbb{A}_{\mathrm{red}} = \ker(R_\Phi)^\perp / \ker(R_\Phi), \quad \mathbb{A}'_{\mathrm{red}} = \operatorname{ran}(R_\Phi)/\operatorname{ran}(R_\Phi)^\perp,$$

so that R_Φ descends to an isomorphism $\mathbb{A}_{\mathrm{red}} \to \mathbb{A}'_{\mathrm{red}}$ taking Π_{red} to Π'_{red}. Since $x \sim_{R_\Phi} x'$ implies $(\mathrm{d}\Phi)(\mathsf{a}(x)) = \mathsf{a}'(x')$, we see that $\mathsf{a}'(\ker(R_\Phi)^\perp) = 0$ and $\mathsf{a}'(\operatorname{ran}(R_\Phi)) \subset \operatorname{ran}(\mathrm{d}\Phi)$. Similarly $\mathsf{a}(\ker(R_\Phi)) \subset \ker(\mathrm{d}\Phi)$. Hence a, a' descend to define the vertical maps of the following commutative diagram,

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{red}} & \xrightarrow[\cong]{} & \mathbb{A}'_{\mathrm{red}} \\ \big\downarrow{\scriptstyle \mathsf{a}_{\mathrm{red}}} & & \big\downarrow{\scriptstyle \mathsf{a}'_{\mathrm{red}}} \\ TM/\ker(\mathrm{d}\Phi) & \xrightarrow[\cong]{} & \operatorname{ran}(\mathrm{d}\Phi) \end{array}$$

The bottom map takes $\pi_{\mathrm{red}} = \mathsf{a}_{\mathrm{red}}(\Pi_{\mathrm{red}})$ to $\pi'_{\mathrm{red}} = \mathsf{a}'_{\mathrm{red}}(\Pi'_{\mathrm{red}})$. On the other hand, Lemma 3.10 shows that π' is the image of π'_{red} under the inclusion $\operatorname{ran}(\mathrm{d}\Phi) \to TM'$, while π_{red} is the image of π under the projection $TM \to TM/\ker(\mathrm{d}\Phi)$. This proves that $\mathrm{d}\Phi\colon TM \to TM'$ takes π to π'. $\square$

4. Courant algebroids of the form $M \times \mathfrak{g}$

An *action of a Lie algebra* $\mathfrak{g}$ on a manifold M is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$. The range of the action map $\mathsf{a}\colon M \times \mathfrak{g} \to TM$ defines an integrable generalized distribution; its leaves are called the $\mathfrak{g}$-*orbits*. The action map makes $M \times \mathfrak{g}$ into a Lie algebroid over M, with anchor map a and Lie bracket $[\cdot,\cdot]_{\mathfrak{X}}$ extending the Lie bracket on constant sections. Explicitly,

$$[x_1, x_2]_{\mathfrak{X}} = [x_1, x_2]_{\mathfrak{g}} + \mathcal{L}_{\mathsf{a}(x_1)} x_2 - \mathcal{L}_{\mathsf{a}(x_2)} x_1 \tag{26}$$

where $[x_1, x_2]_{\mathfrak{g}}$ is the pointwise bracket.

4.1. **$\mathfrak{g}$-actions.** Let $\mathfrak{g}$ be a quadratic Lie algebra, with inner product $\langle\cdot,\cdot\rangle$. Given a $\mathfrak{g}$-action on a manifold M, all of whose stabilizer are co-isotropic, Example 2.14 describes a Courant algebroid structure on the product $\mathbb{A}_M = M\times\mathfrak{g}$, with anchor map $\mathsf{a}\colon M\times\mathfrak{g}\to TM$ given by the action. Recall that the inner product $\langle\cdot,\cdot\rangle$ and Courant bracket $[\![\cdot,\cdot]\!]$ on $\mathbb{A}_M$ extend the inner product and Lie bracket on constant sections $\mathfrak{g}\subset\Gamma(\mathbb{A}_M)$.

Lemma 4.1. *The Courant bracket and the Lie algebroid bracket on $\Gamma(\mathbb{A}_M) = C^\infty(M,\mathfrak{g})$ are related as follows:*

$$[\![x,y]\!] = [x,y]_{\mathfrak{X}} + \mathsf{a}^*\langle dx,y\rangle,\quad x,y\in C^\infty(M,\mathfrak{g}). \tag{27}$$

Proof. The Courant bracket is uniquely determined by its property of extending the Lie bracket on constant sections. It hence suffices to note that the right hand side restricts to the pointwise bracket on constant sections, and that it satisfies c3) and p1) from 1.1. $\square$

4.2. **Examples.**

Example 4.2. Let $\mathfrak{g}$ be a quadratic Lie algebra, and $\overline{\mathfrak{g}}$ the same Lie algebra with the opposite bilinear form. Then $\mathfrak{d}=\mathfrak{g}\oplus\overline{\mathfrak{g}}$ is a quadratic Lie algebra. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and denote by u^L resp. u^R the left-invariant resp. right-invariant vector fields on G defined by $u\in\mathfrak{g}$. The action of $D=G\times G$ on G, given by $(a,b).g=agb^{-1}$, defines an action of $\mathfrak{d}$,

$$\mathsf{a}\colon\mathfrak{d}\to TG,\ (u,v)\mapsto v^L-u^R$$

The stabilizers for the infinitesimal action are co-isotropic: In fact,

$$\ker(\mathsf{a}_g)=\mathfrak{d}_g=\{(u,v)|\ u=\mathrm{Ad}_g(v)\}$$

is Lagrangian. Thus $\mathbb{A}_G=G\times\mathfrak{d}$ is a Courant algebroid. This is extensively studied in [1], where an explicit isomorphism $\mathbb{A}_G\cong\mathbb{T}G^{(\eta)}$ is constructed (cf. Definition b), with $\eta=\frac{1}{12}\theta^L\cdot[\theta^L,\theta^L]\in\Omega^3(G)$ the Cartan 3-form (where $\theta^L\in\Omega^1(G,\mathfrak{g})$ is the left-invariant Maurer-Cartan form.

Example 4.3 (De Concini-Procesi compactification). Let G be a complex semisimple Lie group of adjoint type, and $\mathfrak{d}=\mathfrak{g}\oplus\overline{\mathfrak{g}}$ as above. Let M be the complex manifold given as its de Concini-Procesi 'wonderful compactification' [11]. The action of $D=G\times G$ on G (given by $(a,b).g=agb^{-1}$) extends to an action on M, and $\mathbb{A}_M=M\times\mathfrak{d}$ is a Courant algebroid. (The fact that elements in $\mathrm{ran}(\mathsf{a}^*)$ are isotropic follows by continuity.)

Example 4.4 (Homogeneous spaces G/H). Let $\mathfrak{g}$ be a quadratic Lie algebra, with corresponding Lie group G, and suppose that $H\subset G$ is a closed subgroup whose Lie algebra $\mathfrak{h}$ is coisotropic. Then the stabilizer algebras for the induced G-action on G/H are coisotropic. (The stabilizer algebra at the coset of $g\in G$ is $\mathrm{Ad}_g(\mathfrak{h})$.) Hence we obtain a Courant algebroid $\mathbb{A}_{G/H}=G/H\times\mathfrak{g}$. In case $\mathfrak{h}$ is a Lagrangian subalgebra, this example is discussed by Alekseev-Xu in [5], see also [7].

Example 4.5 (The variety of Lagrangian subalgebras). A quadratic Lie algebra $\mathfrak{d}$ will be called *split quadratic* if its bilinear form is split, i.e. if there exist Lagrangian subspaces. For instance the Lie algebra $\mathfrak{g}\oplus\overline{\mathfrak{g}}$ from Example 4.2 is split quadratic, as is the semi-direct product $\mathfrak{k}^*\rtimes\mathfrak{k}$ for any Lie algebra $\mathfrak{k}$. Given a split quadratic Lie algebra $\mathfrak{d}$ let $\mathrm{Lag}(\mathfrak{d})$ be the manifold of Lagrangian subspaces of $\mathfrak{d}$, and $M\subset\mathrm{Lag}(\mathfrak{d})$ the subset of Lagrangian Lie subalgebras. Then M is not a manifold, but (working over $\mathbb{C}$) it is a (complex) variety. Let D be a Lie group exponentiating $\mathfrak{d}$, with its natural action on M. Then the stabilizer algebra at $m \in M$ contains the Lagrangian subalgebra labelled by m, and hence is co-isotropic. Consequently $\mathbb{A}_M = M \times \mathfrak{d}$ is a Courant algebroid. A similar argument applies more generally to the *variety of co-isotropic subalgebras* of any given dimension.

These examples are related: For instance, as shown by Evens-Lu [12] the de Concini-Procesi compactification of G is one of the irreducible components of the variety of Lagrangian subalgebras of $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$. Example 4.2 is a special case of 4.4, taking the quotient of $D = G \times G$ by the diagonal subgroup.

Example 4.6. Given a manifold N with an action of $\mathfrak{g}$, define a section γ of the bundle $S^2(TN)$ by $\gamma(\mu,\nu) = \langle \mathsf{a}^*(\mu), \mathsf{a}^*(\nu)\rangle$. Then $M = \gamma^{-1}(0)$ is $\mathfrak{g}$-invariant, and (assuming for simplicity that $\gamma^{-1}(0)$ is smooth) $\mathbb{A}_M = M \times \mathfrak{g}$ is a Courant algebroid.

4.3. **Courant morphisms.** We will now consider morphisms between Courant algebroids of the form $M \times \mathfrak{g}$.

Proposition 4.7. *Suppose $\mathfrak{g}, \mathfrak{g}'$ are quadratic Lie algebras, acting on M, M' with co-isotropic stabilizers. Let $\mathbb{A}_M = M \times \mathfrak{g},\ \mathbb{A}_{M'} = M' \times \mathfrak{g}'$ be the resulting Courant algebroids, with anchor map the action maps a, a'. Assume $\Phi\colon M \to M'$ is a smooth map and $R \subset \mathfrak{g}' \oplus \overline{\mathfrak{g}}$ is a Lagrangian subalgebra, with the property*

$$x \sim_R x' \ \Rightarrow\ \mathsf{a}(x) \sim_\Phi \mathsf{a}'(x'). \tag{28}$$

Then $R_\Phi = \mathrm{Graph}(\Phi) \times R$ defines a Courant morphism

$$R_\Phi\colon \mathbb{A}_M \dashrightarrow \mathbb{A}_{M'}, \quad (m,x) \sim_{R_\Phi} (\Phi(m), x').$$

If $\mathfrak{g} = \mathfrak{d},\ \mathfrak{g}' = \mathfrak{d}'$ are split quadratic, then Lagrangian splittings $\mathfrak{d} = \mathfrak{e} \oplus \mathfrak{f},\ \mathfrak{d}' = \mathfrak{e}' \oplus \mathfrak{f}'$ with $(\mathfrak{e}, \mathfrak{f}) \sim_R (\mathfrak{e}', \mathfrak{f}')$ give R_Φ-related Lagrangian splittings of the Courant algebroids,

$$(E, F) \sim_{R_\Phi} (E', F') \tag{29}$$

with $E = M \times \mathfrak{e},\ F = M \times \mathfrak{f},\ E' = M \times \mathfrak{e}',\ F' = M \times \mathfrak{f}'$. Hence the corresponding bivector fields are Φ-related: $\pi \sim_\Phi \pi'$.

Proof. Since R is a Lagrangian subalgebra, the direct product $(M' \times M) \times R \subset \mathbb{A}_M \times \overline{\mathbb{A}_{M'}}$ is a Dirac structure. Condition (28) guarantees that $(\mathsf{a} \times \mathsf{a}')(R)$ is tangent to the graph of Φ. The statements about Lagrangian splittings are obvious since the relation (29) is defined pointwise. ☐

We stress that the relation $R\colon \mathfrak{g} \dashrightarrow \mathfrak{g}'$ need not come from a Lie algebra homomorphism.

Remark 4.8. The Proposition 4.7 may be generalized to the set-up from Section 2.2, replacing $R\colon \mathfrak{g} \dashrightarrow \mathfrak{g}'$ with a Courant morphism $\mathbb{A} \dashrightarrow \mathbb{A}'$.

Example 4.9 (The Courant algebroid $\mathbb{A}_G$). Let $\mathfrak{g}$ be a quadratic Lie algebra, with invariant inner product $B_\mathfrak{g}$. The space $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ may be viewed as a pair groupoid over $\mathfrak{g}$, with source and target map $s(x,y) = y,\ t(x,y) = x$ and with groupoid multiplication

$$z = z' \circ z'' \quad \text{if} \quad s(z) = s(z''),\ t(z) = t(z'),\ s(z') = t(z'').$$

Let

$$R\colon \mathfrak{d}\oplus\mathfrak{d} \dashrightarrow \mathfrak{d},\ (z',z'')\sim_R z'\circ z''.$$

In more detail, $R\subset \mathfrak{d}\oplus\overline{\mathfrak{d}\oplus\mathfrak{d}}$ is the subspace of elements (z,z',z'') such that $z=z'\circ z''$. Then R is a Lagrangian subalgebra, with $\mathrm{ran}(R)=\mathfrak{d}$ and $\ker(R)=\{(z',z'')|\ z'\circ z''=0\}$. Let G be a Lie group integrating $\mathfrak{g}$, with the $\mathfrak{d}$-action $\mathsf{a}\colon \mathfrak{d}\to\mathfrak{X}(G)$ and Courant algebroid $\mathbb{A}_G$ from Example 4.2. Let $\mathrm{Mult}\colon G\times G\to G$ be group multiplication. One easily checks (cf. [1, Proposition 3.8]) that

$$z=z'\circ z''\quad\Rightarrow\quad (\mathsf{a}(z'),\mathsf{a}(z''))\sim_{\mathrm{Mult}}\mathsf{a}(z).$$

Hence the conditions from Proposition 4.7 hold, and we obtain a Courant morphism

$$R_{\mathrm{Mult}}\colon \mathbb{A}_G\times\mathbb{A}_G\dashrightarrow\mathbb{A}_G. \tag{30}$$

There is also a Courant morphism $U_{\mathrm{Inv}}\colon \mathbb{A}_G\dashrightarrow\overline{\mathbb{A}_G}$ lifting the inversion map $\mathrm{Inv}\colon G\to G,\ g\mapsto g^{-1}$. It is given by the direct product of $\mathrm{Graph}(\mathrm{Inv})\times U$, where

$$U\colon \mathfrak{d}\to\overline{\mathfrak{d}},\quad z\sim_U z^{-1}$$

with $z^{-1}=(v,u)$ the groupoid inverse of $z=(u,v)$. The associativity property of group multiplication, and the property $\mathrm{Inv}\circ\mathrm{Mult}=\mathrm{Mult}\circ(\mathrm{Inv}\times\mathrm{Inv})$ lift to the Courant morphisms. Thus G with the Courant algebroid $\mathbb{A}_G$ could justifiably be called a *Courant Lie group.*

Example 4.10 (The action map $G\times M\to M$)*.* Let $\mathfrak{g}$ be a quadratic Lie algebra, G a corresponding Lie group and $\mathbb{A}_G$ the Courant algebroid from Example 4.2. Suppose G acts on a manifold M, with co-isotropic stabilizer algebras. We claim that the action map $\Phi\colon G\times M\to M$ lifts to a morphism of Courant algebroids

$$S_\Phi\colon \mathbb{A}_G\times\mathbb{A}_M\dashrightarrow\mathbb{A}_M.$$

The relation

$$S\colon \mathfrak{d}\oplus\mathfrak{g}\dashrightarrow\mathfrak{g},\quad ((z,z'),z')\sim_S z.$$

defines a Lagrangian subalgebra of $\mathfrak{g}\oplus\overline{\mathfrak{d}\oplus\mathfrak{g}}$, and it has the required property (28). The action property $\Phi\circ(\mathrm{Mult}_G\times\mathrm{id}_M)=\Phi\circ(\mathrm{id}_G\times\Phi)$ lifts to a similar property of the Courant morphisms, so that one might call this a *Courant Lie group action.*

Example 4.11*.* Let $\mathfrak{g}$ be a quadratic Lie algebra, with corresponding Lie group G. Suppose $H\subset G$ is a closed subgroup whose Lie algebra $\mathfrak{h}\subset\mathfrak{g}$ is Lagrangian, and let $\mathbb{A}_{G/H}=G/H\times\mathfrak{g}$ be the resulting Courant algebroid. Then the quotient map $\Psi\colon G\to G/H$ lifts to a Courant morphism

$$T_\Psi\colon \mathbb{A}_G\dashrightarrow\mathbb{A}_{G/H},$$

by taking

$$T\colon \mathfrak{d}\dashrightarrow\mathfrak{g},\quad (z,u)\sim_T z,\ z\in\mathfrak{g},\ u\in\mathfrak{h}.$$

If the subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is only co-isotropic, but contains a Lagrangian subalgebra $\mathfrak{k}\subset\mathfrak{h}$, one obtains a Courant morphism T_Ψ by replacing $u\in\mathfrak{h}$ with $u\in\mathfrak{k}$ in the definition of T.

5. Lagrangian splittings

Throughout this Section, we assume that $\mathfrak{d}$ is a split quadratic Lie algebra.

5.1. **Lagrangian splittings of** $\mathbb{A}_M$**.** Let $\mathbb{A}_M = M \times \mathfrak{d}$ be the Courant algebroid defined by a $\mathfrak{d}$-action with co-isotropic stabilizers. For any Lagrangian subspace $\mathfrak{l} \subset \mathfrak{d}$, the subbundle $E = M \times \mathfrak{l}$ is Lagrangian. Let $\Upsilon^{\mathfrak{l}} \in \wedge^3 \mathfrak{l}^*$ be defined by $\Upsilon^{\mathfrak{l}}(u_1, u_2, u_3) = \langle u_1, [u_2, u_3]\rangle$, $u_i \in \mathfrak{l}$. Then $\Upsilon^{\mathfrak{l}}$ vanishes if and only if $\mathfrak{l}$ is a Lie subalgebra.

Proposition 5.1. *The Courant tensor of the Lagrangian subbundle $E = M \times \mathfrak{l}$ is $\Upsilon^{\mathfrak{l}}$ viewed as a constant section. In particular, it vanishes if and only if $\mathfrak{l}$ is a Lagrangian subalgebra.*

Proof. This is immediate from the definition of Υ^E, since the Courant bracket on constant sections coincides with the pointwise Lie bracket. □

Hence, if $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair (i.e. $\mathfrak{g}$ is a Lagrangian Lie subalgebra), then $(\mathbb{A}_M,\ M \times \mathfrak{g})$ is a Manin pair over M. As a special case of Theorem 3.9 we obtain:

Theorem 5.2. *Suppose $\mathfrak{d}$ acts on M with co-isotropic stabilizers. Let $\mathfrak{d} = \mathfrak{e} \oplus \mathfrak{f}$ be a decomposition into two Lagrangian subspaces, and let $E = M \times \mathfrak{e},\ F = M \times \mathfrak{f}$. Define a bi-vector field on M by*

$$\pi = \tfrac{1}{2} \sum_i \mathsf{a}(e_i) \wedge \mathsf{a}(f^i)$$

where e_i, f^i are dual bases of $\mathfrak{e}, \mathfrak{f}$. Then

$$\tfrac{1}{2}[\pi, \pi] = \mathfrak{a}(\Upsilon^{\mathfrak{e}}) + \mathfrak{a}(\Upsilon^{\mathfrak{f}}).$$

The rank of π at $m \in M$ is given by

$$\mathrm{rk}(\pi_m) = \dim(\mathsf{a}_m(\mathfrak{g}_2)) - \dim(\mathfrak{l}_m \cap \mathfrak{g}_1) \tag{31}$$

with the Lagrangian subalgebra $\mathfrak{l}_m = \mathrm{ran}(\mathsf{a}_m^*) + \ker(\mathsf{a}_m) \cap \mathfrak{g}_2 \subset \mathfrak{d}$. If $\mathfrak{e} = \mathfrak{g}_1$ and $\mathfrak{f} = \mathfrak{g}_2$ are Lagrangian subalgebras, then $(\mathbb{A}_M,\ E,\ F)$ is a Manin triple over M and hence π is a Poisson structure. The symplectic leaves of π are contained in the intersections of the $\mathfrak{g}_1$-orbits and $\mathfrak{g}_2$-orbits, with equality if and only if $\ker(\mathsf{a}) = \mathrm{ran}(\mathsf{a}^*) + (\ker(\mathsf{a}_1) \oplus \ker(\mathsf{a}_2))$*. Here $\mathsf{a}_i \colon \mathfrak{g}_i \to TM$ are the restrictions of the action.*

As remarked in the introduction, the Poisson structure π on M given by Theorem 5.2, in the case that $\mathfrak{g}_i$ are Lagrangian sub-algebras, is due to Lu-Yakimov [19].

Remark 5.3. Note that for $\mathfrak{g}_2$ a Lagrangian subalgebra, $m \mapsto \mathfrak{l}_m = \mathrm{ran}(\mathsf{a}_m^*) + \ker(\mathsf{a}_m) \cap \mathfrak{g}_2$ gives a map from M into the variety of Lagrangian subalgebras of $\mathfrak{d}$ – a generalization of the *Drinfeld map*. In general, this map need not be smooth.

Remark 5.4. If the stabilizer algebras for the $\mathfrak{d}$-action are Lagrangian, then necessarily $\ker(\mathsf{a}) = \mathrm{ran}(\mathsf{a}^*)$. In this case, if $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ is a Manin triple, the symplectic leaves of the Manin triple $(\mathbb{A}_M,\ E,\ F)$ are the intersections of $\mathfrak{g}_1$-orbits with $\mathfrak{g}_2$-orbits.

Example 5.5. If $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ as in Example 4.2, the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ is a Lagrangian subalgebra, and the anti-diagonal $\mathfrak{g}_{-\Delta} = \{(x, -x) |\ x \in \mathfrak{g}\}$ is a Lagrangian complement. The anti-diagonal is not a subalgebra unless $\mathfrak{g}$ is Abelian. In fact, letting $\Xi \in \wedge^3 \mathfrak{g}$ be the structure constants tensor for $\mathfrak{g}$, with normalization $\Xi(x_1, x_2, x_3) = \frac{1}{4} B(x_1, [x_2, x_3])$, the Courant tensor of the anti-diagonal is

$$\Upsilon^{\mathfrak{g}_{-\Delta}}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \Xi(x_1, x_2, x_3), \quad x_i \in \mathfrak{g}, \quad \tilde{x}_i = \tfrac{1}{2}(x_i, -x_i).$$

Given a $\mathfrak{d}$-action on a manifold M, with co-isotropic stabilizers, the resulting Lagrangian splitting of the Courant algebroid $\mathbb{A}_M = M \times \mathfrak{d}$ defines a bi-vector field $\pi \in \mathfrak{X}^2(M)$ with

$$\tfrac{1}{2}[\pi, \pi] = \mathsf{a}(\Xi).$$

Up to an irrelevant constant this is the definition of a *quasi-Poisson manifold* as in [3]. In particular, the group G, and the variety of Lagrangian subalgebras in $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ carry natural quasi-Poisson structures.

Example 5.6. Suppose $\mathfrak{g}$ is complex semi-simple, with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and let $\mathfrak{d} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$. Then another choice of a complement to the diagonal $\mathfrak{g}_\Delta$ is

$$\mathfrak{l} = \mathfrak{h}_{-\Delta} + (\mathfrak{n}_- \oplus \mathfrak{n}_+) \subset \mathfrak{g} \oplus \overline{\mathfrak{g}}.$$

Since $\mathfrak{l}$ is a Lie subalgebra, the Manin triple $(\mathfrak{d}, \mathfrak{g}_\Delta, \mathfrak{l})$ defines a Poisson structure on M. The Poisson structure on G obtained in this way is due to Semenov-Tian-Shansky [24]. Its extension to a Poisson structure on the variety of Lagrangian subalgebras (and hence in particular the de Concini-Procesi compactification) is due to Evens-Lu [12].

Example 5.7. Let $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ be a Manin triple, and D a Lie group integrating $\mathfrak{d}$. Given a closed subgroup $Q \subset D$ whose Lie algebra $\mathfrak{q}$ is co-isotropic in D (cf. Example 4.4). The resulting Poisson structure $\pi \in \mathfrak{X}^2(D/Q)$ was studied in detail in the work of Lu-Yakimov [19]. At the coset $[d] = dQ \in D/Q$, we have $\ker(\mathsf{a}_{[d]}) = \mathrm{Ad}_d(\mathfrak{q})$, hence $\mathrm{ran}(\mathsf{a}^*_{[d]}) = \mathrm{Ad}_d(\mathfrak{q}^\perp)$. Furthermore, as shown in [19, Proposition 2.5] the space $\mathfrak{l}_d = \mathrm{Ad}_d(\mathfrak{q}^\perp) + \mathrm{Ad}_d(\mathfrak{q}) \cap \mathfrak{g}_2$ is the Lagrangian subalgebra given by the Drinfeld homomorphism for the Poisson homogeneous space D/Q. Hence (31) reduces to the following formula [19, (2.11)]

$$\mathrm{rank}(\pi_{[d]}) = \dim(\mathsf{a}_{[d]}(\mathfrak{g}_2)) - \dim(\mathfrak{l}_d \cap \mathfrak{g}_1).$$

5.2. **Lagrangian splittings of $\mathbb{A}_D$.** Suppose $\mathfrak{d}$ is a split quadratic Lie algebra, and D a Lie group integrating $\mathfrak{d}$. Then $\mathbb{A}_D = D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is a Courant algebroid, as a special case of Example 4.2 (with $\mathfrak{d}$ playing the role of $\mathfrak{g}$ in that example). If $(\mathfrak{d}, \mathfrak{g}_1, \mathfrak{g}_2)$ is a Manin triple, we obtain two Manin triples for $\mathfrak{d} \oplus \overline{\mathfrak{d}}$:

$$(\mathfrak{d} \oplus \overline{\mathfrak{d}},\ \mathfrak{e}_+,\ \mathfrak{f}_+), \qquad (\mathfrak{d} \oplus \overline{\mathfrak{d}},\ \mathfrak{e}_-,\ \mathfrak{f}_-)$$

where

$$\mathfrak{e}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathfrak{f}_+ = \mathfrak{g}_2 \oplus \mathfrak{g}_1, \quad \mathfrak{e}_- = \mathfrak{g}_1 \oplus \mathfrak{g}_1, \quad \mathfrak{f}_- = \mathfrak{g}_2 \oplus \mathfrak{g}_2.$$

Letting $E_\pm = D \times \mathfrak{e}_\pm$, $F_\pm = D \times \mathfrak{f}_\pm$ both (D, E_+, F_+) and (D, E_-, F_-) are Manin triples over D, defining Poisson structures $\pi_+,\ \pi_- \in \mathfrak{X}^2(D)$. Since the Courant algebroid $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is exact, Theorem 5.2 gives a simple description of the symplectic leaves: Let G_1, G_2 be subgroups of D integrating $\mathfrak{g}_1, \mathfrak{g}_2$, then the symplectic leaves of π_+ are the components of the intersections of the $G_1 - G_2$ double cosets with the $G_2 - G_1$ double cosets, while the leaves of π_- are the components of the intersections of the $G_1 - G_1$ double cosets with the $G_2 - G_2$ double cosets. Note that the symplectic leaf of the group unit $e \in D$ for the Poisson structure π_+ is an open neighbourhood of e, whereas relative to the Poisson structure π_- the leaf of e is a point. Hence only π_- is a possible candidate for a Poisson

Lie group structure on D. Under the morphism $R\colon \mathfrak{d}\oplus\mathfrak{d} \dashrightarrow \mathfrak{d}$, we find

$$\begin{aligned}
(\mathfrak{e}_-\times\mathfrak{e}_-, \mathfrak{f}_-\times\mathfrak{f}_-) &\sim_R (\mathfrak{e}_-, \mathfrak{f}_-),\\
(\mathfrak{e}_+\times\mathfrak{f}_+, \mathfrak{f}_+\times\mathfrak{e}_+) &\sim_R (\mathfrak{e}_-, \mathfrak{f}_-),\\
(\mathfrak{e}_+\times\mathfrak{f}_-, \mathfrak{f}_+\times\mathfrak{e}_-) &\sim_R (\mathfrak{e}_+, \mathfrak{f}_+),\\
(\mathfrak{e}_-\times\mathfrak{e}_+, \mathfrak{f}_-\times\mathfrak{f}_+) &\sim_R (\mathfrak{e}_+, \mathfrak{f}_+)
\end{aligned}$$

Note in particular that in each case, $\ker(R)$ is a direct sum of its intersections with the two Lagrangian subspaces, as required in Definition 3.13.

Hence, we obtain similar relations $(E_-\times E_-, F_-\times F_-) \sim_{R_{\mathrm{Mult}}} (E_-, F_-)$ etc., and therefore the following relations of the Poisson bivector fields:

$$\begin{aligned}
\pi_-\times\pi_- &\sim_{\mathrm{Mult}} \pi_-\\
\pi_+\times(-\pi_+) &\sim_{\mathrm{Mult}} \pi_-\\
\pi_+\times(-\pi_-) &\sim_{\mathrm{Mult}} \pi_+\\
\pi_-\times\pi_+ &\sim_{\mathrm{Mult}} \pi_+
\end{aligned}$$

In particular (D,π_-) is a Poisson Lie group, and its action on (D,π_+) is a Poisson action.

Remark 5.8. The Poisson Lie group (D,π_-) is the well-known *Drinfeld double* (of the Poisson Lie group G_1, see below), while (D,π_+) is known as the *Heisenberg double*. One has the explicit formulas $\pi_\pm = \mathfrak{r}^R \pm \mathfrak{r}^L$, where the r-matrix $\mathfrak{r}\in\wedge^2\mathfrak{d}$ is given in dual bases of $\mathfrak{g}_1,\mathfrak{g}_2$ by $\mathfrak{r}=\frac{1}{2}\sum_i e_i\wedge f^i$, and the superscripts L,R indicate the extensions to left, right-invariant vector fields. The description of the symplectic leaves of (D,π_+), as intersections of double cosets, was obtained by Alekseev-Malkin [4] using a direct calculation.

Remark 5.9. Given a Manin triple $(\mathfrak{d},\mathfrak{g}_1,\mathfrak{g}_2)$, one may also consider the Manin triples $(\mathfrak{d}\oplus\overline{\mathfrak{d}}, \mathfrak{d}_\Delta, \mathfrak{e}_+)$ and $(\mathfrak{d}\oplus\overline{\mathfrak{d}}, \mathfrak{d}_\Delta, \mathfrak{f}_+)$. These define yet other Poisson structures on D, whose symplectic leaves are the intersections of conjugacy classes in D with G_2-G_1 double cosets, respectively the G_1-G_2 double cosets.

Consider now a D-manifold M as in Example 4.10. Let $\pi\in\mathfrak{X}^2(M)$ be the Poisson bivector defined by the Manin triple $(\mathfrak{d},\mathfrak{g}_1,\mathfrak{g}_2)$. Under the morphism $S\colon(\mathfrak{d}\oplus\overline{\mathfrak{d}})\oplus\mathfrak{d}\dashrightarrow\mathfrak{d}$ from that example,

$$\begin{aligned}
(\mathfrak{e}_+\times\mathfrak{g}_2, \mathfrak{f}_+\times\mathfrak{g}_1) &\sim_S (\mathfrak{g}_1,\mathfrak{g}_2),\\
(\mathfrak{e}_-\times\mathfrak{g}_1, \mathfrak{f}_-\times\mathfrak{g}_2) &\sim_S (\mathfrak{g}_1,\mathfrak{g}_2).
\end{aligned}$$

It follows that relative to the action map $\Phi\colon D\times M\to M$,

$$\begin{aligned}
\pi_+\times(-\pi) &\sim_\Phi \pi,\\
\pi_-\times\pi &\sim \pi.
\end{aligned}$$

In particular, the (D,π_-)-action is a Poisson Lie group action. Finally, let $G\subset D$ be a closed subgroup such that $\mathfrak{g}\subset\mathfrak{d}$ is a Lagrangian subalgebra, and consider the quotient map $\Psi\colon D\to D/G$ and the morphism $T\colon\mathfrak{d}\oplus\overline{\mathfrak{d}}\dashrightarrow\mathfrak{d}$ as in Example 4.11. Let D/G carry the Poisson structure π defined by a Manin triple $(\mathfrak{d},\mathfrak{g}_1,\mathfrak{g}_2)$. Assuming that

$$\mathfrak{g} = (\mathfrak{g}\cap\mathfrak{g}_1)\oplus(\mathfrak{g}\cap\mathfrak{g}_2) \tag{32}$$

(which is automatic if $\mathfrak{g}=\mathfrak{g}_1$ or $\mathfrak{g}=\mathfrak{g}_2$) one finds

$$(\mathfrak{e}_+,\mathfrak{f}_+)\sim_T(\mathfrak{g}_1,\mathfrak{g}_2)$$
$$(\mathfrak{e}_-,\mathfrak{f}_-)\sim_T(\mathfrak{g}_1,\mathfrak{g}_2)$$

and consequently $\pi_+\sim_\Psi\pi,\ \pi_-\sim_\Psi\pi$. Here (32) ensures that $\ker(T)=0\oplus\mathfrak{g}$ is the direct sum of its intersections with $\mathfrak{e}_+,\mathfrak{f}_+$, respectively with $\mathfrak{e}_-,\mathfrak{f}_-$.

6. Poisson Lie groups

In the last Section, we explained how the choice of a Manin triple $(\mathfrak{d},\mathfrak{g}_1,\mathfrak{g}_2)$ defines Poisson structures $\pi_\pm$ on the Lie group D corresponding to $\mathfrak{d}$, where π_- is multiplicative. Let $\Phi\colon G_1\to D$ be a map exponentiating the inclusion $\mathfrak{g}_1\to\mathfrak{d}$ to the level of Lie groups. In this Section, we show that the pull-back Courant algebroid $\Phi^!\mathbb{A}_D$ has the form $G_1\times\overline{\mathfrak{d}}$ for a suitable $\mathfrak{d}$-action on G_1 (the *dressing action*). We fix the following notation: For any Lie group G and any $\xi\in C^\infty(G,\mathfrak{g})$, we denote by $\xi^L\in\mathfrak{X}(G)$ the vector field corresponding to ξ under left trivialization $G\times\mathfrak{g}\cong TG$, and by ξ^R the vector field corresponding to ξ under right trivialization. If $\xi\in\mathfrak{g}$ (viewed as a constant function $G\to\mathfrak{g}$), this agrees with our earlier notation for left-invariant and right-invariant vector fields.

6.1. The Courant algebroid $G_1\times\mathfrak{d}$.

Let $p_1,p_2\colon\mathfrak{d}\to\mathfrak{g}_i$ be the projection to the two summands of the Manin triple $(\mathfrak{d},\mathfrak{g}_1,\mathfrak{g}_2)$.

Theorem 6.1. *Each of the following two maps $\mathfrak{d}\to\mathfrak{X}(G_1)$*

$$\zeta\mapsto p_1(\mathrm{Ad}_{\Phi(g)}\zeta)^R,\quad \zeta\mapsto -p_1(\mathrm{Ad}_{\Phi(g^{-1})}\zeta)^L$$

defines a Lie algebra action of $\mathfrak{d}$, with co-isotropic stabilizers. Let $\mathbb{A}^R=G_1\times\mathfrak{d},\ \mathbb{A}^L=G_1\times\mathfrak{d}$ be the Courant algebroids for these actions. Then

$$\mathbb{A}^L\cong\overline{\mathbb{A}^R}\cong\Phi^!\mathbb{A}_D.$$

Proof. Let C^R be the subbundle of $\mathbb{A}_D$ consisting of elements x with $\mathsf{a}(x)$ tangent to the foliation $\mathfrak{g}_1^R\subset TD$ spanned by the right-invariant vector fields ξ^R, $\xi\in\mathfrak{g}_1$. Then C^R contains the Lagrangian subbundle $\ker\mathsf{a}$, and hence is co-isotropic. We will show that $C^R/(C^R)^\perp$ is a Courant algebroid $D\times\overline{\mathfrak{d}}$ defined by the following $\mathfrak{d}$-action $\zeta\mapsto p_1(\mathrm{Ad}_d\zeta)^R\in\mathfrak{X}(D)$. The isomorphism $\overline{\mathbb{A}^R}\cong\Phi^!\mathbb{A}_D$ and the description of the resulting action on G_1 are then immediate. Since $\mathsf{a}(x)=\mathsf{a}(u,v)=v^L-u^R=(\mathrm{Ad}_d v-u)^R$ for $x=(u,v)$, the fiber of C^R at $d\in D$ is given by

$$C^R|_d=\{(u,v)\in\mathfrak{d}\oplus\overline{\mathfrak{d}}|\ p_2(\mathrm{Ad}_d v-u)=0\}.$$

We read off that C^R has rank $\dim\mathfrak{d}+\dim\mathfrak{g}_1$, and hence $(C^R)^\perp$ has rank $\dim\mathfrak{g}_1$. If $x\in C^R$, i.e. $\iota_{\mathsf{a}(x)}\theta^R\in\mathfrak{g}_1$ where $\theta^R\in\Omega^1(D,\mathfrak{d})$ is the right-invariant Maurer Cartan form, it follows that for all $\xi\in\mathfrak{g}_1$

$$\langle x,\mathsf{a}^*\langle\theta^R,\xi\rangle\rangle=\langle\iota_{\mathsf{a}(x)}\theta^R,\xi\rangle=0.$$

Hence $(C^R)^\perp$ contains all $\mathsf{a}^*\langle\theta^R,\xi\rangle=-(\xi,\mathrm{Ad}_{d^{-1}}\xi)$ with $\xi\in\mathfrak{g}_1$, and for dimensional reasons such elements span all of $(C^R)^\perp|_d$. Hence, $(C^R)^\perp|_d$ consists of all those elements $(u,v)\in C^R|_d$ for which $u\in\mathfrak{g}_1$ and $\mathrm{Ad}_d v-u=0$. The set of elements $(u,v)\in C^R|_d$ with $u\in\mathfrak{g}_2$ is hence a complement to $(C^R)^\perp|_d$ in $C^R|_d$. Note that for any element of this form, $u = p_2(u) = p_2(\mathrm{Ad}_d\, v)$. We conclude that the trivial bundle $D \times \overline{\mathfrak{d}}$, embedded in $\mathbb{A}_D = D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ by the map

$$\zeta \mapsto \phi^R(\zeta) = (p_2(\mathrm{Ad}_d\, \zeta), \zeta), \tag{33}$$

is a complement to $(C^R)^\perp$ in C^R. To compute the Courant bracket of two such sections let $\zeta, \tilde{\zeta} \in \mathfrak{d}$ and put $u = p_2(\mathrm{Ad}_d\, \zeta),\ \tilde{u} = p_2(\mathrm{Ad}_d\, \tilde{\zeta}) \in C^\infty(D, \mathfrak{d})$. We have

$$\begin{aligned}[\![\phi^R(\zeta), \phi^R(\tilde{\zeta})]\!] &= [\phi^R(\zeta), \phi^R(\tilde{\zeta})]_{\mathfrak{x}} + \mathsf{a}^*\langle \mathrm{d}\phi^R(\zeta), \phi^R(\tilde{\zeta})\rangle \\ &= \Big([u, \tilde{u}] + \mathcal{L}_{\mathsf{a}(\phi^R(\zeta))}\tilde{u} - \mathcal{L}_{\mathsf{a}(\phi^R(\tilde{\zeta}))} u,\ [\zeta, \tilde{\zeta}]\Big) + \mathsf{a}^*\langle \mathrm{d}u, \tilde{u}\rangle.\end{aligned}$$

But $\langle \mathrm{d}u, \tilde{u}\rangle = 0$ since $u, \tilde{u}$ take values in the isotropic subalgebra $\mathfrak{g}_2$. Hence the Courant bracket takes on the form $(?, [\zeta, \tilde{\zeta}])$ where '?' indicates some function with values in $\mathfrak{g}_2$. Since this Courant bracket is a section of C^R, we know without calculation that $? = p_2(\mathrm{Ad}_d[\zeta, \tilde{\zeta}])$. We have thus shown

$$[\![\phi^R(\zeta), \phi^R(\tilde{\zeta})]\!] = \phi^R([\zeta, \tilde{\zeta}]). \tag{34}$$

Finally, the anchor map for $C^R/(C^R)^\perp \cong D \times \overline{\mathfrak{d}}$ is obtained from

$$\mathsf{a}(\phi^R(\zeta)) = \zeta^L - (p_2(\mathrm{Ad}_d\, \zeta))^R = p_1(\mathrm{Ad}_d\, \zeta)^R.$$

This gives the desired description of $C^R/(C^R)^\perp$. Similarly, to prove $\mathbb{A}^L = \Phi^!\mathbb{A}_D$ we define C^L to be the set of all x with $\mathsf{a}(x) \in \mathfrak{g}_1^L$. Arguing as above, one finds that $C^L/(C^L)^\perp$ is isomorphic to $D \times \mathfrak{d}$, where the isomorphism is defined by sections $\phi^L(\zeta) = (\zeta, p_2(\mathrm{Ad}_{d^{-1}}\, \zeta))$. Again, we find that ϕ^L is a Lie algebra homomorphism relative to the Courant bracket. The anchor map for $D \times \mathfrak{d}$ is given by $\zeta \mapsto -p_1(\mathrm{Ad}_{d^{-1}}\, \zeta)^L$. ☐

It is worthwhile to summarize the main properties of the sections ϕ^L, ϕ^R:

Proposition 6.2. *The two sections*

$$\phi^L, \phi^R \colon \mathfrak{d} \to \Gamma(\mathbb{A}_D) = C^\infty(D, \mathfrak{d} \oplus \overline{\mathfrak{d}})$$

given by $\phi^L(\zeta) = (\zeta, p_2(\mathrm{Ad}_{d^{-1}}\, \zeta))$ *and* $\phi^R(\zeta) = (p_2(\mathrm{Ad}_d\, \zeta), \zeta)$ *are both homomorphisms relative to the Courant bracket. They satisfy* $\langle \phi^L(\zeta), \phi^L(\zeta)\rangle = -\langle \phi^R(\zeta), \phi^R(\zeta)\rangle = \langle \zeta, \zeta\rangle$, *and their images under the anchor map define* $\mathfrak{d}$*-actions on* D,

$$\zeta \mapsto -p_1(\mathrm{Ad}_{d^{-1}}\, \zeta)^L \quad \textit{resp.} \quad \zeta \mapsto p_1(\mathrm{Ad}_d\, \zeta)^R. \tag{35}$$

One has, for all $\zeta, \xi \in \mathfrak{d}$,

$$\begin{aligned}[\![\phi^L(\zeta), \mathsf{a}^*\langle \theta^L, \xi\rangle]\!] &= -\mathsf{a}^*\langle \theta^L,\ [p_1(\mathrm{Ad}_{d^{-1}}\, \zeta), \xi]\rangle, \\ [\![\phi^R(\zeta), \mathsf{a}^*\langle \theta^R, \xi\rangle]\!] &= -\mathsf{a}^*\langle \theta^R,\ [p_1(Ad_d\zeta), \xi]\rangle.\end{aligned}$$

Proof. The Courant bracket $[\![\phi^L(\zeta), \mathsf{a}^*\langle \theta^L, \xi\rangle]\!]$ is computed using p4), together with

$$\mathcal{L}_{\mathsf{a}(\phi^L(\zeta))}\langle \theta^L, \xi\rangle = -\langle \theta^L, [p_1(\mathrm{Ad}_{d^{-1}}\, \zeta), \xi]\rangle.$$

The computation of $[\![\phi^R(\zeta), \mathsf{a}^*\langle \theta^R, \xi\rangle]\!]$ is similar, and the other properties of ϕ^L, ϕ^R had already been established in the proof of Theorem 6.1. ☐

Remark 6.3. The fact that (35) defines $\mathfrak{d}$-actions holds true for any Lie algebra $\mathfrak{d}$, with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ such that $\mathfrak{d} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces. If $\mathfrak{d}$ carries an invariant inner product for which $\mathfrak{g}_2$ is co-isotropic, the $\mathfrak{d}$-space G_1 (for each of these actions) defines a Courant algebroid $G_1 \times \mathfrak{d}$, and similarly $D \times \mathfrak{d}$.

Let us denote $\mathbb{A}_{G_1} = \Phi^! \mathbb{A}_D$. From now on, we will work with the identification

$$\mathbb{A}_{G_1} = \overline{\mathbb{A}^R} = G_1 \times \overline{\mathfrak{d}}$$

from Theorem 6.1. By Proposition 2.10, the map $\Phi \colon G_1 \to D$ lifts to a Courant morphism

$$P_\Phi \colon \mathbb{A}_{G_1} \dashrightarrow \mathbb{A}_D, \quad (g, \zeta) \sim_{P_\Phi} (\Phi(g), \varphi(\xi, \zeta)), \quad \zeta \in \mathfrak{d}, \ \xi \in \mathfrak{g}_1$$

with

$$\varphi(\xi, \zeta) = (p_2(\mathrm{Ad}_{\Phi(g)} \zeta) - \xi, \zeta - \mathrm{Ad}_{\Phi(g^{-1})} \xi). \tag{36}$$

6.2. **Multiplicative properties.** As discussed in Example 4.9 the multiplication map Mult for the group D lifts to a Courant morphism $R_{\mathrm{Mult}} \colon \mathbb{A}_D \times \mathbb{A}_D \dashrightarrow \mathbb{A}_D$.

Proposition 6.4. *The multiplication map* Mult: $G_1 \times G_1 \to G_1$ *lifts to a Courant morphism*

$$Q_{\mathrm{Mult}} \colon \mathbb{A}_{G_1} \times \mathbb{A}_{G_1} \dashrightarrow \mathbb{A}_{G_1}, \tag{37}$$

with $R_{\mathrm{Mult}} \circ (P_\Phi \times P_\Phi) = P_\Phi \circ Q_{\mathrm{Mult}}$. *For all* $g', g'' \in G_1$ *with product* $g = g'g''$, *the fiber* $Q_{\mathrm{Mult}}|_{(g,g',g'')}$ *consists of all elements* $(\zeta, \zeta', \zeta'') \in \mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \overline{\mathfrak{d}}$ *such that*

$$\zeta = \zeta'' + \mathrm{Ad}_{\Phi((g'')^{-1})} p_1(\zeta'), \quad p_2(\mathrm{Ad}_{\Phi(g'')} \zeta'') = p_2(\zeta'). \tag{38}$$

At any given point (g, g', g''), $\mathrm{ran}(Q_{\mathrm{Mult}}) = \overline{\mathfrak{d}}$ *while*

$$\ker(Q_{\mathrm{Mult}}) = \{(-\mathrm{Ad}_{\Phi((g'')^{-1})} \xi, \xi) | \ \xi \in \mathfrak{g}_1\}.$$

Proof. We will obtain Q_{Mult} as the 'pull-back' of R_{Mult}, in the sense that

$$Q_{\mathrm{Mult}} = \Phi^*(R_{\mathrm{Mult}} \cap (C \times C \times C))/\Phi^*(R_{\mathrm{Mult}} \cap (C^\perp \times C^\perp \times C^\perp)). \tag{39}$$

with $C = C^R$ as in the proof of Theorem 6.1. Recall that R_{Mult} consists of elements of the form $(x, z; x, y, y, z) \subset \mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \overline{\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}}$, while C consists of elements $(u, v) \in \mathfrak{d} \oplus \overline{\mathfrak{d}}$ with $p_2(\mathrm{Ad}_d v - u) = 0$. Taking the intersection of R_{Mult} with $C \times C \times C$ imposes conditions, at $(d, d', d'') \in D \times D \times D$,

$$p_2(\mathrm{Ad}_d z - x) = 0, \quad p_2(\mathrm{Ad}_{d'} y - x) = 0, \quad p_2(\mathrm{Ad}_{d''} z - y) = 0. \tag{40}$$

Since the quotient map $C/C^\perp$ takes (u, v) to $\zeta = v - \mathrm{Ad}_{d^{-1}} p_1(u)$, the quotient on the right hand side of (39) consists of elements (ζ, ζ', ζ'') where

$$\zeta = z - \mathrm{Ad}_{d^{-1}} p_1(x), \quad \zeta' = y - \mathrm{Ad}_{(d')^{-1}} p_1(x), \quad \zeta'' = z - \mathrm{Ad}_{(d'')^{-1}} p_1(y).$$

with $x, y, z \in \mathfrak{d}$ subject to the conditions (40). It is straightforward to check that these are related by (38) if d', d'', d are the images of $g', g'', g = g'g'' \in G_1$. Suppose now that $(\zeta', \zeta'') \in \ker(Q_{\mathrm{Mult}})$ (at a given point $(g', g'') \in G_1 \times G_1$). The first equation in (38) (with $\zeta = 0$) shows that $\zeta'' \in \mathfrak{g}_1$, and then the second equation shows that $\zeta' \in \mathfrak{g}_1$ also, with $\zeta'' + \mathrm{Ad}_{\Phi((g'')^{-1})} p_1(\zeta') = 0$. This gives the desired description of $\ker(Q_{\mathrm{Mult}})$, while $\mathrm{ran}(Q_{\mathrm{Mult}}) = \overline{\mathfrak{d}}$ follows by dimension count using (21). $\square$

6.3. **Lagrangian splittings.** Let $\mathfrak{e}_- = \mathfrak{g}_1 \oplus \mathfrak{g}_1$, $\mathfrak{f}_- = \mathfrak{g}_2 \oplus \mathfrak{g}_2$ be the Lagrangian splitting of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ discussed in Section 5.2. It defines the Manin triple $(\mathbb{A}_D, E_-, F_-)$ with $E_- = D \times \mathfrak{e}_-$ and $F_- = D \times \mathfrak{f}_-$, and the corresponding Poisson structure π_-. On the other hand, the splitting $\mathfrak{d} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ defines a Manin triple $(\mathbb{A}_{G_1},\ E,\ F)$ over G_1, with $E = G_1 \times \mathfrak{g}_1$ and $F = G_1 \times \mathfrak{g}_2$. Let $\pi \in \mathfrak{X}^2(G_1)$ be the corresponding Poisson structure.

Proposition 6.5. *The backward images of E_-, F_- under the Courant morphism P_Φ are E, F respectively. In fact,*

$$(E, F) \sim_{P_\Phi} (E_-, F_-).$$

so that $\Phi\colon G_1 \to D$ is a Poisson map: $\pi \sim_\Phi \pi_-$.

Proof. For any Lagrangian subspace $\mathfrak{l} \subset (\mathfrak{d} \oplus \overline{\mathfrak{d}})$, the backward image $\mathfrak{l} \circ P_\Phi|_{(\Phi(g),g)}$ consists of all $\zeta \in \mathfrak{d}$ such that there exists $\xi \in \mathfrak{g}_1$ such that $\varphi(\xi, \zeta)$ (cf. (36)) lies in $\mathfrak{l}$. For $\mathfrak{l} = \mathfrak{e}_- = g_1 \oplus \mathfrak{g}_1$, these conditions say that $\zeta \in \mathfrak{g}_1$. For $\mathfrak{l}_- = \mathfrak{f}_- = \mathfrak{g}_2 \oplus \mathfrak{g}_2$, the conditions say that $\xi = 0$ and $\zeta \in \mathfrak{g}_2$. This gives the desired description of the backward image. It remains to show that $\operatorname{ran}(P_\Phi)$ is the direct sum of its intersections with E_-, F_-, or equivalently that $\operatorname{pr}_E(\operatorname{ran}(P_\Phi)) \subset \operatorname{ran}(P_\Phi)$. By definition, $\operatorname{ran}(R_\Phi)$ consists of all elements $\varphi(\xi, \zeta)$ with $\xi \in \mathfrak{g}_1,\ \zeta \in \mathfrak{d}$. For any such element, the projection to E_- is given by

$$\operatorname{pr}_E(\varphi(\xi, \zeta)) = (-\xi, p_1(\zeta) - \operatorname{Ad}_{\Phi(g^{-1})} \xi) = \varphi(\xi, p_1(\zeta))$$

since $p_2(\operatorname{Ad}_{\Phi(g)} p_1(\zeta)) = 0$. Thus $\operatorname{pr}_E(\varphi(\xi, \zeta)) \in \operatorname{ran}(R_\Phi)$. $\square$

Remark 6.6. By contrast, the backward images of E_+, F_+ under P_Φ are *both* equal to E.

Proposition 6.7. *The forward image of $E \times E$ under the Courant morphism Q_{Mult} is E. Similarly the forward image of $F \times F$ is F. Indeed one has*

$$(E \times E, F \times F) \sim_{Q_{\mathrm{Mult}}} (E,\ F);$$

in particular Mult: $G_1 \times G_1 \to G_1$ *is a Poisson map.*

Proof. By (38), we see that the forward image $Q_{\mathrm{Mult}}|_{(g,g',g'')} \circ \mathfrak{l}$ (for $g', g'' \in G_1,\ g = g'g''$) of any Lagrangian subspace $\mathfrak{l} \subset \mathfrak{d} \oplus \mathfrak{d}$ consists of all $\zeta = \zeta'' + \operatorname{Ad}_{\Phi((g'')^{-1})} p_1(\zeta')$, such that $(\zeta', \zeta'') \in \mathfrak{l}$ satisfying the condition $p_2(\operatorname{Ad}_{\Phi(g'')} \zeta'') = p_2(\zeta')$. If $\mathfrak{l} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$ the extra condition is automatic, and we find that $\zeta \in \mathfrak{g}_1$. If $\mathfrak{l} = \mathfrak{g}_2 \oplus \mathfrak{g}_2$ the formula for ζ simplifies to $\zeta = \zeta'' \in \mathfrak{g}_2$, while the extra condition fixes ζ' as $\zeta' = p_2(\operatorname{Ad}_{\Phi(g'')} \zeta'')$ This shows $(E \times E, F \times F) \sim_{Q_{\mathrm{Mult}}} (E,\ F)$: Indeed the conditions (a),(b) from Definition 3.13 on $\ker(Q_{\mathrm{Mult}}), \operatorname{ran}(Q_{\mathrm{Mult}})$ are automatic since $\ker(Q_{\mathrm{Mult}}) \subset E \times E$ and $\operatorname{ran}(Q_{\mathrm{Mult}}) = G_1 \times \overline{\mathfrak{d}}$. $\square$

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University of Toronto, Department of Mathematics, 40 St George Street, Toronto, Ontario M4S2E4, Canada

E-mail address: `dbland@math.toronto.edu`

University of Toronto, Department of Mathematics, 40 St George Street, Toronto, Ontario M4S2E4, Canada

E-mail address: `mein@math.toronto.edu`