VERLINDE SUMS AND $[Q,R] = 0$

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ABSTRACT. We prove a formula for the quantization of a proper Hamiltonian LG-space $\mathcal{M}$ with terms indexed by the components of the critical set of the norm-square of the moment map. Combined with a certain Lie-theoretic inequality, the formula implies that the multiplicity of the minimal irreducible positive energy representation at a given level is a quasi-polynomial function of the power of the prequantum line bundle. This is closely related to the $[Q,R] = 0$ Theorem for Hamiltonian LG-spaces. Our approach is based on the theory of quasi-Hamiltonian G-spaces, as well as the approach of Paradan and Szenes-Vergne to the $[Q,R] = 0$ Theorem for compact Hamiltonian G-spaces.

CONTENTS

1. INTRODUCTION

Let G be a compact connected Lie group, and let M be a compact symplectic manifold equipped with a Hamiltonian G-action, with moment map $\phi\colon M \to \mathfrak{g}^*$. Given a G-equivariant prequantum line bundle L, the equivariant index of the Dolbeault-Dirac operator twisted by L is an element in the representation ring,

$$Q(M) = \mathrm{index}(\mathsf{D}^L) \in R(G).$$

The Guillemin-Sternberg *quantization commutes with reduction* principle states that the multiplicity of the trivial representation in $Q(M)$ equals the index of the similarly-defined operator on the symplectic quotient $M_{\mathrm{red}} = \phi^{-1}(0)/G$, assuming the latter is non-singular:

$$(1) \qquad Q(M)^G = Q(M_{\mathrm{red}}).$$

This was first proved in [26] for regular symplectic quotients; in the singular case (1) still holds true after a suitable desingularization [29].

An early approach [24] to the $[Q, R] = 0$ Theorem involved replacing L with $\mathcal{L} = L^k$ and studying the asymptotics of $Q(M, k) = \mathrm{index}(\mathsf{D}^{\mathcal{L}})$ as $k \to \infty$ using the stationary phase approximation. An important theme is that the family $Q(M, k)$ has good behavior as k varies. By Ehrhart's Theorem and the fixed-point formula, one shows that, for sufficiently large k, the multiplicity of the trivial representation in $Q(M, k)$ is a quasi-polynomial function of k. This implies the $o(k^{-\infty})$ errors in the stationary phase approximation must vanish, leading to a proof of (1) for $\mathcal{L} = L^k$, assuming k is sufficiently large. To deduce the result for $k = 1$, one needs a separate argument showing that $Q(M, k)^G$ is a quasi-polynomial function of k for all $k \geq 1$. This was proved in [24] for the case that G is abelian. An argument for the general case was achieved many years later by Szenes-Vergne [38].

The Szenes-Vergne argument was motivated by a different approach to the $[Q, R] = 0$ Theorem, due to Paradan [32]. Using a deformation argument for transversally elliptic operators, Paradan proved a formula for the multiplicities in $Q(M)$, with terms labelled by the components of the critical set of the norm-square of the moment map $\|\phi\|^2$. The leading term of Paradan's formula comes from the zero level of the moment map, all other terms are 'correction terms' which, as shown by Paradan, do not contribute to the multiplicity of the trivial representation. Szenes-Vergne [38] observed that Paradan's norm-square localization formula for $Q(M, k)$ could be derived from the Atiyah-Bott fixed-point formula using a combinatorial argument. The resulting formula leads to a proof that $Q(M, k)^G$ is a quasi-polynomial function of $k \geq 1$.

The goal of this paper is to carry out a similar strategy in the case of Hamiltonian loop group spaces. A Hamiltonian LG-space $\mathcal{M}$ is an infinite dimensional symplectic manifold equipped with a Hamiltonian action of the loop group LG (the precise definition is given in Section 3). In addition we assume throughout that the moment map $\Phi_{\mathcal{M}} \colon \mathcal{M} \to L\mathfrak{g}^*$ is proper. An important example of a proper Hamiltonian LG-space is the moduli space of flat connections on a compact oriented surface with one boundary component, with framing along the boundary; the moment map is induced by restriction of the connection to the boundary, cf. [30] for details.

There have been several approaches to define the 'quantization' of a Hamiltonian LG-space, as an element in the Verlinde ring of projective positive energy representations of LG. In the earliest approach [3], an Atiyah-Singer-type fixed-point formula for the multiplicities was taken as the definition of the quantization of $\mathcal{M}$. A symplectic cutting argument was used to show that this definition satisfies quantization-commutes-with-reduction. Specialized to the moduli space examples, this proves the Verlinde formulas: indeed the fixed-point formula in these examples is the Verlinde formula, while the quantization of the reduced space $\mathcal{M}_{\mathrm{red}} = \Phi_{\mathcal{M}}^{-1}(0)/G$ is the Euler characteristic of the cohomology of $\mathcal{L}_{\mathrm{red}}$.

A more conceptual understanding of the fixed-point formula proposed in [3] was developed in [28, 2]. In this approach one works with the finite dimensional quasi-Hamiltonian G-space $M = \mathcal{M}/\Omega G$ associated to $\mathcal{M}$ [1]. The quantization of M is defined in terms of a pushforward in twisted K-homology from M to G, and the twisted K-homology of the latter is known to be isomorphic to the Verlinde ring by the celebrated Freed-Hopkins-Teleman theorem. The fixed-point formula of [3] is recovered by Atiyah-Segal localization.

A third approach to the quantization of $\mathcal{M}$ was developed in [21, 22] and proved to yield the same result as the twisted K-homology approach in [19]. In this approach one defines

the quantization in terms of the equivariant index of a Dirac-type operator on a non-compact finite-dimensional submanifold of $\mathcal{M}$. In [23] a norm-square localization formula for the index was proved using analytic techniques.

In this article we return to the fixed-point formula for the quantization $Q(\mathcal{M}, k)$, and implement the Szenes-Vergne strategy directly. Let us give an overview of the article. In Section 2 we prove a Lie theoretic inequality. The inequality is difficult to motivate in general, but in the very special case $\mathfrak{g} = \mathfrak{su}_n$ and $\sigma = \{\xi\}$ a vertex of the Stiefel diagram, the inequality is simply $\|\xi - \frac{\rho}{n}\| \geq \|\frac{\rho}{n}\|$, where ρ is half the sum of the positive roots and $\|\cdot\|$ is the norm on $\mathfrak{t} \simeq \mathfrak{t}^*$ induced by the basic inner product. This inequality reappears at the end of the article, where we find that our approach to the quasi-polynomiality theorem hinges on its truth.

Section 3 provides a brief introduction to Hamiltonian LG-spaces $\mathcal{M}$ and the associated quasi-Hamiltonian G-spaces M. The norm-square of the moment map $\|\Phi_{\mathcal{M}}\|^2$ as well as its perturbation $\|\Phi_{\mathcal{M}}\|_\epsilon^2$ by an element $\epsilon \in \mathfrak{t}$ are briefly studied. In particular the discrete subset $\mathfrak{B}_\epsilon = \{\xi \in \mathfrak{t} | \mathcal{M}^{\xi-\epsilon} \cap \Phi_{\mathcal{M}}^{-1}(\xi) \neq \emptyset\}$ indexing the components of the critical set of $\|\Phi_{\mathcal{M}}\|_\epsilon^2$ is introduced.

The quantization of a Hamiltonian LG-space $\mathcal{M}$ (or the associated quasi-Hamiltonian space M) is described in Section 4. The fixed-point formula for the quantization [3] is given towards the end of this section; it has an appearance similar to the Atiyah-Singer fixed-point formula for the index of a Spin_c Dirac operator on M, even though M does not have a Spin_c structure in general. An introduction to the Atiyah-Singer integrand is contained in Appendix A. The quantization $Q(\mathcal{M}, k)$ at level $k > 0$ can be described in terms of its multiplicity function $N(-, k) \colon \Lambda^* \to \mathbb{Z}$, where Λ^* is the weight lattice of a maximal torus $T \subseteq G$; $N(-, k)$ is antisymmetric under an action of the affine Weyl group $W_{\mathrm{aff}} = \Lambda \rtimes W$, and $N(0, k)$ is the multiplicity of the minimal level k positive energy representation in $Q(\mathcal{M}, k)$. Let $\mathrm{h}^\vee$ be the dual Coxeter number of $\mathfrak{g}$. Here and throughout the article $\ell = k + \mathrm{h}^\vee$. The fixed-point formula for N is

$$(2) \qquad N(\lambda, k) = \frac{1}{\#T_\ell} \sum_{t \in T_\ell^{\mathrm{reg}}} \sum_{F \subseteq M^t} t^{-\lambda} \prod_{\alpha \in \mathfrak{R}_-} (1 - t^\alpha) \int_F \mathcal{AS}^t(\nu_{\widetilde{F}}),$$

where $T_\ell = \ell^{-1}\Lambda^*/\Lambda$ (a finite subgroup of T), and $\mathcal{AS}^t(\nu_{\widetilde{F}})$ is a suitable Atiyah-Singer integrand depending on k.

The heart of the argument is contained in Sections 5 and 6. The first step in Section 5 is to reverse the order of the summations in (2). The sum over T_ℓ (intersect a subtorus of T stabilizing some fixed-point set F) is then shown to be essentially an example of a *Verlinde sum* or *rational trigonometric sum*. This type of sum first appeared in the Verlinde formula itself. For example if $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m)$ is a list of weights, then

$$V_{\boldsymbol{\alpha}}(\lambda, \ell) = {\sum_{t \in T_\ell}}' \frac{t^{-\lambda}}{\prod_j (1 - t^{\alpha_j})}$$

is an example of a Verlinde sum, where the prime next to the summation means to omit terms from the sum such that the denominator vanishes. The algebraic and combinatorial properties of Verlinde sums were studied by Szenes, who proved a remarkable residue formula for them [36]. One should also see recent work of Szenes and Trapeznikova [37] related to the residue formula, the Verlinde formula and wall-crossing.

In [20], Szenes' result along with methods pioneered by Boysal-Vergne [9] for closely-related Bernoulli series, were used to derive a 'decomposition formula' for Verlinde sums. The latter is a purely combinatorial analogue of the norm-square localization formula; it expresses a Verlinde sum in terms of an infinite collection of contributions from affine subspaces of different dimensions in $\mathfrak{t} \simeq \mathfrak{t}^*$. When applied to (2), one obtains a formula

$$(3) \qquad N = \sum_{\Delta \in \mathcal{S}} N_{\Delta}^{\mathrm{qpol}}$$

where $\mathcal{S}$ is essentially the collection of all affine subspaces of $\mathfrak{t} \simeq \mathfrak{t}^*$ generated by the image of some T-orbit-type stratum under the moment map. For a subset $I \subseteq \mathfrak{t}^*$ and vector $\delta \in \mathfrak{t}^*$, let

$$C_{I,\delta} = \{(t\xi + \delta, t) \in \mathfrak{t}^* \times \mathbb{R} | \xi \in I, t > 0\}$$

be the cone generated by $I \times \{1\}$, shifted by δ. The contributions $N_{\Delta}^{\mathrm{qpol}}$ are given by rather complicated explicit formulae (equation (46)), but more importantly, they have the following properties (Proposition 5.7):

(a) For each $\delta \in \mathfrak{t}^*$, $N_{\Delta}^{\mathrm{qpol}}$ restricts to a quasi-polynomial function of (λ, ℓ) on $C_{\Delta,\delta}$. In particular $N_{\mathfrak{t}^*}^{\mathrm{qpol}}$ is a quasi-polynomial function.

(b) For $\Delta \neq \mathfrak{t}^*$ and at any fixed ℓ, the function $N_{\Delta}^{\mathrm{qpol}}$ is supported in a half space.

Thus to prove that $k \mapsto N(0, k)$ is quasi-polynomial, it suffices to show that for each Δ not containing the origin, the half space containing the support of $N_{\Delta}^{\mathrm{qpol}}$ does not contain the origin.

In Section 6 we prove the following theorem giving an alternative characterization of the terms $N_{\Delta}^{\mathrm{qpol}}$ in (3).

Theorem 1.1 (Norm-square localization formula). *Let $N_{\Delta}(-, k)$ denote the Fourier transform of the measure*

$$(4) \qquad Q_{\Delta}(t, k) = \delta_{T_{\Delta} T_{\ell}}(t) \sum_{F \in \mathfrak{F}_{\Delta}^t} \int_F \mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_{\Delta}}) \mathcal{AS}^t(\nu_{\tilde{F},\Delta}) \mathrm{Ch}^t(\mathrm{Sym}_{\tau_{\Delta}}(\nu_{\tilde{F},\Delta}^{\perp}) \otimes \wedge \mathfrak{n}_-) \in \mathcal{D}'(T).$$

The multiplicity function N admits a decomposition

$$(5) \qquad N = \sum_{\Delta \in \mathcal{S}} N_{\Delta}^{\mathrm{qpol}}$$

where $N_{\Delta}^{\mathrm{qpol}}$ is the unique function such that (i) $N_{\Delta}^{\mathrm{qpol}}$ is quasi-polynomial on all subsets $C_{\Delta,\delta}$, $\delta \in \mathfrak{t}^*$, and (ii) there is a constant K and an open neighborhood b of ϵ_{Δ} in $\mathfrak{t}$ such that $N_{\Delta}^{\mathrm{qpol}}(\lambda, k) = N_{\Delta}(\lambda, k)$ for $\lambda \in \ell \cdot \mathsf{b}$, $k > K$.

In other words the contribution $N_{\Delta}^{\mathrm{qpol}}$ is obtained by passing to what one might call the 'quasi-polynomial germ at ϵ_{Δ} in the Δ-directions' of the Fourier transform of (58). Here $\epsilon_{\Delta} = \mathrm{pr}_{\Delta}(\epsilon)$ is the orthogonal projection of the generic small perturbation $\epsilon \in \mathfrak{t}$ onto Δ. Towards the end of Section 6 we use a stationary phase argument to prove that $N_{\Delta}^{\mathrm{qpol}}$ vanishes unless $\epsilon_{\Delta} \in \mathfrak{B}_{\epsilon}$, indicating that Theorem 1.1 is a norm-square localization formula.

Section 7 contains examples of the norm-square localization formula, including that illustrated in Figure 1. In Section 8 we complete the last step in the strategy indicated above, in order to prove the quasi-polynomiality theorem. The key ingredients are the norm-square
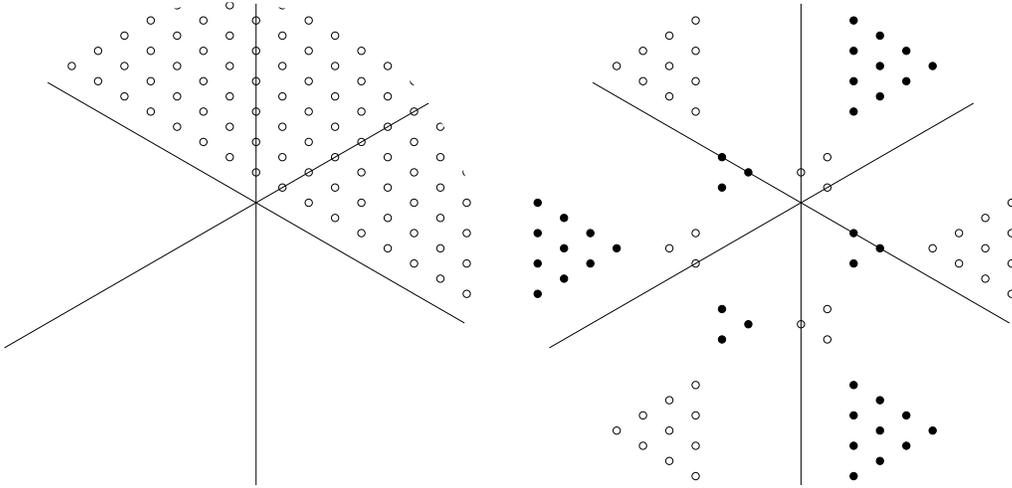
FIGURE 1. The left image shows a single contribution to the norm-square localization formula for a multiplicity-free q-Hamiltonian $SU(3)$-space (at level $k = 2$). The right image shows the sum of the first 6 contributions.

localization formula, the description of the T action on the 'canonical bundle' [31] involved in the fixed-point formula, and of course the inequality from Section 2.

Theorem 1.2 (Quasi-polynomiality theorem)**.** *The function $k \mapsto N(0, k)$ is quasi-polynomial for all $k \geq 1$. In other words, the multiplicity of the minimal irreducible level k positive energy representation in the quantization $Q(\mathcal{M}, k)$, is a quasi-polynomial function of k.*

The $[Q, R] = 0$ theorem for Hamiltonian LG-spaces can be deduced from an additional argument using the stationary phase expansion, as in [24]. Specialized to the moduli space examples, this provides a new proof of the Verlinde formula.

This is a preliminary version of the article. The results described here appeared a few years ago in the PhD thesis of the first author [16], based on joint work of both authors. In the intervening years the authors have attempted to simplify the presentation and proofs of the results.

2. AN INEQUALITY FOR STIEFEL DIAGRAMS

We begin by discussing the key Lie-theoretic inequality alluded to in the introduction.

2.1. **Notation.** Let G be a compact, 1-connected, simple Lie group with maximal torus T. The Lie algebras will be denoted $\mathfrak{g}, \mathfrak{t}$. Let $\Lambda = \ker(\exp \colon \mathfrak{t} \to T)$ be the integral lattice and $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbb{Z})$ its dual, the (real) weight lattice. Denote by $\mathfrak{R} \subseteq \Lambda^*$ the set of roots α, and by $\mathfrak{R}^\vee \subseteq \Lambda$ the set of coroots $\alpha^\vee$. Recall that the coroots generate Λ.

Let $\mathfrak{R}_{\mathrm{aff}} = \mathfrak{R} \times \mathbb{Z}$ be the set of affine roots. The *Stiefel diagram* of G is defined by the affine root hyperplanes

$$H_{\alpha,k} = \{\xi \in \mathfrak{t}|\ \langle \alpha, \xi \rangle + k = 0\}$$

for $(\alpha, k) \in \mathfrak{R}_{\mathrm{aff}}$. The closures of the connected components of $\mathfrak{t} - \bigcup_{\alpha,k} H_{\alpha,k}$ are simple polytopes called the (closed) *alcoves*. The open faces σ of these polytopes will be referred to as the *open faces* of the Stiefel diagram; thus $\mathfrak{t}$ is a disjoint union over the open faces. A zero-dimensional face is also called a *vertex*. Let W be the Weyl group of (G, T). The affine Weyl group $W_{\mathrm{aff}} = \Lambda \rtimes W$ acts on $\mathfrak{t}$ by reflections across affine root hyperplanes; any of the alcoves is a fundamental domain for this action.

Fix a fundamental Weyl chamber $\mathfrak{t}_+$, and let $\mathfrak{R}_+$ be the corresponding set of positive roots, taking on nonnegative values on $\mathfrak{t}_+$. The highest root $\theta \in \mathfrak{R}_+$ defines the *fundamental alcove*

$$\mathsf{a} = \{\xi \in \mathfrak{t}_+ | \langle \theta, \xi \rangle \le 1\}.$$

Let $\mathfrak{R}_{\mathrm{aff},+}$ be the positive *affine* roots, taking on nonnegative values on a:

$$\mathfrak{R}_{\mathrm{aff},+} = \{(\alpha, k) \in \mathfrak{R}_{\mathrm{aff}}|\ \forall \xi \in \mathsf{a}\colon \langle \alpha, \xi \rangle + k \ge 0\}.$$

Thus, either $k = 0$ and $\alpha \in \mathfrak{R}_+$, or $k > 0$ and α arbitrary. The half-sum of the positive roots is denoted

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha,$$

and the dual Coxeter number is

$$\mathrm{h}^\vee = 1 + \langle \rho, \theta^\vee \rangle.$$

The *basic inner product* $\cdot$ on $\mathfrak{g}$ is the unique invariant inner product on $\mathfrak{g}$ such that the coroot $\theta^\vee$ satisfies $\theta^\vee \cdot \theta^\vee = 2$. The associated norm will be denoted $\| \cdot \|$. Throughout this paper, we often identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{t} \cong \mathfrak{t}^*$ using the basic inner product.

2.2. **The inequality.** Given an open face σ of the Stiefel diagram, let

$$\mathfrak{R}_{\mathrm{aff}}(\sigma) = \{(\alpha, k) \in \mathfrak{R}_{\mathrm{aff}}|\ \sigma \subseteq H_{\alpha,k}\},$$

the set of affine roots vanishing on σ. The quotient map $\mathfrak{R}_{\mathrm{aff}} \to \mathfrak{R}$, $(\alpha, k) \mapsto \alpha$ restricts to a bijection of this set with the root system $\mathfrak{R}(\sigma)$ of the centralizer $G_{\exp(\xi)} \subseteq G$ of the element $\exp(\xi)$, for any $\xi \in \sigma$. Its rank is equal to the codimension of σ in $\mathfrak{t}$. There are two natural choices for systems of positive roots in $\mathfrak{R}(\sigma)$, both of which will play a role in what follows. One choice is

$$\mathfrak{R}_+(\sigma) = \mathfrak{R}(\sigma) \cap \mathfrak{R}_+;$$

let $\mathfrak{t}_{\sigma,+}$ be the corresponding positive Weyl chamber, and put $\rho_\sigma = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+(\sigma)} \alpha$. Another choice is the image of

$$\mathfrak{R}_{\mathrm{aff},+}(\sigma) = \mathfrak{R}_{\mathrm{aff}}(\sigma) \cap \mathfrak{R}_{\mathrm{aff},+}$$

under $(\alpha, k) \mapsto \alpha$; we will denote this choice by $\mathfrak{R}'_+(\sigma)$. Let $\mathfrak{t}'_{\sigma,+}$ be the associated Weyl chamber, and ρ'_σ the corresponding half-sums of positive roots.

Lemma 2.1. *For $\xi \in \sigma$,*

$$\rho'_\sigma = -\rho_\sigma + \sum_{\alpha \in \mathfrak{R}_+\colon\ \langle \alpha, \xi \rangle \in \mathbb{Z}_{\le 0}} \alpha.$$

Proof. Note that $\rho'_\sigma + \rho_\sigma$ is the sum over all $\alpha \in \mathfrak{R}_+(\sigma) \cap \mathfrak{R}'_+(\sigma)$. The roots in $\mathfrak{R}(\sigma)$ are those $\alpha \in \mathfrak{R}$ for which there exists $k \in \mathbb{Z}$ with $\langle \alpha, \xi \rangle + k = 0$. The affine root (α, k) is positive if and only if either $k = 0$ and $\alpha \in \mathfrak{R}_+$, or $k > 0$. Hence, the condition that α lies in $\mathfrak{R}_+(\sigma) \cap \mathfrak{R}'_+(\sigma)$ just means that $\langle \alpha, \xi \rangle = -k$ is a non-positive integer. $\qquad\square$

Recall next that the interior of the fundamental alcove contains a distinguished element [14]

$$\zeta_{\mathring{\mathsf{a}}} = \frac{1}{\mathrm{h}^\vee} \rho.$$

For the open faces σ of the fundamental alcove we define a distinguished element $\zeta_\sigma \in \sigma$ to be the orthogonal projection of $\zeta_{\mathring{\mathsf{a}}}$ onto σ; we extend this definition to all open faces of the Stiefel diagram by requiring that $\zeta_{w\sigma} = w\zeta_\sigma$ for all $w \in W_{\mathrm{aff}}$.

Theorem 2.2 (Key inequality). *Let σ be an open face of the Stiefel diagram, and $\xi = \zeta_\sigma$ its distinguished element. Then*

(6) $$\mathrm{h}^\vee \|\xi\|^2 - \langle \rho - \rho'_\sigma, \xi \rangle \geq 0,$$

with equality if and only if σ is a face of the fundamental alcove.

Remark 2.3.   (a) Equation (6) can be written

$$\|\xi - \tau_\sigma\| \geq \|\tau_\sigma\|, \qquad \tau_\sigma = \frac{\rho - \rho'_\sigma}{2\mathrm{h}^\vee}.$$

     (b) Combining (6) with Lemma 2.1, we obtain

(7) $$\mathrm{h}^\vee \|\xi\|^2 - \langle \rho + \rho_\sigma, \xi \rangle \geq 0.$$

This inequality will be used later in the paper, for the special case that σ is a vertex $\{\xi\}$ of the Stiefel diagram. If ξ is furthermore in the co-weight lattice, so that it exponentiates to an element of the center of G, then $\rho_\sigma = \rho$, hence the inequality says that

$$\|\xi - \zeta_{\mathring{\mathsf{a}}}\| \geq \|\zeta_{\mathring{\mathsf{a}}}\|.$$

We will prepare the proof of Theorem 2.2 with several lemmas. Denote by $W_{\mathrm{aff}}(\sigma)$ the stabilizer of points in σ under the action of the affine Weyl group. It is generated by the affine reflections for affine roots vanishing on σ; its image under the quotient map $W_{\mathrm{aff}} \to W$ is the Weyl group $W(\sigma)$ of $\mathfrak{R}(\sigma)$. In particular, $W_{\mathrm{aff}}(\sigma)$ acts simply transitively on the set of *affine Weyl chambers at σ*, by which we mean the Weyl chambers of $\mathfrak{R}(\sigma)$, shifted by $\xi = \zeta_\sigma$. For instance, $\xi + \mathfrak{t}'_{\sigma,+}$ is an affine Weyl chamber at σ.

Lemma 2.4. *Given any open face σ of the Stiefel Diagram, there is a unique shortest element $w \in W_{\mathrm{aff}}$ such that $\sigma \subseteq w\mathsf{a}$. With this choice of w,*

$$w\mathsf{a} \subseteq \xi + \mathfrak{t}'_{\sigma,+}$$

for $\xi \in \sigma$. Alternatively, w is uniquely characterized by the properties that $\tau := w^{-1}\sigma \subseteq \mathsf{a}$ and $\overline{w}\mathfrak{t}'_{\tau,+} = \mathfrak{t}'_{\sigma,+}$, where $\overline{w} \in W$ is the image of $w \in W_{\mathrm{aff}}$.

Proof. Recall that length of the unique Weyl group element taking a given alcove a_1 to the fundamental alcove a equals the number of affine root hyperplanes that are crossed by a line segment from a point in the interior of a to a point in the interior of a. Hence, the length of the shortest shortest Weil group element w taking σ to a face of a is the number of affine root

hyperplanes from a point in the interior of a to a point in σ. The corresponding alcove $\mathsf{a}_1 = w\mathsf{a}$ contains σ, and is the unique alcove containing ξ and contained in $\xi + \mathfrak{t}'_{\sigma,,+}$. [....]

The alternative characterization of w is obvious if $\sigma \in \mathsf{a}$ (so that $w = 1$), the general case follows because $w\mathsf{a} \subseteq w(\zeta_\tau + \mathfrak{t}'_{\tau,+}) = \zeta_{w\tau} + \overline{w}\mathfrak{t}'_{w\tau}$ for $w \in W_{\mathrm{aff}}$. $\square$

Lemma 2.5. *Suppose σ is an open face of the Stiefel diagram, and let $w \in W_{aff}$ be the unique shortest element such that $w^{-1}\sigma \subseteq \mathsf{a}$. (cf. Lemma 2.4). Then the distinguished element $\zeta_\sigma \in \sigma$ is given by*

$$\zeta_\sigma = w\Big(\frac{1}{\mathrm{h}^\vee}\rho\Big) - \frac{1}{\mathrm{h}^\vee}\rho'_\sigma$$

Proof. For the case $\sigma \subseteq \mathsf{a}$, $w = 1$ this is the statement of [31, Lemma 3.1]. In the general case, write $\tau = w^{-1}\sigma \subseteq \mathsf{a}$, so that $\zeta_\sigma = w\zeta_\tau$, and $\mathfrak{t}'_{\sigma,+} = \overline{w}\mathfrak{t}'_{\tau,+}$, hence also $\rho'_\sigma = \overline{w}\rho'_\tau$. Since $\zeta_\tau = \frac{1}{\mathrm{h}^\vee}\rho - \frac{1}{\mathrm{h}^\vee}\rho'_\tau$ by [31], we obtain

$$\zeta_\sigma = w\zeta_\tau = w\Big(\frac{1}{\mathrm{h}^\vee}(\rho - \rho'_\tau)\Big) = w\Big(\frac{1}{\mathrm{h}^\vee}\rho\Big) - \overline{w}\Big(\frac{1}{\mathrm{h}^\vee}\rho'_\tau\Big) = w\Big(\frac{1}{\mathrm{h}^\vee}\rho\Big) - \frac{1}{\mathrm{h}^\vee}\rho'_\sigma. \qquad \square$$

This following is a version of a standard fact about affine root systems [15, Corollary 1.3.22].

Lemma 2.6. *For all $w \in W_{aff}$, we have that*

$$\frac{1}{\mathrm{h}^\vee}\rho - w\Big(\frac{1}{\mathrm{h}^\vee}\rho\Big) = \frac{1}{\mathrm{h}^\vee}\sum \alpha$$

where the sum is over all positive affine $(\alpha, k) \in \mathfrak{R}_{aff,+}$ such that the alcoves $\mathsf{a}, w\mathsf{a}$ are on opposite sides of the affine hyperplane $H_{\alpha,k}$.

Proof. If w is a reflection corresponding to one of the simple roots, this is well known. If w is the affine reflection $\xi \mapsto \xi - (\langle\theta, \xi\rangle - 1)\theta$ corresponding to the affine root $(-\theta, 1)$, it follows by direct computation:

$$w\Big(\frac{1}{\mathrm{h}^\vee}\rho\Big) = \frac{1}{\mathrm{h}^\vee}\rho - \Big(\langle\theta, \frac{1}{\mathrm{h}^\vee}\rho\rangle - 1\Big)\theta = \frac{1}{\mathrm{h}^\vee}(\rho + \theta).$$

This proves the lemma for all simple affine reflections. By applying the action of W_{aff}, it follows more generally that if $\mathsf{a}_1, \mathsf{a}_2$ are two adjacent alcoves of the Stiefel diagram, separated by an affine root hyperplane $H_{\alpha,k}$, with $\langle\alpha, \cdot\rangle + k$ positive on $\mathring{\mathsf{a}}_1$ and negative on $\mathring{\mathsf{a}}_2$, then the difference of their distinguished elements is

$$(8) \qquad\qquad\qquad\qquad \zeta_{\mathring{\mathsf{a}}_1} - \zeta_{\mathring{\mathsf{a}}_2} = \frac{1}{\mathrm{h}^\vee}\alpha.$$

For the general case of the Lemma, we may choose the points in $\mathring{\mathsf{a}}$ and $w\mathring{\mathsf{a}}$ in such a way that the line segment between these points crosses the various affine root hyperplanes $H_{(\alpha,k)}$, $(\alpha, k) \in \mathfrak{R}_{\mathrm{aff},+}$ at distinct times $0 < t_1 < \ldots < t_N < 1$. Note that this line segement always crosses from positive sides to negative sides. Letting $\mathsf{a}_0 = \mathsf{a}, \ldots, \mathsf{a}_N = w\mathsf{a}$ be the sequence of alcoves through which this line segment passes, write

$$\frac{1}{\mathrm{h}^\vee}\rho - w\Big(\frac{1}{\mathrm{h}^\vee}\rho\Big) = \sum_{i=0}^{N-1}(\zeta_{\mathring{\mathsf{a}}_i} - \zeta_{\mathring{\mathsf{a}}_{i+1}})$$

and apply (8) at each step. $\square$

Proof of Theorem 2.2. Let $\xi = \zeta_\sigma$. Combining Lemmas 2.4, 2.5 and 2.6, we obtain

$$
(9) \qquad \xi - \frac{1}{h^\vee}(\rho - \rho'_\sigma) = w\left(\frac{1}{h^\vee}\rho\right) - \frac{1}{h^\vee}\rho,
$$

hence

$$
(10) \qquad \|\xi\|^2 - \frac{1}{h^\vee}\langle \rho - \rho'_\sigma, \xi \rangle = \left\langle w\left(\frac{1}{h^\vee}\rho\right) - \frac{1}{h^\vee}\rho, \ \xi \right\rangle = -\frac{1}{h^\vee}\sum_{(\alpha,k)}\langle \alpha, \xi \rangle
$$

where the sum is over all positive affine roots $(\alpha, k) \in \mathfrak{R}_{\mathrm{aff},+}$ such that the line segment from $\zeta_{\mathring{\mathsf{a}}}$ to $\xi = \zeta_\sigma$ crosses $H_{\alpha,k}$. In other words, there exists $t \in [0, 1]$ such that

$$
\langle \alpha, (1-t)\zeta_{\mathring{\mathsf{a}}} + t\xi \rangle + k = 0.
$$

Note that for $k \geq 0$, this can only happen if $\langle \alpha, \xi \rangle < 0$. Hence, the right hand side of (10) is ≥ 0, with equality if and only if σ is a face of a (so that the line segment does not cross any affine root hyperplanes). $\qquad\square$

2.3. The level k fusion ring.

Recall that we use the basic inner product on $\mathfrak{g}$ to identify $\mathfrak{g} = \mathfrak{g}^*$, $\mathfrak{t} = \mathfrak{t}^*$. Under this identification, Λ is identified with a sublattice of Λ^*. For $\ell \in \mathbb{N}$, the quotient

$$
T_\ell := \left(\tfrac{1}{\ell}\Lambda^*\right)/\Lambda \subseteq \mathfrak{t}/\Lambda = T,
$$

is a finite subgroup of $T = \mathfrak{t}/\Lambda$. Let $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}_+$ be the *dominant weights*, and $\Lambda^*_k = \Lambda^* \cap k\mathsf{a}$ for $k \in \mathbb{N} \cup \{0\}$ the *level k weights*. Let χ_λ, $\lambda \in \Lambda^*_+$ be the characters of the irreducible representations of highest weight λ; these span $R(G) \subseteq C^\infty(G)$. The *level k fusion ring* (Verlinde algebra) is the quotient

$$
R_k(G) = R(G)/I_k(G),
$$

where $I_k(G)$ is the ideal of characters vanishing at all points of $T^{\mathrm{reg}}_{k+h^\vee} = T_{k+h^\vee} \cap G^{\mathrm{reg}}$. It has an additive basis given by the images τ_λ of the characters χ_λ for $\lambda \in \Lambda^*_k$. For elements $\tau \in R_k(G)$, the evaluation at points $t \in T^{\mathrm{reg}}_{k+h^\vee}$ is well-defined, and τ is recovered from these values by finite Fourier transform. In fact, writing $\tau = \sum_{\mu \in \Lambda^*_k} N(\mu)\tau_\mu$ we obtain the multiplicities as

$$
(11) \qquad N(\mu) = \frac{1}{\#T_\ell}\sum_{t \in T^{\mathrm{reg}}_\ell} t^{-\mu}\,\tau(t)\prod_{\alpha \in \mathfrak{R}_-}(1 - t^\alpha)
$$

where $\ell = k + h^\vee$. It is convenient to extend the domain of the multiplicity function N from Λ^*_k to the entire weight lattice, by requiring the following anti-symmetry under the shifted Weyl group action at level ℓ:

$$
N(w \bullet_\ell \mu) = (-1)^{l(w)}N(\mu),
$$

where

$$
w \bullet_\ell \mu = \ell\, w\left(\frac{\mu + \rho}{\ell}\right) - \rho.
$$

With this definition, formula (11) applies to all $\mu \in \Lambda^*$.

Remark 2.7. The fusion ring can be regarded as the Grothendieck group of projective level k positive energy representations of LG. For $\tau \in R_k(G)$, the character of the corresponding positive energy representation is described by the Weyl-Kac formula. The function N is the multiplicity function for the distributional character (on T) given as the Weyl-Kac enumerator.

3. Hamiltonian LG-spaces

Throughout this section, G is compact, 1-connected, and simple, although many of the results discussed here hold in much greater generality. We take $\cdot$ to be the basic inner product on $\mathfrak{g}$.

3.1. The coadjoint LG-action.

Let LG denote the Banach Lie group consisting of maps $S^1 \to G$ of some fixed Sobolev class $\mathsf{s} > \frac{1}{2}$, and $L\mathfrak{g} = \Omega_{\mathsf{s}}^0(S^1, \mathfrak{g})$ its Lie algebra. We define the smooth dual of the Lie algebra to be $L\mathfrak{g}^* = \Omega_{\mathsf{s}-1}^1(S^1, \mathfrak{g})$, with the pairing given by the inner product on $\mathfrak{g}$ followed by integration: $\langle \mu, \xi \rangle = \int_{S^1} \mu \cdot \xi$. The loop group LG acts smoothly on $L\mathfrak{g}^*$ by gauge transformations:

$$\text{(12)} \qquad\qquad \mu \mapsto \mathrm{Ad}_g \mu - g^*\theta^R;$$

we refer to (12) as the coadjoint LG-action. The generating vector fields for this action are $\xi_{L\mathfrak{g}^*}(\mu) = \mathrm{d}_\mu \xi$, where

$$\mathrm{d}_\mu = \mathrm{d} + \mathrm{ad}_\mu \colon \Omega_{\mathsf{s}}^0(S^1, \mathfrak{g}) \to \Omega_{\mathsf{s}-1}^1(S^1, \mathfrak{g})$$

is the covariant derivative. We shall regard $\mathfrak{g}$ as the submanifold of $L\mathfrak{g}^*$ consisting of constant loops. It is well-known that the intersection of a coadjoint LG-orbit with $\mathfrak{t} \subseteq \mathfrak{g} \subseteq L\mathfrak{g}^*$ is an orbit of the affine Weyl group W_{aff} on $\mathfrak{t}$.

The coadjoint LG-action has canonical slices, as follows. Let $\sigma \subseteq \mathfrak{t}$ be an open face of the Stiefel diagram. The stabilizer of elements $\xi \in \sigma$ under the coadjoint LG-action depends only on σ, and is denoted by LG_σ. The set

$$\text{(13)} \qquad\qquad \mathcal{U}_\sigma = LG_\sigma \cdot \bigcup_{\overline{\tau} \supseteq \sigma} \tau$$

(union is over the set of faces whose closure contains σ) is an open subset of the affine subspace $\xi + \mathrm{ran}(\mathrm{d}_\xi^*) \subseteq L\mathfrak{g}^*$ for $\xi \in \sigma$, and is a slice at ξ for the coadjoint action. For $\sigma \subseteq \overline{\tau}$ we have $LG_\tau \subseteq LG_\sigma$, hence $G_{\exp \tau} \subseteq G_{\exp \sigma}$, and equivariant embeddings

$$LG_\sigma \times_{LG_\tau} \mathcal{U}_\tau \hookrightarrow \mathcal{U}_\sigma$$

as dense open subsets.

The slices for the coadjoint LG-action are closely related to slices for the conjugation action. The exponential map $\exp \colon \mathfrak{g} \to G$ extends equivariantly to the *holonomy map*

$$\mathrm{Hol} \colon L\mathfrak{g}^* \to G,$$

with $\mathrm{Hol}(g \cdot \mu) = \mathrm{Ad}_{g(0)} \mathrm{Hol}(\mu)$. It maps coadjoint LG-orbits to conjugacy classes; the map $LG \to G$, $g \mapsto g(0)$ restricts to isomorphisms of stabilizers $LG_\mu \to G_{\mathrm{Hol}(\mu)}$. For $\mu \in \sigma$, the centralizer $G_{\exp(\mu)}$ depends only on σ, and will be denoted $G_{\exp(\sigma)}$. The sets $U_{\exp \sigma} = G_{\exp \sigma} \cdot \bigcup_{\overline{\tau} \supseteq \sigma} \exp \tau$ are slices for the conjugation action, and Hol restricts to diffeomorphisms $\mathcal{U}_\sigma \to U_{\exp \sigma}$, intertwining the actions of $LG_\sigma \cong G_{\exp \sigma}$.

3.2. The central extension of the loop group.

Let $\widehat{LG}$ be the basic central extension of LG. We will need some facts regarding the restriction of these central extension to various subgroups of the loop group. Over $G \subseteq LG$, the central extension $\widehat{LG}$ is canonically trivial (as is any central extension of a simply connected group). Hence the restriction to $T \subseteq G$ is trivial as well. On the other hand, the restriction to the lattice $\Lambda \subseteq LT \subseteq LG$ of 'exponential loops) may be non-trivial. It may be described, up to isomorphism, in terms of the corresponding

group commutator: If $\widehat{\lambda}_1, \widehat{\lambda}_2 \in \widehat{\Lambda}$ are elements lifting λ_1, λ_2, then the commutator is expressed in terms of the basic inner product $\lambda_1 \cdot \lambda_2 \in \mathbb{Z}$ as

$$\widehat{\lambda}_1 \widehat{\lambda}_2 \widehat{\lambda}_1^{-1} \widehat{\lambda}_2^{-1} = (-1)^{\lambda_1 \cdot \lambda_2}, \tag{14}$$

see [33, 39]. Over the product $\Lambda \times T \subseteq LG$, the central extension restricts to a semi-direct product

$$\widehat{LG}|_{\Lambda \times T} = \widehat{\Lambda} \rtimes T \tag{15}$$

with $\widehat{\Lambda} = \widehat{LG}|_\Lambda$. Here the T-action on $\widehat{\Lambda}$ is given, on the fiber of a given element $\lambda \in \Lambda$, as scalar multiplication by t^λ, where the basic inner product is used to regard Λ as a sublattice of Λ^*; see [3, Proposition 4.1].

For $\ell \in \mathbb{Z}$, we use a superscript (ℓ) to indicate the ℓ-th power of the extension of LG (and its subgroups). Equation (14) shows that $\widehat{\Lambda}^{(\ell)}$ for *even* ℓ can be trivialized, by choosing lifts of a basis of Λ. Furthermore, by (15)

$$\widehat{LG}^{(\ell)}|_{\Lambda \times T_\ell} = \widehat{\Lambda} \times T_\ell$$

(a direct product) for any $\ell \in \mathbb{N}$.

3.3. Hamiltonian LG-spaces.
The basic theory of Hamiltonian loop group spaces was developed in [30, 3, 7, 10].

Definition 3.1. A *proper Hamiltonian LG-space* $(\mathcal{M}, \omega_\mathcal{M}, \Phi_\mathcal{M})$ is a Banach manifold $\mathcal{M}$ equipped with a smooth LG-action, an LG-invariant weakly symplectic form $\omega_\mathcal{M}$, and a proper LG-equivariant smooth *moment map*

$$\Phi_\mathcal{M} \colon \mathcal{M} \to L\mathfrak{g}^*$$

satisfying $\iota(\xi_\mathcal{M})\omega = -\mathrm{d}\langle \Phi_\mathcal{M}, \xi \rangle$ for all $\xi \in L\mathfrak{g}$.

In [1], it was observed that proper Hamiltonian loop group spaces are in 1-1 correspondence with finite-dimensional quasi-Hamiltonian spaces, as follows. Denote by $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ the left-invariant, right-invariant Maurer-Cartan forms, and by $\eta \in \Omega^3(G)$ the Cartan 3-form

$$\eta = \tfrac{1}{12}\langle \theta^L, [\theta^L, \theta^L] \rangle.$$

It has an equivariant extension, for G acting on itself by conjugation,

$$\eta_G(\xi) = \eta - \tfrac{1}{2}\langle \theta^L + \theta^R, \xi \rangle, \qquad \xi \in \mathfrak{g}.$$

Definition 3.2. [1] A *q-Hamiltonian G-space* (M, ω, Φ) is a G-manifold M, together with an invariant 2-form ω and a smooth equivariant map $\Phi \colon M \to G$ (the *group-valued moment map*) satisfying

$$\mathrm{d}_G\omega = -\Phi^*\eta_G, \tag{16}$$

and such that $\ker(\omega_m) \cap \ker(T_m\Phi) = \{0\}$ for all $m \in M$. This last condition is referred to as 'minimal degeneracy'; if it is not satisfied, then we refer to M as a *degenerate* q-Hamiltonian G-space.

As shown in [1], the quotient of $\mathcal{M}$ by the based loop group $L_0 G$ is a q-Hamiltonian G-space M; in turn, $\mathcal{M}$ is recovered from M as the pullback of the holonomy fibration $L\mathfrak{g}^* \to G$ under the moment map $\Phi \colon M \to G$. Let $p \colon \mathcal{M} \to M$ be the quotient map:

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\ \Phi_{\mathcal{M}}\ } & L\mathfrak{g}^* \\
{\scriptstyle p}\downarrow & & \downarrow \\
M & \xrightarrow[\ \Phi\ ]{} & G
\end{array}
$$

The 2-form $\omega_{\mathcal{M}}$ is related to ω by an equality

$$\omega_{\mathcal{M}} = p^*\omega + \Phi_{\mathcal{M}}^* \varpi,$$

where $\varpi \in \Omega^2(L\mathfrak{g}^*)$ is a distinguished LG-invariant primitive of the pullback of η under the holonomy fibration $L\mathfrak{g}^* \to G$. Since G is connected, the q-Hamiltonian G-space M is even-dimensional [1], and since G simply connected it comes with a canonical volume form [4], defining in particular an orientation.

Given a face σ of the Stiefel diagram, the pre-image of the corresponding slice is a finite-dimensional symplectic manifold $\mathcal{Y}_\sigma = \Phi_{\mathcal{M}}^{-1}(\mathcal{U}_\sigma)$ called the *symplectic cross-section*. Its image under the quotient map to M is a q-Hamiltonian $G_{\exp\sigma}$-space $Y_{\exp\sigma} = \Phi^{-1}(U_{\exp\sigma})$.

Here is one way of describing the canonical orientation of M. One first observes that the symplectic structure on $Y_{\exp\sigma}$ induced by the diffeomorphism with $\mathcal{Y}_\sigma$ depends only on $\exp\sigma$ (not on σ). In particular, $Y_{\exp\sigma}$ has a canonical orientation. The normal bundle to $Y_{\exp\sigma}$ inside M is $G_{\exp\sigma}$-equivariantly isomorphic to the trivial bundle with fibers $\mathfrak{g}/\mathfrak{g}_{\exp\sigma}$. This space has an orientation, obtained as the quotient of the orientations on $\mathfrak{g}/\mathfrak{t}$ and on $\mathfrak{g}_{\exp\sigma}/\mathfrak{t} \cong L\mathfrak{g}_\sigma/\mathfrak{t}$. Here, the latter orientation is defined by the Weyl chamber $\mathfrak{t}'_{+,\sigma}$ discussed in Section 2.2., or equivalently by the complex structure on $L\mathfrak{g}_\sigma/\mathfrak{t} \subseteq L\mathfrak{g}/\mathfrak{t}$. This explains the orientation of M along $Y_{\exp\sigma}$, hence also on its flow-out under the conjugation action; one finds that the local orientations agree on overlaps.

3.4. **The norm-square of the moment map.** For $\mathsf{s} > 1$, the elements of $L\mathfrak{g}^*$ have Sobolev class > 0, hence the *energy functional*

$$\| \cdot \|^2 \colon L\mathfrak{g}^* \to \mathbb{R}, \quad \mu \mapsto \|\mu\|^2 = \int_{S^1} \mu \cdot *\mu$$

(where $*(\mathrm{d}s) = 1$, $*(1) = \mathrm{d}s$) is a well-defined smooth function, invariant under the action of $G \subseteq LG$. More generally, for given $\epsilon \in L\mathfrak{g}^*$ we consider the *ϵ-shifted energy functional* $\| \cdot \|_\epsilon^2$ defined as

$$\|\mu\|_\epsilon^2 = \|\mu - \epsilon\|^2.$$

This satisfies $\|g \cdot \mu\|_\epsilon^2 = \|\mu\|_{g^{-1}\cdot\epsilon}^2$, in particular, $\| \cdot \|_\epsilon^2$ is LG_ϵ-invariant. From now on, we will take ϵ in $\mathfrak{t} \subseteq L\mathfrak{g}^*$ (mostly $\epsilon \in \mathfrak{a}$). The following is a straightforward extension of a result of Bott (for $\epsilon = 0$):

Proposition 3.3. *Let $\epsilon \in \mathfrak{t}$. For any coadjoint LG-orbit $\mathcal{O} \subseteq L\mathfrak{g}^*$, the set of critical points of the restriction of $\| \cdot \|_\epsilon^2$ to $\mathcal{O}$ is*

$$LG_\epsilon.(\mathcal{O} \cap \mathfrak{t}).$$

In particular, if ϵ is regular (with stabilizer T) the critical set is the intersection of $\mathcal{O} \cap \mathfrak{t}$, while for $\epsilon = 0$ it is the intersection $\mathcal{O} \cap \mathfrak{g}$.

Proof. Every tangent vector at a given point $\mu \in \mathcal{O}$ is realized as the derivative, at $t = 0$, of a path $\mu_t = g_t \cdot \mu = \mathrm{Ad}_{g_t}(\mu - g_t^* \theta^L)$, with $g_t \in LG$, $g_0 = e$. Let $\xi = \frac{\partial}{\partial t}\big|_{t=0} g_t \in L\mathfrak{g}$. Then μ is a critical point if and only if

$$\frac{d}{dt}\Big|_{t=0} \|\mu_t\|_\epsilon^2 = -2 \int_{S^1} (\mu - \epsilon) \cdot *\mathrm{d}_\epsilon \xi = 2 \int_{S^1} \mathrm{d}_\epsilon(*\mu) \cdot \xi$$

(noting that $\mathrm{d}_\epsilon * \epsilon = 0$) vanishes for all ξ, which is the case if and only if $\mathrm{d}_\epsilon(*\mu) = 0$. But this is equivalent to $\mathrm{Ad}_{\exp s\epsilon}\, \mu$ being constant, i.e. in $\mathfrak{g} \subseteq L\mathfrak{g}^*$. Thus, $\mu_s = \mathrm{Ad}_{\exp(-s\epsilon)}\, \nu$ for some $\nu \in \mathfrak{g}$. The condition that μ is a loop means that ν is in the Lie algebra of the centralizer of $\exp(\epsilon)$. Equivalently, $\nu \in \mathrm{Ad}_k \mathfrak{t}$ for some $k \in G_{\exp(\epsilon)}$, i.e. $\mu \in g \cdot \mathfrak{t}$, where $g \in LG_\epsilon$ is the loop $s \mapsto \mathrm{Ad}_{\exp(-s\epsilon)}(k)$, i.e., the pre-image of k under the isomorphism $LG_\epsilon \cong G_{\exp(\epsilon)}$. $\qquad\square$

Given a proper Hamiltonian LG-space $(\mathcal{M}, \omega_\mathcal{M}, \Phi_\mathcal{M})$, the composition of the moment map with the shifted energy functional for $\epsilon \in \mathsf{a}$ gives a smooth LG_ϵ-equivariant map,

$$\|\Phi_\mathcal{M}\|_\epsilon^2 \colon \mathcal{M} \to \mathbb{R}.$$

The following description of its critical set is the loop group counterpart of a result of Kirwan [12]; it was proved in [8, 17] for the case $\epsilon = 0$. The description is in terms of the subset

(17) $$\mathfrak{B}_\epsilon = \{\xi \in \mathfrak{t} \colon \mathcal{M}^{\xi - \epsilon} \cap \Phi_\mathcal{M}^{-1}(\xi) \neq \emptyset\}.$$

where $\mathcal{M}^\xi \subseteq \mathcal{M}$ denotes the fixed point set of $\xi_\mathcal{M}$.

Proposition 3.4. *Let $(\mathcal{M}, \omega_\mathcal{M}, \Phi_\mathcal{M})$ be a proper Hamiltonian LG-space, and $\epsilon \in \mathfrak{t}$. Then the set of critical points of the shifted energy functional is*

$$\mathrm{crit}(\|\Phi_\mathcal{M}\|_\epsilon^2) = LG_\epsilon \cdot \bigcup_{\xi \in \mathfrak{B}_\epsilon} \mathcal{M}^{\xi - \epsilon} \cap \Phi_\mathcal{M}^{-1}(\xi).$$

In particular, the set of critical values is $LG_\epsilon.(\mathfrak{B}_\epsilon)$.

Proof. For $\epsilon = 0$, the proof may be found in [8, 17]. The argument in the general case is similar: We first observe that by equivariance of the moment map, if $x \in \mathcal{M}$ is a critical point for $\|\Phi_\mathcal{M}\|_\epsilon^2$, then $\xi = \Phi_\mathcal{M}(x) \in LG_\epsilon \cdot \mathfrak{t}$ is a critical point for the restriction of $\|\cdot\|_\epsilon^2$ to the orbit $\mathcal{O}$ containing ξ; hence Proposition 3.3 shows that $\xi \in LG_\epsilon \cdot \mathfrak{t}$. Using the action of LG_ϵ, we may therefore assume $\xi \in \mathfrak{t} \subseteq L\mathfrak{g}^*$. Let σ be the open face of the Stiefel diagram containing ξ, and $\mathcal{Y}_\sigma$ the corresponding symplectic cross-section containing x. Then x is a critical point for the function $\|\Phi_\mathcal{M}\|_\epsilon^2$ if and only if it is a critical point for the restriction to the finite dimensional symplectic manifold $\mathcal{Y}_\sigma$. Thus as in Kirwan [12], x is in the critical set if and only if the vector field generated by $\xi - \epsilon$ vanishes at x, or in other words $x \in (\mathcal{Y}_\sigma)^{\xi-\epsilon} \cap \Phi_\mathcal{M}^{-1}(\xi) = \mathcal{M}^{\xi-\epsilon} \cap \Phi_\mathcal{M}^{-1}(\xi)$. $\quad\square$

As an aside, we note that the critical set of $\|\Phi_\mathcal{M}\|_\epsilon^2$ also admits an interpretation in terms of the associated q-Hamiltonian space $M = \mathcal{M}/L_0 G$:

Proposition 3.5. *The quotient map $\mathcal{M} \to M$ restricts to bijections, for all $\xi \in \mathfrak{t}$,*

$$\mathcal{M}^{\xi-\epsilon} \cap \Phi_\mathcal{M}^{-1}(\xi) \xrightarrow{\cong} M^{\xi-\epsilon} \cap \Phi^{-1}(\exp(\xi)).$$

Proof. Regarding $\mathcal{M}$ as the fiber product $\mathcal{M} \subseteq M \times L\mathfrak{g}^*$ over G, with moment map given by projection to the second factor, the elements in $\Phi_\mathcal{M}^{-1}(\xi)$ for $\xi \in \mathfrak{t}$ are exactly those of the form $x = (m, \xi) \in \mathcal{M}$ with $\Phi(m) = \exp(\xi)$. Furthermore, x is $(\xi - \epsilon)$-fixed if and only if m is $(\xi - \epsilon)$-fixed. $\qquad\square$

To get a better understanding of the set $\mathfrak{B}_\epsilon$, one uses some basic facts about orbit type decompositions of Hamiltonian spaces. Consider the collection $\mathfrak{S} = \{\Delta\}$ of all affine subspaces of the form

$$(18) \qquad \Delta = \Phi_{\mathcal{M}}(x) + (\mathfrak{t}_x)^\perp \subseteq \mathfrak{t}$$

for elements $x \in \Phi_{\mathcal{M}}^{-1}(\mathfrak{t})$, where $\mathfrak{t}_x$ is the infinitesimal stabilizer under the action of $T \subseteq LG$, and $(\mathfrak{t}_x)^\perp$ its orthogonal complement inside $\mathfrak{t}$.

Lemma 3.6. *The collection $\mathfrak{S}$ is W_{aff}-invariant and locally finite, in the sense that every bounded subset of $\mathfrak{t}$ meets only finitely many elements of $\mathfrak{S}$.*

Proof. The pre-image $\Phi_{\mathcal{M}}^{-1}(\mathfrak{t})$ is covered by the collection of cross sections $\mathcal{Y}_\sigma$, where σ ranges over the open faces of the Stiefel diagram. By well-known results in symplectic geometry (see e.g. [34]), letting $\mathcal{Y}_\sigma$ be the cross-section containing x, the image of the T-orbit type stratum of x in $\mathcal{Y}_\sigma$ is an open subset of the symplectic submanifold $\mathcal{Y}_\sigma^{T_x}$, and its image under the moment map is an open subset of the affine subspace $\Delta = \Phi_{\mathcal{M}}(x) + (\mathfrak{t}_x)^\perp$. Since $\mathcal{Y}_\sigma$ has only finitely many orbit type strata, there are only finitely many such affine subspaces, all passing through the union of alcoves adjacent to σ. Hence $\mathfrak{S}$ is locally finite. The second claim follows since W_{aff} permutes the collection of symplectic cross-sections. $\qquad\square$

Lemma 3.7. *For all $\epsilon \in \mathfrak{t}$,*

$$\mathfrak{B}_\epsilon \subseteq \{\epsilon_\Delta := \mathrm{pr}_\Delta(\epsilon)| \ \Delta \in \mathfrak{S}\},$$

where $\mathrm{pr}_\Delta \colon \mathfrak{t} \to \Delta$ is the orthogonal projection. In particular, the set $\mathfrak{B}_\epsilon$ is discrete.

Proof. By definition, $\xi \in \mathfrak{B}_\epsilon$ if and only if there exists $x \in \mathcal{M}$ with $\Phi_{\mathcal{M}}(x) = \xi$ and $\xi - \epsilon \in \mathfrak{t}_x$. Letting Δ be the affine subspace $\Phi_{\mathcal{M}}(x) + (\mathfrak{t}_x)^\perp$, this means that $\xi \in \Delta$, with $\xi - \epsilon$ orthogonal to Δ. Equivalently, $\xi = \mathrm{pr}_\Delta(\epsilon)$. $\qquad\square$

4. QUANTIZATION OF HAMILTONIAN LG-SPACES

4.1. **The canonical twisted Spin_c-structure.** A Spin_c-structure on a finite-dimensional even-rank vector bundle $V \to M$ is given by Euclidean metric on V, together with a $\mathbb{Z}_2$-graded unitary spinor module $\mathsf{S} \to M$ over the Clifford bundle $\mathrm{Cl}(V)$, identifying the latter with $\mathrm{End}(\mathsf{S})$. For any spinor module S, the dual bundle is again a spinor module in a canonical way; the line bundle $\mathcal{Z} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S}) \to M$ is called the anti-canonical line bundle associated to the Spin_c-structure. Finite-dimensional symplectic manifolds have canonical Spin_c-structures, up to isomorphism, defined by compatible almost complex structures. By contrast, q-Hamiltonian G-spaces need not be Spin_c, in general. However, we have the following fact. Suppose as before that G is simple and 1-connected, and let $\widehat{LG}$ denote the basic central extension of the loop group. Given a $\widehat{LG}$-equivariant vector bundle $\mathcal{E} \to \mathcal{M}$, we say that $\mathcal{E}$ is at level $\ell \in \mathbb{Z}$ if the central elements $z \in \mathrm{U}(1) \subseteq \widehat{LG}$ act as multiplication by z^ℓ.

Theorem 4.1. [2, 21] *For any q-Hamiltonian G-space M, with associated proper Hamiltonian loop group space $\mathcal{M}$, the pull-back bundle*

$$p^* TM \to \mathcal{M}$$

has an $\widehat{LG}$-equivariant Spin_c-structure, which is canonical up to equivariant isomorphism. The associated $\mathbb{Z}_2$-graded spinor bundle $\mathsf{S} \to \mathcal{M}$ is $\widehat{LG}$-equivariant at level $\mathrm{h}^\vee$.

The anti-canonical line bundle $\mathcal{Z}$ associated to the Spin_c-structure is equivariant at level $2\mathrm{h}^\vee$; its dual is the 'canonical line bundle' $\mathcal{K} \to \mathcal{M}$, constructed in [31].

4.2. Prequantization. The triple

$$(19) \qquad\qquad (\mathcal{M}, k\omega_{\mathcal{M}}, k\Phi_{\mathcal{M}})$$

can be thought of as a Hamiltonian $\widehat{LG}$-space, where the central circle acts trivially, with moment map k. In particular, the moment map $(k\Phi_{\mathcal{M}}, k)$ is equivariant for the usual coadjoint action on

$$\widehat{L\mathfrak{g}}^* \cong L\mathfrak{g}^* \times \mathbb{R}.$$

A *level k prequantization* of $(\mathcal{M}, \omega_{\mathcal{M}}, \Phi_{\mathcal{M}})$ is given by a $\widehat{LG}$-equivariant line bundle $\mathcal{L} \to \mathcal{M}$ for (19), equipped with an invariant connection ∇ such that

$$\frac{\mathrm{i}}{2\pi} \operatorname{curv}(\nabla) = k\omega,$$

and the lift of the group action is given by Kostant's moment map condition for the moment map $(k\Phi_{\mathcal{M}}, k)$. In particular, $\mathcal{L}$ is equivariant at level k. Note that a level k prequantization determines prequantizations for all multiples of k.

Given a level k pre-quantization, we may tensor the spinor bundle $\mathsf{S} \to \mathcal{M}$ from Theorem 4.1 by $\mathcal{L}$ to obtain a new $\widehat{LG}$-equivariant spinor bundle $\mathcal{L} \otimes \mathsf{S}$ over $p^*\mathrm{Cl}(TM)$, which is equivariant at level $k + \mathrm{h}^\vee$. The associated anti-canonical line bundle

$$\mathcal{L}^2 \otimes \mathcal{Z} \to \mathcal{M}$$

is thus equivariant at level $2(k + \mathrm{h}^\vee)$.

4.3. Quantization. A level k prequantization $\mathcal{L}$ gives rise to a canonically defined element of the level k fusion ring $R_k(G)$, denoted

$$(20) \qquad\qquad \mathcal{Q}(\mathcal{M}, k) \in R_k(G)$$

and called the *quantization*. (This element may well depend on the choice of $\mathcal{L}$, but we will not indicate this dependence in the notation.) In [3], (20) was defined through an 'Atiyah-Bott'-like fixed point formula; in [28] it was defined as a K-homology push-forward and the fixed point formula was obtained as a theorem. We will recall this formula in Section 4.4 below. Equivalently, (20) is described in terms of the associated multiplicity function (see Section 2.3)

$$N(\cdot, k) \colon \Lambda^* \to \mathbb{Z},$$

which is anti-invariant under the shifted affine Weyl group action at level $k + \mathrm{h}^\vee$.

Remark 4.2 (Interpretation of the W_{aff}-anti-invariant extension). If the moment map $\Phi_{\mathcal{M}}$ is transverse to $\mathfrak{t} \subseteq L\mathfrak{g}^*$, then $\mathcal{X} = \Phi_{\mathcal{M}}^{-1}(\mathfrak{t})$ is a finite-dimensional submanifold, with a T-equivariant Spin_c-structure induced from that on p^*TM, and $\mathcal{L}_{\mathcal{X}} = \mathcal{L}|_{\Phi_{\mathcal{M}}^{-1}(\mathfrak{t})}$ is a pre-quantum line bundle for the presymplectic form given as the pullback of $k\omega_{\mathcal{M}}$. Even though $\mathcal{X}$ is non-compact, the corresponding Spin_c-Dirac operator with coefficients in $\mathcal{L}_{\mathcal{X}}$ has a well-defined T-equivariant index, with finite multiplicities. The function $N(\cdot, k)$ is the corresponding multiplicity function. See [22] for details.

Remark 4.3 (Tensor powers of the line bundle). A level k_0 prequantization of $(\mathcal{M}, \omega_{\mathcal{M}}, \Phi_{\mathcal{M}})$ canonically induces a level $k = k_0 n$ prequantization for all $n \in \mathbb{N}$, by taking powers of the pre-quantum line bundle $\mathcal{L}$. We hence obtain a sequence of quantizations $\mathcal{Q}(M, k) \in R_k(G)$ for $k = k_0 n$, where $n = 1, 2, \ldots$, with associated multiplicity functions $\mu \mapsto N(\mu, k)$ on Λ^*. We extend the definition to all $k \in \mathbb{N}$, by putting $N(\mu, k) = 0$ if k is not a multiple of k_0.

Example 4.4. For $G = \mathrm{SU}(2)$, we have that $\mathrm{h}^\vee = 2$, and $\Lambda^* = \mathbb{Z}$ with $1 \in \mathbb{Z}$ corresponding to the element ρ. The affine Weyl group action at level $k + 2$ is generated by affine reflections across the points -1 and $k + 2$. The multiplicity function $N(\cdot, k)$ for any level k prequantized Hamiltonian $L\,\mathrm{SU}(2)$-space (or equivalently, the associated q-Hamiltonian $\mathrm{SU}(2)$-space) is anti-invariant under this action, and so is determined by the values $N(\mu, k)$ for $\mu = 0, \ldots, k$. Some examples of such multiplicity functions are worked out in [27]. For the 'multiplicity-free' q-Hamiltonian $\mathrm{SU}(2)$-space $M = S^4$, one finds

$$N(\mu, k) = 1$$

for all $k \in \mathbb{N}$ and $\mu = 0, \ldots, k$. Another example is the 'fused double' $M = D(\mathrm{SU}(2))$, with corresponding Hamiltonian LG-space $\mathcal{M}$ the moduli space of flat connections on a surface of genus 1 with one boundary component. Here

$$N(\mu, k) = \begin{cases} 0 & \mu \text{ odd,} \\ k + 1 - \mu & \mu \text{ even.} \end{cases}$$

for $0 \leq \mu \leq k$.

4.4. **Fixed point formula.** We will state the fixed point formula from [3] and explain its ingredients; our eventual aim is to derive from it a 'norm-square' localization formula. Recall that for elements $\tau \in R_k(G)$, the evaluation $\tau(t)$ at regular elements $t \in T^{\mathrm{reg}}_{k+\mathrm{h}^\vee}$ is defined, and these values determine τ. Suppose $(\mathcal{M}, \omega_{\mathcal{M}}, \Phi_{\mathcal{M}})$ is a proper Hamiltonian LG-manifold, prequantized at level k, and let $Q(\mathcal{M}) \in R_k(G)$ be its quantization. Let M be the associated q-Hamiltonian G-space. The fixed point formula for (20) reads as

$$(21) \qquad Q(\mathcal{M})(t) = \sum_{F \subseteq M^t} \int_F \mathcal{AS}^t(\nu_{\widetilde{F}}), \quad t \in T^{\mathrm{reg}}_{k+\mathrm{h}^\vee}$$

a sum over connected components F of the fixed point set of t in M, where the differential form

$$(22) \qquad \mathcal{AS}^t(\nu_{\widetilde{F}}) \in \Omega(F)$$

is 'essentially' the integrand for the fixed point formula (Atiyah-Segal-Singer) of a Spin_c-Dirac operator – even though M does not have a Spin_c-structure, in general. To explain the definition, let

$$\widetilde{F} = F \times_T \mathfrak{t} \subseteq F \times \mathfrak{t}$$

be the fiber product of F and $\mathfrak{t}$ over T, so that Λ acts on $\widetilde{F}$ with quotient F.

Lemma 4.5. *There is a $\Lambda \times T$-equivariant embedding of $\widetilde{F}$ as a finite-dimensional symplectic submanifold*

$$\widetilde{F} \hookrightarrow \mathcal{M},$$

taking values in $\mathcal{M}^t \cap \Phi_{\mathcal{M}}^{-1}(\mathfrak{t})$, *in such a way that the quotient map* $\mathcal{M} \to M$ *restricts to the covering map* $\widetilde{F} \to F = \widetilde{F}/\Lambda$.

Proof. The inclusion is given by the natural map of fibered products, $F \times_T \mathfrak{t} \to M \times_G L\mathfrak{g}^* \cong \mathcal{M}$. For any face σ of the Stiefel diagram, the intersection with the corresponding cross-section, $\widetilde{F} \cap \mathcal{Y}_\sigma$ is a union of components of the fixed point set $\mathcal{Y}_\sigma^t$. We hence see that $\widetilde{F}$ is a union of symplectic submanifolds. $\square$

Let ν_F be the normal bundle to F inside M. We define

(23) $$\nu_{\widetilde{F}} := p^* TM|_{\widetilde{F}}/T\widetilde{F} \simeq p^* \nu_F \to \widetilde{F}.$$

The spinor bundle S for $p^* TM \to \mathcal{M}$ restricts to a $\widehat{\Lambda} \rtimes T$-equivariant spinor bundle $\mathsf{S}|_{\widetilde{F}}$ for

(24) $$p^* TM|_{\widetilde{F}} \simeq T\widetilde{F} \oplus \nu_{\widetilde{F}} \to \widetilde{F}$$

at level $\mathrm{h}^\vee$. On the other hand, the prequantum line bundle $\mathcal{L}$ restricts to a $\widehat{\Lambda} \rtimes T$-equivariant equivariant line bundle $\mathcal{L}_{\widetilde{F}}$ at level k; their tensor product $\mathsf{S}|_{\widetilde{F}} \otimes \mathcal{L}_{\widetilde{F}}$ is a Spin_c-structure at level $k + \mathrm{h}^\vee$. After choosing $\widehat{\Lambda} \rtimes T$-invariant bundle metrics and connections, the Atiyah-Singer integrand for the given element $t \in T_{k+\mathrm{h}^\vee}^{\mathrm{reg}}$ and the t-equivariant Spin_c-structure $\mathsf{S}|_{\widetilde{F}} \otimes \mathcal{L}_{\widetilde{F}}$ on $T\widetilde{F} \oplus \nu_{\widetilde{F}}$,

(25) $$\mathcal{AS}^t(\nu_{\widetilde{F}}) \in \Omega(\widetilde{F})$$

is a well-defined differential form. The general definition and several equivalent formulae for $\mathcal{AS}^t(\nu_{\widetilde{F}})$ are recalled in Appendix A. However in this case there is a simpler expression for $\mathcal{AS}^t(\nu_{\widetilde{F}})$.

Since $\widetilde{F}$ is symplectic, the 2-out-of-3 principle for Spin_c-structures gives a $\widehat{\Lambda} \rtimes T$-equivariant spinor bundle $\mathsf{S}_{\nu_{\widetilde{F}}}$ for $\nu_{\widetilde{F}}$. For this case the Atiyah-Singer integrand takes on the simpler form

(26) $$\mathcal{AS}^t(\nu_{\widetilde{F}}) = \frac{\mathrm{Td}(\widetilde{F}) \, \mathrm{Ch}^t(\mathcal{L}_{\widetilde{F}})}{\mathrm{Ch}^t(\mathsf{S}_{\nu_{\widetilde{F}}}^*)}.$$

where the denominator is the equivariant (super) Chern character of the dual spinor module $\mathsf{S}_{\nu_{\widetilde{F}}}^*$. Under the action of $\lambda \in \Lambda$, the factor $\mathrm{Ch}^t(\mathcal{L}_{\widetilde{F}})$ in (26) is preserved up to a phase factor $t^{k\lambda}$, while the denominator $\mathrm{Ch}^t(\mathsf{S}_{\nu_{\widetilde{F}}}^*)$ transforms similarly with a phase factor $t^{-\mathrm{h}^\vee \lambda}$. Since $t^{(k+\mathrm{h}^\vee)\lambda} = 1$ for $t \in T_{k+\mathrm{h}^\vee}$, we see that the product is Λ-invariant, and so descends to F. This is our form (22).

Remark 4.6. Our definition of (25) works more generally for elements $t \in T$ and F a connected component of M^t, provided that the subgroup $T_F \subseteq T$ fixing F pointwise contains *some* regular element. (We will soon need this more general situation.) Indeed, this assumption guarantees that $F \subseteq \Phi^{-1}(T)$ and Lemma 4.5 is unchanged. If furthermore $t \in T_{k+\mathrm{h}^\vee}$, then (25) descends to F.

Remark 4.7. If $k + \mathrm{h}^\vee$ is even, then the $\widehat{\Lambda}$-action on $p^* TM|_{\widetilde{F}}$ descends to a Λ-action, and we obtain a $T_{k+\mathrm{h}^\vee}$-equivariant Spin_c-structure on the bundle $TM|_F$ (and equivalently, on the total space of ν_F). In this case, the integrand (22) is directly defined as in the Atiyah-Singer fixed point formula, without the need to work on the cover $\widetilde{F}$. If $k + \mathrm{h}^\vee$ is odd, one can still define

an equivariant Spin_c-structure on $TM|_F$, but it depends on a choice and results in a slightly different formula. See [28] for details.

Remark 4.8. One option is to interpret $\mathcal{AS}^t(\nu_{\tilde{F}})$ as the Atiyah-Singer fixed point integrand for the Spin_c structure on the ambient space $\mathrm{Tot}(\nu_{\tilde{F}})$. Alternatively it is possible to construct an 'ambient space' that works for all components $\tilde{F}$ simultaneously, as follows. Let

$$U \cong T \times \mathfrak{g}/\mathfrak{t} \hookrightarrow G$$

be an $N(T)$-equivariant tubular neighbourhood embedding, with pre-image $X = \Phi^{-1}(U)$. Let $\widetilde{U} \cong \mathfrak{t} \times_T U$ and $\widetilde{X} \cong \mathfrak{t} \times_T X$ be the fiber products over T. There exists a $\Lambda \rtimes N_G(T)$-equivariant embedding

$$\widetilde{U} \cong \mathfrak{t} \times \mathfrak{g}/\mathfrak{t} \hookrightarrow L\mathfrak{g}^*,$$

restricting to the inclusion of $\mathfrak{t}$, such that the holonomy map $L\mathfrak{g}^* \to G$ restricts to the covering map $\widetilde{U} \to U$. (See [21] for details.) Its pre-image under $\Phi_{\mathcal{M}}$ defines an embedding of $\widetilde{X}$ into $\mathcal{M}$. Regarding $\widetilde{X}$ as a submanifold of $\mathcal{M}$ in this way, its tangent bundle is identified with $p^*TM|_{\widetilde{X}}$, and so it has a level $\mathrm{h}^\vee$ spinor bundle $\mathsf{S}|_{\widetilde{X}}$. The normal bundle of $\tilde{F}$ inside $\widetilde{X}$ is identified with $\nu_{\tilde{F}}$, and $\mathcal{AS}^t(\nu_{\tilde{F}})$ is the corresponding Atiyah-Singer integrand. This point of view will become important in Section 6.3.

5. Rearrangement of the fixed point contributions and the decomposition formula

In this section we re-write the fixed-point formula for $Q(\mathcal{M}, k)$ in terms of *Verlinde sums* (see Section 5.4), and then apply the Decomposition formula proved in [20].

5.1. **Orbit type rearrangement.** Let $(\mathcal{M}, \omega_{\mathcal{M}}, \Phi_{\mathcal{M}}, \mathcal{L}_0)$ be a level k_0 prequantized Hamiltonian LG-space. As in Remark 4.3, $\mathcal{M}$ acquires a prequantization $\mathcal{L} = \mathcal{L}_0^{\otimes n}$ at each level $k = nk_0$ for $n \in \mathbb{Z}_{>0}$. Let $Q(\mathcal{M}, k) \in R_k(G)$ be its quantization.

By (11), the associated multiplicity function $N(\cdot, k) \colon \Lambda^* \to \mathbb{Z}$ is expressed in terms of the values $\mathcal{Q}(M, k)(t)$ for $t \in T_\ell^{\mathrm{reg}}$, which are given, in turn, by (21):

$$N(\lambda, k) = \frac{1}{\#T_\ell} \sum_{t \in T_\ell^{\mathrm{reg}}} \sum_{F \subseteq M^t} t^{-\lambda} \prod_{\alpha \in \mathfrak{R}_-} (1 - t^\alpha) \int_F \mathcal{AS}^t(\nu_{\tilde{F}}).$$

Note that $N(\lambda, k)$ is a function of $k \in k_0\mathbb{Z}_{>0}$. The right hand side of this expression depends on k due to the explicit appearance of $\ell = k + \mathrm{h}^\vee$, as well as through the line bundle $\mathcal{L}$ (defined for $k \in k_0\mathbb{Z}_{>0}$) that plays a role in the definition of $\mathcal{AS}^t(\nu_{\tilde{F}})$.

We will now exchange the two summations, by first summing over all possible F. Let

(27) $$\mathfrak{F} = \{F \subseteq M | \ \exists t \in T^{\mathrm{reg}} \colon F \text{ is a component of } M^t\}.$$

Each $F \in \mathfrak{F}$ is a T-invariant submanifold. Since F is fixed by at least one regular element t, the equivariance of the map Φ shows that $\Phi(F) \subseteq G^t = T$. Let $T_F \subseteq T$ be the subgroup (possibly disconnected) fixing F pointwise. Some points of F will have a larger stabilizer, but basic properties of orbit type stratifications (see e.g. [34, Section 2]) show that the principal stratum of F (i.e, the subset of elements in F with stabilizer exactly equal to T_F) is open, dense and connected in F. The set $\mathfrak{F}$ has a partial ordering given by

$$F' \preceq F \quad \Leftrightarrow \quad F' \subseteq F,$$

and in this case $T_F \subseteq T_{F'}$. In terms of this partial ordering, the principal stratum of F is $F \setminus \cup_{F' \prec F} F'$.

We now rewrite the multiplicities by first summing over all $F \in \mathfrak{F}$, and then over all $t \in T_\ell$ having F as a fixed point component. Note for a given $t \in T$, the submanifold $F \in \mathfrak{F}$ is a component of M^t if and only if $t \in T_F$ and the action of t on the normal bundle does not fix any non-zero vectors. The formula for the multiplicities becomes

$$N(\lambda, k) = \frac{1}{\# T_\ell} \sum_{F \in \mathfrak{F}} \sideset{}{'}\sum_{t \in T_F \cap T_\ell} t^{-\lambda} \prod_{\alpha \in \mathfrak{R}_-} (1 - t^\alpha) \int_F \mathcal{AS}^t(\nu_{\widetilde{F}}),$$

where the prime indicates that we leave out any t whose action on ν_F has a non-trivial fixed point set. Observe that we dropped the superscript 'reg' from T_ℓ, since the product over negative roots is zero when t is not regular; on the other hand, in Remark 4.6 we had defined $\mathcal{AS}^t(\nu_{\widetilde{F}})$ even for non-regular t, so long as T_F contains *some* regular element.

Thinking of this sum as a finite Fourier transform, we may move the factor $\prod_{\alpha \in \mathfrak{R}_-} (1 - t^\alpha)$ outside the sum, replacing it with finite difference operators $\prod_{\alpha \in \mathfrak{R}_-} \nabla_\alpha$. We hence arrive at the expression

$$(28) \qquad N(\cdot, k) = \frac{1}{\# T_\ell} \Big(\prod_{\alpha \in \mathfrak{R}_-} \nabla_\alpha \Big) \overline{N}(\cdot, k)$$

with

$$(29) \qquad \overline{N}(\lambda, k) = \sum_{F \in \mathfrak{F}} \sideset{}{'}\sum_{t \in T_F \cap T_\ell} t^{-\lambda} \int_F \mathcal{AS}^t(\nu_{\widetilde{F}}).$$

5.2. **Action of components of T_F.** In general, the subgroups T_F for $F \in \mathfrak{F}$ may be disconnected. Let $T_{F,0}$ denote the identity component. For $a \in T_F / T_{F,0}$, let $T_{F,a}$ be the corresponding component of T_F, let $\ell_{F,a} \in \mathbb{Z}_{>0}$ be the smallest natural number such that $T_{F,a} \cap T_{\ell_{F,a}} \neq \emptyset$, and choose an element $t_{F,a}$ in this intersection.

Lemma 5.1. *For $\ell \in \mathbb{Z}_{>0}$, we have that $T_{F,a} \cap T_\ell \neq \emptyset$ if and only if ℓ is a multiple of $\ell_{F,a}$. Furthermore, in this case*

$$T_{F,a} \cap T_\ell = t_{F,a} \cdot (T_{F,0} \cap T_\ell).$$

Proof. The condition $T_{F,a} \cap T_\ell \neq \emptyset$ is equivalent to

$$(30) \qquad \ell \exp^{-1}(T_{F,a}) \cap \Lambda^* \neq \emptyset$$

It is immediate that the set of $\ell \in \mathbb{Z}$ with this property is a subgroup of $\mathbb{Z}$ which implies the first claim. The second claim follows from $T_{F,a} = t_{F,a} \cdot T_{F,0}$. $\qquad \square$

Suppose $\ell = k + h^\vee$ is a multiple of $\ell_{F,a}$, so that $t_{F,a} \in T_\ell$. We are interested in the fixed point contribution from elements in $T_{F,a} \cap T_\ell$. Hence we will give a description of the Atiyah-Singer integrand $\mathcal{AS}^{t_{F,a} t}(\nu_{\widetilde{F}},)$ for $t \in T_{F,0}$.

For any $F \in \mathfrak{F}$, the normal bundle ν_F splits T_F-equivariantly as a direct sum

$$\nu_F = \nu_F' \oplus \nu_F'',$$

where ν_F'' is the subbundle fixed by the identity component $T_{F,0}$. Pick a 'Kronecker generator' $\upsilon \in \mathfrak{t}$, i.e., such that $\exp(\upsilon)$ generates a dense subgroup of T. The bundle ν_F' has a unique T-invariant orthogonal complex structure, with the property that the pairings of the complex

weights of the T_F-action on ν'_F with $\mathrm{pr}_{\mathfrak{t}_F}(v)$ are all > 0. (Indeed, $\mathrm{pr}_{\mathfrak{t}_F}(v) \in \mathfrak{t}_F$ acts as an invertible skew-adjoint transformation of ν'_F; dividing by its 'absolute value' we obtain a skew-adjoint operator with the same eigenspaces and with eigenvalues $\pm i$.) The complex structure determines a T-equivariant Spin_c-structure for ν'_F. On the other hand, the choice of an ω_F-compatible, T-invariant, almost complex structures on F gives a T-equivariant Spin_c-structure for TF. Pulling back to $\widetilde{F}$ and using the $\Lambda \times T$-equivariant decomposition

$$p^* TM|_{\widetilde{F}} = T\widetilde{F} \oplus \nu'_{\widetilde{F}} \oplus \nu''_{\widetilde{F}}$$

where $\nu'_{\widetilde{F}} = p^*\nu'_F|_{\widetilde{F}}$, $\nu''_{\widetilde{F}} = p^*\nu''_F|_{\widetilde{F}}$, the bundle $\nu''_{\widetilde{F}}$ inherits a $\widehat{\Lambda} \rtimes T$-equivariant Spin_c-structure at level $\mathrm{h}^\vee$, by quotient. For all $t \in T_F$,

$$\mathrm{Ch}^t(\mathsf{S}^*_{\nu_{\widetilde{F}}}) = \mathrm{Ch}^t(\mathsf{S}^*_{\nu'_{\widetilde{F}}})\,\mathrm{Ch}^t(\mathsf{S}^*_{\nu''_{\widetilde{F}}}) = \mathcal{D}^t_{\mathbb{C}}(\nu'_{\widetilde{F}})\,\mathrm{Ch}^t(\mathsf{S}^*_{\nu''_{\widetilde{F}}}),$$

where

$$\mathcal{D}^t_{\mathbb{C}}(\nu'_{\widetilde{F}}) = \det_{\mathbb{C}}(1 - t^{-1} e^{-iR(\nu'_{\widetilde{F}})/2\pi}),$$

the complex determinant being taken in the fibres of the complex vector bundle $\nu'_{\widetilde{F}}$ (see also Proposition A.8). The Atiyah-Singer integrand takes on the form

$$\mathcal{AS}^t(\nu_{\widetilde{F}}) = \frac{\mathrm{Td}(T\widetilde{F})\,\mathrm{Ch}^t(\mathcal{L}_{\widetilde{F}})}{\mathcal{D}^t_{\mathbb{C}}(\nu'_{\widetilde{F}})\,\mathrm{Ch}^t(\mathsf{S}^*_{\nu''_{\widetilde{F}}})}.$$

Let $\Phi_{\widetilde{F}}\colon \widetilde{F} \to \mathfrak{t}^*$ be the restriction of the moment map. For $t \in T_{F,0}$, we have that

$$\mathrm{Ch}^{t_{F,a}t}(\mathcal{L}_{\widetilde{F}}) = t^{\mathrm{pr}^*_{\mathfrak{t}_F}(k\Phi_{\widetilde{F}})}\,\mathrm{Ch}^{t_{F,a}}(\mathcal{L}_{\widetilde{F}})$$

since the $T_{F,0}$-action fixes the base, and so the locally constant phase factor describing its action on the pre-quantum line bundle $\mathcal{L}_{\widetilde{F}}$ is described in terms of the moment map (the corresponding weight for $T_{F,0}$ is the projection $\mathrm{pr}_{\mathfrak{t}^*_F}(k\Phi_{\widetilde{F}})$). (By contrast, it is usually not possible to determine the $t_{F,a}$-action on $\mathcal{L}_{\widetilde{F}}$ from the moment map alone.)

Similarly, the T-action on the anti-canonical line bundle $\mathcal{Z}_{\nu''_{\widetilde{F}}}$ defines a (Berline-Vergne) moment map, which we write as $2\mathrm{h}^\vee \Psi''_{\widetilde{F}}\colon \widetilde{F} \to \mathfrak{t}^*$ to take into account the level $2\mathrm{h}^\vee$-equivariance with respect to the Λ-action on $\widetilde{F}$, $\mathfrak{t}^*$. Since $T_{F,0}$ acts trivially on the base $\widetilde{F}$, as well as on $\nu''_{\widetilde{F}}$, the $\mathrm{pr}_{\mathfrak{t}^*_F}$-component of $\Psi''_{\widetilde{F}}$ is locally constant, hence $\mathrm{h}^\vee \mathrm{pr}_{\mathfrak{t}^*_F} \Psi''_{\widetilde{F}}\colon \widetilde{F} \to \mathfrak{t}^*_F$ is a locally constant weight for the T_F-action such that

$$\mathrm{Ch}^{t_{F,a}t}(\mathsf{S}^*_{\nu''_{\widetilde{F}}}) = t^{-\mathrm{h}^\vee\,\mathrm{pr}_{\mathfrak{t}^*_F}\Psi''_{\widetilde{F}}}\,\mathrm{Ch}^{t_{F,a}}(\mathsf{S}^*_{\nu''_{\widetilde{F}}}).$$

We hence obtain

(31) $$\mathcal{AS}^{t_{F,a}t}(\nu_{\widetilde{F}}) = Q_{\widetilde{F},a,k}\,\frac{t^{\sigma_{\tilde{F},\ell}}}{\mathcal{D}_{\mathbb{C}}(\nu'_{\widetilde{F}}, t_{F,a}t)}$$

where $\sigma_{\tilde{F},\ell}$ is the affine-linear function of ℓ given by

$$\sigma_{\tilde{F},\ell} = \mathrm{pr}_{\mathfrak{t}^*_F}(k\Phi_{\widetilde{F}} + \mathrm{h}^\vee\Psi''_{\widetilde{F}}) = \ell\,\mathrm{pr}_{\mathfrak{t}^*_F}(\Phi_{\widetilde{F}}) + \mathrm{h}^\vee\,\mathrm{pr}_{\mathfrak{t}^*_F}(\Psi''_{\widetilde{F}} - \Phi_{\widetilde{F}}).$$

We also define $\sigma_{\tilde{F}} = \sigma_{\tilde{F},0} = \mathrm{h}^\vee \mathrm{pr}_{\mathfrak{t}_{\tilde{F}}^*}(\Psi''_{\tilde{F}} - \Phi_{\tilde{F}})$. In equation (31) we have also collected the t-independent factors into a single differential form $Q_{\tilde{F},a,k}$, defined by

$$(32) \qquad Q_{\tilde{F},a,k} = \frac{\mathrm{Td}(\tilde{F})\,\mathrm{Ch}^{t_{F,a}}(\mathcal{L}_{\tilde{F}})}{\mathrm{Ch}^{t_{F,a}}(\mathsf{S}^*_{\nu''_{\tilde{F}}})}$$

when ℓ is a multiple of $\ell_{F,a}$, and $Q_{\tilde{F},a,k} = 0$ when ℓ is not a multiple of $\ell_{F,a}$. Note that $\sigma_{\tilde{F},\ell}$ is locally constant on components of $\tilde{F}$, and takes values in the weight lattice $\Lambda_{\tilde{F}}^* = \mathrm{pr}_{\mathfrak{t}_{\tilde{F}}^*}(\Lambda^*)$ of $T_{F,0}$; its values on different components differ by elements of the lattice $\mathrm{pr}_{\mathfrak{t}_{\tilde{F}}^*}(\ell\Lambda)$. Consequently, if $t \in T_{F,0} \cap T_\ell$, the factor $t^{\sigma_{\tilde{F},\ell}}$ descends to F. Likewise, the form (32) descends by Λ-invariance to a form $Q_{F,a,k} \in \Omega(F)$. Equation (29) becomes

$$(33) \qquad \overline{N}(\lambda,k) = \sum_{F\in\mathfrak{F}} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} {\sum_{t\in T_{F,0}\cap T_\ell}}' \frac{t^{\sigma_{\tilde{F},\ell}-\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda)}}{\mathcal{D}_{\mathbb{C}}^{t_{F,a}t}(\nu'_F)}.$$

The summation

$$(34) \qquad {\sum_{t\in T_{F,0}\cap T_\ell}}' \frac{t^{\sigma_{\tilde{F},\ell}-\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda)}}{\mathcal{D}_{\mathbb{C}}^{t_{F,a}t}(\nu'_F)}$$

is an example of a (differential-form valued) *Verlinde sum* for the torus $T_{F,0}$. The next step in our analysis involves a combinatorial 'decomposition formula' for Verlinde sums.

5.3. Quasi-polynomials and shifted cones $C_{I,\delta}$.
Let Γ be a finite rank lattice. A complex-valued function on Γ of the form $\gamma \mapsto e^{2\pi\mathrm{i}\langle\mu,\gamma\rangle} \in U(1)$ for some $\mu \in \mathrm{Hom}(\Gamma,\mathbb{Q})$ is called a *rational character*. The algebra of *quasi-polynomials* on Γ is the algebra generated by polynomials on Γ together with the rational characters. Equivalently, a function $q\colon \Gamma \to \mathbb{C}$ is quasi-polynomial if there is a sublattice $\Gamma' \subseteq \Gamma$ of finite index such that the restriction of q to each coset $\gamma + \Gamma' \subseteq \Gamma$ is a polynomial. More generally, a function from Γ to some vector space V is quasi-polynomial if its components (in any basis of V) are quasi-polynomial. A function defined on a subset $S \subseteq \Gamma$ is called quasi-polynomial if it is the restriction of a quasi-polynomial function on Γ (clearly this only has content if S is an infinite subset). Similarly if the restriction of a function to $S \subseteq \Gamma$ equals the restriction of a quasi-polynomial function to S, then the function will be said to be quasi-polynomial on S. Finally when there is no risk of confusion, we use all the same terminology when S is in fact a subset of the ambient real vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, with the understanding that we are referring to functions defined on the intersection $S \cap \Gamma$.

The definitions above apply to the special case $\Gamma = \Lambda^* \times \mathbb{Z} \subseteq \mathfrak{t}^* \times \mathbb{R}$, with elements $\gamma = (\lambda,\ell)$. We shall encounter partially-defined functions of $(\lambda,\ell) \in \Lambda^* \times \mathbb{Z}$. For example, the multiplicity function N for a Hamiltonian LG-space prequantized at level $k_0 > 0$ is defined for $\lambda \in \Lambda^*$ and $\ell \in \mathrm{h}^\vee + k_0 \cdot \mathbb{Z}_{>0} \subseteq \mathbb{Z}$. Frequently these functions will be quasi-polynomial on the intersection of their domain with a shifted cone in $\mathfrak{t}^* \times \mathbb{R}$. It is convenient to introduce the following notation. For any subset $I \subseteq \mathfrak{t}^*$ and $\delta \in \mathfrak{t}^*$, let

$$(35) \qquad C_{I,\delta} = \{(t\xi + \delta, t)|\xi \in I, t > 0\} \subseteq \mathfrak{t}^* \times \mathbb{R}.$$

For $\delta = 0$ we also write $C_I = C_{I,0}$. If I is invariant under translation by elements of a rational subspace $\mathfrak{h}^*$ then for $\delta \in \mathfrak{t}^*/\mathfrak{h}^*$ we define $C_{I,\delta}$ to be $C_{I,\hat{\delta}}$ where $\hat{\delta} \in \mathfrak{t}^*$ is any lift of δ.

5.4. **The decomposition formula for Verlinde sums.** In this section we provide a minimal introduction to Verlinde sums and the decomposition formula. Instances of Verlinde sums first appeared in the Verlinde formula [40]. They were studied by Szenes [36], under the name *rational trigonometric sums*. Szenes proved a remarkable residue formula, implying that Verlinde sums exhibit a piecewise quasi-polynomial behavior. Verlinde sums are discrete analogues of *multiple Bernoulli series* [35]. Boysal-Vergne [9] proved a 'decomposition formula' for multiple Bernoulli series as an infinite sum consisting of a polynomial contribution, plus corrections supported in half-spaces. In [20] we provided a proof of the discrete counterpart of this formula; we refer the reader there for details.

Consider a torus $T = \mathfrak{t}/\Lambda$, with a unitary representation

$$(36) \qquad A\colon T \to \mathrm{U}(E)$$

on a Hermitian vector space E, with $E^T = \{0\}$. Let $g \in \mathrm{U}(E)$ be a fixed unitary transformation of finite order, commuting with all $A(t)$, $t \in T$. Put

$$\mathcal{D}_{\mathbb{C}}^g(E,t) = \det_{\mathbb{C}}(1 - g^{-1}A(t)^{-1}).$$

Suppose we are given another lattice $\Xi \subseteq \mathfrak{t}$ containing Λ; for $\ell \in \mathbb{Z}_{>0}$, define a finite subgroup

$$T_\ell = \left(\tfrac{1}{\ell}\Xi\right)/\Lambda \subseteq T.$$

Definition 5.2. The *Verlinde sum* $V_E^g\colon \Lambda^* \times \mathbb{Z}_{>0} \to \mathbb{C}$ associated to the T-representation (36), and the element $g \in \mathrm{U}(E)$ is

$$(37) \qquad V_E^g(\lambda,\ell) = \sideset{}{'}\sum_{t \in T_\ell} \frac{t^{-\lambda}}{\mathcal{D}_{\mathbb{C}}^g(E,t)}, \qquad \lambda \in \Lambda^*, \quad \ell \in \mathbb{Z}_{>0}.$$

The prime next to the summation sign means the sum only extends over elements $t \in T_\ell$ such that the denominator does not vanish.

Remark 5.3. Equation (37) differs from the convention in [20] by a substitution $\lambda \rightsquigarrow -\lambda$.

The Verlinde sum is periodic with respect to lattice translations of λ by elements of $\ell\Xi^*$. By ([20], Proposition 2.7) its support is contained in the subspace spanned by the weights of the representation (36), along with its translates under Ξ.

The representation (36) determines a collection $\mathcal{S}(E)$ of affine subspaces, consisting of spans of subsets of weights of the representation (36), along with their translates under the dual lattice $\Xi^* \subseteq \mathfrak{t}^*$. In particular, the elements of Ξ^* are in $\mathcal{S}(E)$ (corresponding to the empty set of weights). The affine subspaces in $\mathcal{S}(E)$ are called *admissible* in the terminology of Boysal-Vergne [9]. For $\Delta \in \mathcal{S}(E)$, let $\mathfrak{t}_\Delta \subseteq \mathfrak{t}$ be the annihilator of the subspace parallel to Δ, and let E_Δ be the subspace fixed by $\mathfrak{t}_\Delta$. The representation (36) restricts to a subrepresentation $A_\Delta\colon T \to \mathrm{U}(E_\Delta)$, and with the restriction of g to E_Δ the Verlinde sum $V_{E_\Delta}^g$ is defined, with support on $\ell\Delta$ together with its $\ell\Xi^*$-translates. Let $V_{E_\Delta,\Delta}^g(\cdot,\ell)$ denote the part of $V_{E_\Delta}^g(\cdot,\ell)$ supported on $\ell\Delta$ (i.e. the Verlinde sum multiplied by the characteristic function of $\ell\Delta$). Then

$$(38) \qquad V_{E_\Delta}^g = \sum_{\Delta'} V_{E_\Delta,\Delta'}^g$$

where the sum is over all distinct Ξ^*-translates Δ' of Δ.

For any $\Delta \in \mathcal{S}(E)$, the connected components $\mathfrak{c}$ of the complement of the admissible subspaces that are strictly contained in Δ are referred to as the (open) *chambers of* Δ. For any such chamber, $\mathfrak{c}$, the function $V^g_{E_\Delta}$ is quasi-polynomial on the subset of (λ, ℓ) with $\lambda \in \ell\mathfrak{c}$ (see ([20], Theorem 2.8); in terms of the notation from Definition 35, $V^g_{E_\Delta}$ is quasi-polynomial on the set of lattice points in contained in the cone

$$C_\mathfrak{c} = \{(t\xi, t) | \xi \in \mathfrak{c}, t > 0\} \subseteq \mathfrak{t}^* \times \mathbb{R}.$$

Let μ be any point in $\mathfrak{c}$.

Definition 5.4. The *quasi-polynomial germ* of $V^g_{E_\Delta}$ at $\mathfrak{c}$ (or μ) is the unique function

$$\mathrm{Ver}^g_{E_\Delta, \mu} = \mathrm{Ver}^g_{E_\Delta, \mathfrak{c}} \colon \Lambda^* \times \mathbb{Z}_{>0} \to \mathbb{C}$$

quasi-polynomial on and with support contained in $(\Lambda^* \times \mathbb{Z}) \cap C_\Delta$, that coincides with V_{E_Δ} on $(\Lambda^* \times \mathbb{Z}) \cap C_\mathfrak{c}$.

The continuous counterpart of the Verlinde sum is the *partition function*. It depends on the choice of a *polarizing vector* for the representation (36), i.e., an element $\tau \in \mathfrak{t}$ that acts without fixed points on E. Using the extension of (36) to a homomorphism $T^\mathbb{C} \to \mathrm{GL}(E)$, we define:

Definition 5.5. The *partition function* $P^g_{E, \tau} \colon \Lambda^* \to \mathbb{C}$ is given by the formula

$$(39) \qquad P^g_{E, \tau} = \lim_{\epsilon \to 0^+} \int_T \frac{t^{-\lambda}}{\mathcal{D}^g_\mathbb{C}(E, t \exp(-i\epsilon\tau))} \, \mathrm{d}t$$

One checks that this is well-defined, and depends only on the connected component of τ in the set of elements of $\mathfrak{t}$ acting without fixed points on E. The partition function $P^g_{E, \tau}$ has support contained in the intersection of Λ^* with the cone spanned by the *polarized weights* α^+ of E, i.e., $\alpha^+ = \pm\alpha$ with the sign chosen such that $\langle \alpha^+, \tau \rangle > 0$.

Fix an integral inner product on $\mathfrak{t}$ identifying $\mathfrak{t} \simeq \mathfrak{t}^*$, and let $\gamma \in \mathfrak{t}$. For every $\Delta \in \mathcal{S}(E)$, let $\gamma_\Delta = \mathrm{pr}_\Delta(\gamma)$ denote the orthogonal projection of γ onto Δ, and put

$$\tau_\Delta = \gamma_\Delta - \gamma.$$

For a generic choice of γ, all of the projections γ_Δ for $\Delta \in \mathcal{S}(E)$ are in open chambers, and all the normal vectors τ_Δ are non-zero polarizing vectors for $E^\perp_\Delta \subseteq E$.

Theorem 5.6 (Decomposition formula for Verlinde sums, [20])**.** *Let $\gamma \in \mathfrak{t}$ be generic. Then the Verlinde sum decomposes as follows:*

$$(40) \qquad V^g_E = \sum_{\Delta \in \mathcal{S}(E)} \mathrm{Ver}^g_{E_\Delta, \gamma_\Delta} \star P^g_{E^\perp_\Delta, \tau_\Delta}$$

(a convolution of functions on Λ^* for fixed $\ell \in \mathbb{Z}_{>0}$).

For our purposes, we need a more general version of the decomposition formula incorporating differential forms. Suppose $E \to N$ is a T-equivariant vector bundle, where the T-action fixes the base N. (We have in mind the bundles $\nu'_F \to F$ with the action of $T_{F,0}$.) The T-action is described by a group homomorphism $A \colon T \to \Gamma(\mathrm{U}(E))$, where $\mathrm{U}(E)$ denotes the unitary bundle. Suppose also that $g \in \Gamma(\mathrm{U}(E))$ has finite order and commutes with all $A(t)$.

Choose a T-invariant Hermitian connection on E, invariant under the action of g, and consider the Chern-Weil forms

$$\mathcal{D}^g_{\mathbb{C}}(E,t) = \det_{\mathbb{C}}\big(1 - g^{-1}A(t^{-1})e^{-\mathrm{i}R/2\pi}\big) \in \Omega(N), \qquad t \in T,$$

where R is the curvature. For all $t \in T$ such that the bundle endomorphism $1 - g^{-1}A(t^{-1}) \in \mathrm{End}(\nu)$ is invertible, the form $\mathcal{D}^g_{\mathbb{C}}(E,t)$ has a well-defined inverse. Using this inverse, we may define the Verlinde sums and partition functions by the same formulas as before, now given as differential forms. The decomposition formula (40) holds for this extended setting, at the level of differential forms [20].

5.5. Application of the decomposition formula.
We return to analysing the multiplicity function $\overline{N}(\lambda, \ell)$, given by (40):

$$(41) \qquad \overline{N}(\lambda, k) = \sum_{F \in \mathfrak{F}} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} \sum_{t \in T_{F,0} \cap T_\ell}' \frac{t^{\sigma_{\tilde{F},\ell} - \mathrm{pr}_{\mathfrak{t}_F^*}(\lambda)}}{\mathcal{D}^{t_{F,a}}_{\mathbb{C}}(\nu'_{\tilde{F}}, t)}.$$

Recall that although $\sigma_{\tilde{F},\ell} \colon \tilde{F} \to \Lambda_F^*$ is only locally constant, $t^{\sigma_{\tilde{F},\ell}} \in U(1)$ is actually constant (for a fixed $t \in T_{F,0} \cap T_\ell$), hence descends to F. At this stage it is convenient to break the symmetry and fix a connected component $\tilde{F}_0 \subseteq \tilde{F}$ and hence a choice of representative $\sigma_{\tilde{F}_0,\ell} \in \Lambda_F^*$ for the coset $\sigma_{\tilde{F},\ell} + \mathrm{pr}_{\mathfrak{t}_F^*}(\ell\Lambda)$. By the preceding comment, the locally constant function $\sigma_{\tilde{F},\ell}$ may be replaced with the constant $\sigma_{\tilde{F}_0,\ell}$ in (41) without changing the result.

With these preparations we re-phrase (41) in terms of Verlinde sums. For the term indexed by F, the role of T is played by the torus $T_{F,0} = \mathfrak{t}_F/\Lambda_F$ where $\Lambda_F := \Lambda \cap \mathfrak{t}_F$, and the role of Ξ by the lattice $\Xi_F = \Lambda^* \cap \mathfrak{t}_F$. The vector bundle E is ν'_F, with auxiliary endomorphism g given by the action of $t_{F,a}$. Using the definition of Verlinde sums, we have

$$\overline{N}(\lambda, k) = \sum_{F \in \mathfrak{F}} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} V^{t_{F,a}}_{\nu'_F}\big(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda) - \sigma_{\tilde{F}_0,\ell}, \ell\big).$$

Applying the decomposition formula for generic elements $\gamma_F \in \mathfrak{t}_F$ yields

$$(42) \qquad \overline{N}(\lambda, k) = \sum_{F \in \mathfrak{F}} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} \sum_{\tilde{\Delta} \in \mathcal{S}(\nu'_F)} \mathrm{Ver}^{t_{F,a}}_{\nu'_{F,\tilde{\Delta}}, \gamma_{F,\tilde{\Delta}}} \star P^{t_{F,a}}_{\nu'^{\perp}_{F,\tilde{\Delta}}, \tau_{F,\tilde{\Delta}}}\big(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda) - \sigma_{\tilde{F}_0,\ell}, \ell\big).$$

To choose the vectors γ_F, recall that our objective is to obtain an expression on the right-hand-side of (42) optimized for studying the quasi-polynomial behavior of $\overline{N}(\lambda, k)$ on the ray $\{(0, \ell)|\ell \in \mathbb{Z}_{>0}\} \subseteq \Lambda^* \times \mathbb{Z}_{>0}$. Since the ℓ-linear component of the shift $\sigma_{\tilde{F}_0,\ell}$ is $\ell\Phi_{\tilde{F}_0}$, the choice suited to this end is

$$(43) \qquad \gamma_F = -\Phi_{\tilde{F}_0} + \mathrm{pr}_{\mathfrak{t}_F^*}(\epsilon),$$

where $\epsilon \in \mathfrak{t}^*$ is close to 0, does not depend on F, and is such that γ_F is generic, for all $F \in \mathfrak{F}$.

The next step in the analysis of (42) is to reverse the order of the summations over F, $\tilde{\Delta}$. This will require new notation. Let

$$\mathcal{S} = \bigcup_{F \in \mathfrak{F}} \{\mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(\tilde{\Delta} + \Phi_{\tilde{F}_0})|\tilde{\Delta} \in \mathcal{S}(\nu'_F)\}.$$

Then $\mathcal{S}$ is a Λ-invariant, locally finite collection of affine subspaces of $\mathfrak{t}^*$. Equivalently $\mathcal{S}$ is the set of all affine subspaces of the form $\Phi_{\mathcal{M}}(x) + \mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(R)$ for $F \in \mathfrak{F}$, $x \in \tilde{F}$ and $R \subseteq \mathfrak{t}_F^*$ the span of a subset of the set of weights for the $T_{F,0}$-action on the normal bundle ν_F.

For $\Delta \in \mathcal{S}$, let

$$\epsilon_\Delta = \mathrm{pr}_\Delta(\epsilon), \qquad \tau_\Delta = \epsilon_\Delta - \epsilon.$$

Note that if $\Delta = \mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(\tilde{\Delta} + \Phi_{\tilde{F}_0})$ then

$$\mathrm{pr}_{\mathfrak{t}_F^*}(\epsilon_\Delta) = \gamma_{F,\tilde{\Delta}} + \Phi_{\tilde{F}_0}, \qquad \mathrm{pr}_{\mathfrak{t}_F^*}(\tau_\Delta) = \tau_{F,\tilde{\Delta}}.$$

For each $\Delta \in \mathcal{S}$, let $\mathfrak{t}_\Delta$ denote the annihilator of the subspace parallel to Δ. Let $\mathfrak{F}_\Delta$ be the subset of those $F \in \mathfrak{F}$ such that Δ appears in the collection of subspaces $\mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(\tilde{\Delta} + \Phi_{\tilde{F}_0})$ for some $\tilde{\Delta} \in \mathcal{S}(\nu_F')$, i.e.,

$$\mathfrak{F}_\Delta = \{F \in \mathfrak{F} | \Delta \in \mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(\Phi_{\tilde{F}_0} + \mathcal{S}(\nu_F'))\}.$$

Define $\nu'_{F,\Delta} = (\nu_F')^{\mathfrak{t}_\Delta}$ and let $\nu_{F,\Delta}'^\perp$ be its orthogonal complement in ν_F'; when $\Delta = \mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(\Phi_{\tilde{F}_0} + \tilde{\Delta})$, these agree with $\nu'_{F,\tilde{\Delta}}$, $\nu_{F,\tilde{\Delta}}'^\perp$ respectively. We also define

$$(44) \qquad V_{\nu'_{F,\Delta},\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}(\lambda, \ell) = V_{\nu'_{F,\tilde{\Delta}}}^{t_{F,a}}(\lambda - \sigma_{\tilde{F}_0,\ell}, \ell),$$

$$(45) \qquad \mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}(\lambda, \ell) = \mathrm{Ver}_{\nu'_{F,\tilde{\Delta}},\gamma_{F,\tilde{\Delta}}}^{t_{F,a}}(\lambda - \sigma_{\tilde{F}_0,\ell}, \ell).$$

(The motivation for the notation is that $V_{\nu'_{F,\Delta},\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}$ and $\mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}$ are supported on $C_{\Delta,\sigma_{\tilde{F}_0}}$, where recall that $\sigma_{\tilde{F}_0} = \sigma_{\tilde{F}_0,0} = \mathrm{h}^\vee \, \mathrm{pr}_{\mathfrak{t}_F^*}(\Psi''_{\tilde{F}_0} - \Phi_{\tilde{F}_0})$.) On reordering the sums, equations (28), (42) become

$$(46\mathrm{a}) \qquad N = \sum_{\Delta \in \mathcal{S}} N_\Delta^{\mathrm{qpol}}, \qquad N_\Delta^{\mathrm{qpol}} = \frac{1}{\#T_\ell}\Big(\prod_{\alpha \in \mathfrak{R}_-} \nabla_\alpha\Big)\overline{N}_\Delta^{\mathrm{qpol}},$$

where

$$(46\mathrm{b}) \qquad \overline{N}_\Delta^{\mathrm{qpol}}(\lambda, k) = \sum_{F \in \mathfrak{F}_\Delta} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} \mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}} \star P_{\nu_{F,\Delta}'^\perp,\tau_\Delta}^{t_{F,a}}\big(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda), \ell\big).$$

For later reference, we record some properties of N_Δ^{qpol}.

Proposition 5.7. *The function N_Δ^{qpol} has the following two properties:*

     (a) For each $\delta \in \Lambda^* \otimes \mathbb{Q}$, the function $(\lambda, \ell) \mapsto N_\Delta^{\mathrm{qpol}}(\lambda, \ell - \mathrm{h}^\vee)$ is quasi-polynomial on the intersection of $C_{\Delta,\delta}$ with its domain of definition. In particular if $0 \in \Delta$ then $k \in \mathbb{Z}_{>0} \mapsto N_\Delta^{\mathrm{qpol}}(0, k)$ is quasi-polynomial.

     (b) For $\Delta \neq \mathfrak{t}^*$, $k \in \mathbb{Z}_{>0}$, the function $\lambda \mapsto N_\Delta^{\mathrm{qpol}}(\lambda, k)$ is supported in a half-space, with the interior-pointing normal to the boundary τ_Δ.

Proof. The function $(\lambda, k) \mapsto t_{F,a}^{-\lambda} Q_{F,a,k}$ is quasi-polynomial. By definition the function $(\lambda, \ell) \mapsto \mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda))$ is quasi-polynomial on subsets of the form $C_{\Delta,\delta}$ for $\delta \in \Lambda^* \otimes \mathbb{Q}$. From basic properties of convolution, it follows that the function $(\lambda, \ell) \mapsto \overline{N}_\Delta^{\mathrm{qpol}}(\lambda, \ell - \mathrm{h}^\vee)$ is

also quasi-polynomial on subsets $C_{\Delta,\delta}$. The latter property is preserved by finite difference operators (which amount to taking finite linear combinations of the quasi-polynomials on different translates of C_Δ). Moreover, if $(\lambda,\ell) \mapsto (\#T_\ell) \cdot N_\Delta^{\mathrm{qpol}}(\lambda, \ell - \mathrm{h}^\vee)$ is quasi-polynomial on $C_{\Delta,\delta}$, then so is $(\lambda,\ell) \mapsto N_\Delta^{\mathrm{qpol}}(\lambda, \ell - \mathrm{h}^\vee)$. This is because $N_\Delta^{\mathrm{qpol}}(\lambda, \ell - \mathrm{h}^\vee)$ is an integer for $\ell - \mathrm{h}^\vee = k \geq 1$, hence $\#T_\ell = (\#T_1)\ell^{\dim(T)}$ must divide this quasi-polynomial, implying $N_\Delta^{\mathrm{qpol}}(\lambda, \ell - \mathrm{h}^\vee)$ is itself quasi-polynomial on $C_{\Delta,\delta}$.

Property (b) is clear because the function $\lambda \mapsto \mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda), \ell)$ is supported on a subspace parallel to Δ, while the partition function $P_{\nu'^\perp_{F,\Delta},\tau_\Delta}^{t_{F,a}}$ is τ_Δ-polarized. $\qquad\square$

6. THE NORM-SQUARE LOCALIZATION FORMULA

In this section we give a new description of the contributions N_Δ^{qpol} in (46). This description is used to prove that N_Δ^{qpol} vanishes identically when $M^{\mathfrak{t}_\Delta} \cap \Phi^{-1}(\exp(\epsilon_\Delta)) = \emptyset$. The surviving terms are indexed by the critical values of the norm-square of the moment map for the Hamiltonian loop group space associated to M. This vanishing result is crucial to establish the quasi-polynomial behavior of $k \mapsto N(0,k)$ in the next section.

Recall that N_Δ^{qpol} is quasi-polynomial on $C_{\Delta,\delta}$ for each $\delta \in \Lambda^* \otimes \mathbb{Q}$. To show N_Δ^{qpol} vanishes on $C_{\Delta,\delta}$, it suffices to show that $|N_\Delta^{\mathrm{qpol}}|$ decays asymptotically on an open cone in $C_{\Delta,\delta}$, as this cannot occur for a quasi-polynomial unless it is identically 0. The first step is to replace N_Δ^{qpol} with a function N_Δ having the same large ℓ asymptotics. The needed decay will follow from a Kirillov-Berline-Vergne-type formula for the Fourier transform of N_Δ.

6.1. **Asymptotic replacements.** Replacing $\mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}$ with $V_{\nu'_{F,\Delta},\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}$ in (46b), leads to closely related functions

$$(47\mathrm{a}) \qquad N_\Delta := \left(\prod_{\alpha \in \mathfrak{R}_-} \nabla_\alpha \right) \overline{N}_\Delta,$$

and

$$(47\mathrm{b}) \qquad \overline{N}_\Delta(\lambda, k) = \sum_{F \in \mathfrak{F}_\Delta} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} V_{\nu'_{F,\Delta},\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}} \star P_{\nu'^\perp_{F,\Delta},\tau_\Delta}^{t_{F,a}} \left(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda), \ell \right).$$

Example 6.1. For $\Delta = \mathfrak{t}^*$, $\mathfrak{F}_\Delta = \mathfrak{F}$ hence $N_\Delta = N$ is the original multiplicity function.

Proposition 6.2. *There is a $K \in \mathbb{Z}_{>0}$ and an open neighborhood b of ϵ_Δ in $\mathfrak{t}^*$, such that*

$$N_\Delta(\lambda, k) = N_\Delta^{\mathrm{qpol}}(\lambda, k), \qquad \lambda \in \ell \cdot \mathsf{b}, \quad k > K.$$

In particular, for each $\lambda \in \Lambda^*$, $N_\Delta(\lambda, k) = N_\Delta^{\mathrm{qpol}}(\lambda, k)$ for $k \gg 0$.

Remark 6.3. Note that given N_Δ, there is at most one function N_Δ^{qpol} satisfying the property in the proposition and that is quasi-polynomial on subsets $C_{\Delta,\eta}$, $\eta \in \Lambda \otimes \mathbb{Q}$.

Proof. By construction, $\mathrm{Ver}_{\nu'_{F,\Delta},\epsilon_\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda), \ell)$ and $V_{\nu'_{F,\Delta},\Delta}^{t_{F,a},\sigma_{\tilde{F}_0}}(\mathrm{pr}_{\mathfrak{t}_F^*}(\lambda), \ell)$ both have support contained in $C_{\Delta,\sigma_{\tilde{F}_0}} \subseteq \mathfrak{t}^* \times \mathbb{R}$. Moreover, there is a relatively open chamber c_F in $\mathrm{pr}_{\mathfrak{t}_F^*}(\Delta)$ containing $\mathrm{pr}_{\mathfrak{t}_F^*}(\epsilon_\Delta)$ such that the two functions agree on the affine cone $C_{\mathrm{pr}_{\mathfrak{t}_F^*}^{-1}(\mathsf{c}_F),\sigma_{\tilde{F}_0}}$.

Recall $P^{t_{F,a}}_{\nu'^{\perp}_{F,\Delta}, \tau_\Delta}$ has support contained in the pointed cone $R_{F,\Delta} \subseteq \mathfrak{t}^*_F$ spanned by the τ_Δ-polarized weights for the action of $T_{F,0}$ on $\nu'^{\perp}_{F,\Delta}$. From the fact that the support of a convolution is the pushforward of the supports under addition, it follows that the convolutions

$$V^{t_{F,a}, \sigma_{\tilde{F}_0}}_{\nu'_{F,\Delta}, \Delta} \star P^{t_{F,a}}_{\nu'^{\perp}_{F,\Delta}, \tau_\Delta}, \qquad \mathrm{Ver}^{t_{F,a}, \sigma_{\tilde{F}_0}}_{\nu'_{F,\Delta}, \epsilon_\Delta} \star P^{t_{F,a}}_{\nu'^{\perp}_{F,\Delta}, \tau_\Delta}$$

agree on the set $C_{\mathfrak{b}'_F, \sigma_{\tilde{F}_0}}$, where $\mathfrak{b}'_F$ is the complement in $\mathfrak{t}^*$ of $(\Delta \setminus \mathrm{pr}^{-1}_{\mathfrak{t}^*_F}(\mathsf{c}_F)) + \mathrm{pr}^{-1}_{\mathfrak{t}^*_F}(R_{F,\Delta})$.

Let $\mathfrak{b}'$ be the intersection over $F \in \mathfrak{F}_\Delta$ of the $\mathfrak{b}'_F$. Notice that $\mathfrak{b}'$ is non-empty and open, since each $\mathfrak{b}'_F$ contains an open neighborhood of ϵ_Δ. Let $\hat{\sigma}_{\tilde{F}_0}$ be any lift of $\sigma_{\tilde{F}_0}$ to $\mathfrak{t}^*$; recall $C_{\mathfrak{b}'_F, \sigma_{\tilde{F}_0}} = C_{\mathfrak{b}'_F, \hat{\sigma}_{\tilde{F}_0}}$. Then N_Δ, N^{qpol}_Δ agree on the set

(48)
$$\bigcap_\eta \bigcap_{F \in \mathfrak{F}_\Delta} C_{\mathfrak{b}', \hat{\sigma}_{\tilde{F}_0} + \eta}$$

where η ranges over the finite set of possible sums of subsets of the negative roots. For small ℓ, the intersection (over η, F) of the sets $(\ell \cdot \mathfrak{b}' + \hat{\sigma}_{\tilde{F}_0} + \eta) \times \{\ell\}$ might be empty, because of the shifts. But if we let $\mathfrak{b} \subseteq \mathfrak{b}'$ be an open subset containing ϵ_Δ such that the closure $\bar{\mathfrak{b}} \subseteq \mathfrak{b}'$, then for $\ell \gg 0$ one has

$$\mathfrak{b} \subseteq \mathfrak{b}' + \ell^{-1}(\hat{\sigma}_{\tilde{F}_0} + \eta)$$

for each pair (F, η) in (48). $\qquad\square$

6.2. The distributional character for N_Δ.
The function $N_\Delta(-, k)$ introduced in Section 6.1 is the Fourier transform of a distributional character $Q_\Delta(-, k) \in \mathcal{D}'(T)$. We introduce additional notation in order to give a compact formula for Q_Δ.

Example 6.4. By Example 6.1, for $\Delta = \mathfrak{t}^*$, $\mathfrak{F}_\Delta = \mathfrak{F}$ and

$$Q_\Delta(t, k) = \delta_{T_\ell}(t) Q(\mathcal{M})(t) \prod_{\alpha \in \mathfrak{R}_-} (1 - t^\alpha),$$

where δ_{T_ℓ} denotes the delta distribution for T_ℓ with total integral 1 (i.e. $(\#T_\ell)^{-1}$ times the counting measure). The formula for Q_Δ below generalizes this to arbitrary $\Delta \in \mathcal{S}$.

To state the formula we introduce some additional notation. For $\Delta \in \mathcal{S}$ and $t \in T$ we define

$$\mathfrak{F}^t_\Delta = \{F \in \mathfrak{F}_\Delta | F \subseteq M^t\}.$$

For $F \in \mathfrak{F}_\Delta$, let $\tilde{F}_\Delta$ be the union of connected components of $\tilde{F}$ whose image under the moment map lies in Δ, or in other words, $\tilde{F}_\Delta$ is the fibre product

$$\tilde{F}_\Delta = \tilde{F} \times_{\mathfrak{t}^*} \Delta.$$

Then

$$F = \tilde{F}_\Delta / (\Lambda \cap \mathfrak{t}^\perp_\Delta)$$

where $\mathfrak{t}^\perp_\Delta$ is the orthogonal complement of $\mathfrak{t}_\Delta$ in $\mathfrak{t}$. Let

$$\nu_{\tilde{F}, \Delta} = \nu^{\mathfrak{t}_\Delta}_{\tilde{F}} |_{\tilde{F}_\Delta},$$

and $\nu^{\perp}_{\tilde{F}, \Delta}$ for the complementary vector bundle over $\tilde{F}_\Delta$. We also have a decomposition

$$\nu_{\tilde{F}, \Delta} = \nu'_{\tilde{F}, \Delta} \oplus \nu''_{\tilde{F}_\Delta}$$

where $\nu''_{\tilde{F}_\Delta} = (\nu_{\tilde{F},\Delta})^{\mathfrak{t}_F} = \nu''_{\tilde{F}}|_{\tilde{F}_\Delta}$.

In the proof of the next result, it will be convenient to consider distinct complex structures on the same underlying real vector bundle, and we introduce the corresponding notation here. If $\xi \in \mathfrak{t}$ and ξ acts on a vector bundle ν with no eigenvalue equal to 0 (we say ξ is polarizing), then we write J_ξ for the complex structure on ν such that the weights are ξ-polarized, and $\wedge_\xi \nu$, $\det_\xi(\nu)$, $\mathrm{Sym}_\xi(\nu)$ for the corresponding complex exterior algebra, determinant and symmetric algebra bundles, respectively.

For example, τ_Δ is polarizing for the action of T_F on $\nu^\perp_{\tilde{F},\Delta}$, and so we have a corresponding complex structure J_{τ_Δ}. Below we will need the symmetric algebra bundle $\mathrm{Sym}_{\tau_\Delta}(\nu^\perp_{\tilde{F},\Delta})$. The latter has a distributional Chern character, defined for $t \in T_F$:

$$\mathrm{Ch}(\mathrm{Sym}_{\tau_\Delta}(\nu^\perp_{\tilde{F},\Delta}), t) = \lim_{\epsilon \to 0^+} \frac{1}{\mathrm{Ch}(\wedge_{\tau_\Delta} \nu^\perp_{\tilde{F},\Delta}, t \exp(\mathrm{i}\epsilon\tau_\Delta))}.$$

On the same vector bundle we also have the complex structure J_υ (strictly speaking, $J_{\mathrm{pr}_{\mathfrak{t}_F^*}(\upsilon)}$), which is not the same in general. For use in the proof of the result below, we introduce the sub-bundles $\nu^{\perp+}_{F,\Delta}$, $\nu^{\perp-}_{F,\Delta}$, where $\nu^{\perp+}_{F,\Delta}$ is the sub-bundle where the complex structures J_{τ_Δ} and J_υ agree, thus

$$(\nu^\perp_{F,\Delta}, J_{\tau_\Delta}) = (\nu^{\perp+}_{F,\Delta}, J_\upsilon) \oplus (\nu^{\perp-}_{F,\Delta}, J_{-\upsilon}).$$

It follows that we have an isomorphism of $\mathbb{Z}_2$-graded spinor modules:

$$\wedge_{\tau_\Delta} \nu^\perp_{F,\Delta} = \wedge_\upsilon \nu^{\perp+}_{F,\Delta} \widehat{\otimes} \wedge_{-\upsilon} \nu^{\perp-}_{F,\Delta} = \wedge_{-\upsilon} \nu^\perp_{F,\Delta} \widehat{\otimes} \det_\upsilon(\nu^{\perp+}_{F,\Delta})$$

where $\det_\upsilon(\nu^{\perp+}_{F,\Delta})$ is regarded as a $\mathbb{Z}_2$-graded line bundle with parity equal to the complex rank of $\nu^{\perp+}_{F,\Delta}$. The Chern characters thus satisfy

(49) $\mathrm{Ch}(\wedge_{\tau_\Delta} \nu^\perp_{F,\Delta}, u) = \mathrm{Ch}(\wedge_{-\upsilon} \nu^\perp_{F,\Delta}, u)\mathrm{Ch}(\det_\upsilon(\nu^{\perp+}_{F,\Delta}), u).$

The $T \ltimes \widehat{\Lambda}$-equivariant spinor modules $\mathsf{S}_{\nu_{\tilde{F}}}|_{\tilde{F}_\Delta}$ for $\nu_{\tilde{F}}|_{\tilde{F}_\Delta}$ and $\wedge_{\tau_\Delta} \nu^\perp_{\tilde{F},\Delta}$ for the complex vector bundle $\nu^\perp_{\tilde{F},\Delta}$ determine a spinor module for $\nu_{\tilde{F},\Delta}$:

(50) $\mathsf{S}_{\nu_{\tilde{F},\Delta}} = \mathrm{Hom}_{\mathrm{Cl}(\nu^\perp_{\tilde{F},\Delta})}(\wedge_{-\tau_\Delta} \nu^\perp_{\tilde{F},\Delta}, \mathsf{S}_{\nu_{\tilde{F}}}|_{\tilde{F}_\Delta})$ $\Rightarrow$ $\mathsf{S}_{\nu_{\tilde{F}}}|_{\tilde{F}_\Delta} = \mathsf{S}_{\nu_{\tilde{F},\Delta}} \widehat{\otimes} \wedge_{-\tau_\Delta} \nu^\perp_{\tilde{F},\Delta}.$

(The minus sign $(-\tau_\Delta)$ is intentional here.)

Theorem 6.5. *The distributional character Q_Δ is given by the expression*

(51) $Q_\Delta(t, k) = \delta_{T_\Delta T_\ell}(t) \sum_{F \in \mathfrak{F}_\Delta^t} \int_F \mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_\Delta}) \mathcal{AS}^t(\nu_{\tilde{F},\Delta}) \mathrm{Ch}^t(\mathrm{Sym}_{\tau_\Delta}(\nu^\perp_{\tilde{F},\Delta}) \otimes \wedge \mathfrak{n}_-),$

where $\delta_{T_\Delta T_\ell}$ is the Dirac delta measure supported on $T_\Delta T_\ell$ with total integral 1.

Remark 6.6. It is not difficult to see that the right hand side of (51) is a well-defined distribution. Consider a product of the form

$$\delta_{sT_\Delta}(t) \cdot \prod_i \sum_{n \geq 0} t^{n\alpha_i}$$

where $t_0 \in T$, $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$ is a list of weights forming a pointed cone $\mathrm{Cone}(\underline{\alpha})$, and the sum over n converges in the sense of generalized functions. The Fourier transform of $\delta_{t_0 T_\Delta}$ is product

of the character $\lambda \in \Lambda^* \mapsto t_0^{-\lambda} \in U(1)$ with the characteristic function of $\Lambda^* \cap \mathrm{ann}(\mathfrak{t}_\Delta)$, while the Fourier transform of the second factor is the partition function for the list $\underline{\alpha}$. The convolution is well-defined when there is an open half-space $H = H(\tau_\Delta > 0)$ with $\mathrm{ann}(\mathfrak{t}_\Delta) \subseteq \partial H$ and $(\mathrm{Cone}(\underline{\alpha}) - \{0\}) \subseteq H$, as is the case in (51). More generally one has

$$\mathrm{Ch}^t(\mathrm{Sym}(\nu_{\tilde{F},\Delta}^\perp)) = \lim_{\epsilon \to 0^+} \frac{1}{\mathrm{Ch}(\wedge_{\tau_\Delta} \nu_{\tilde{F},\Delta}^\perp, t \exp(i\epsilon\tau_\Delta))}.$$

This is a generalized function of $t \in T_F$, with values in $\Omega(F)$ (differential forms on F). It admits a restriction to $T_\Delta T_\ell \cap T_F$, given by the same expression.

Remark 6.7. We claim that the differential form $\mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_\Delta})\mathcal{AS}^t(\nu_{\tilde{F},\Delta})$ on $\tilde{F}_\Delta$ can be taken to be $\Lambda \cap \mathfrak{t}_\Delta^\perp$-invariant, and so descends to give a differential form on

$$\tilde{F}_\Delta/(\Lambda \cap \mathfrak{t}_\Delta^\perp) = F,$$

which may then be integrated over F. One needs to check whether the locally constant phase factor $\zeta_{\tilde{F}_\Delta}(t)^{1/2}$ (see Definition A.4) is $(\Lambda \cap \mathfrak{t}_\Delta^\perp)$-invariant on $\tilde{F}_\Delta$, for each $t \in T_\Delta T_\ell$. In brief: we know that it is for $t \in T_\ell$. On the other hand the phase factor for the *connected* group T_Δ is completely determined by moment maps (composed with the quotient map to $\mathfrak{t}_\Delta^*$): $\mathrm{pr}_{\mathfrak{t}_\Delta^*} \Phi_{\tilde{F}_\Delta}$ (for the prequantum line bundle), and $\mathrm{pr}_{\mathfrak{t}_\Delta}^* \circ \Psi_{\tilde{F}_\Delta}''$ for the spinor module. These are locally constant on $\tilde{F}_\Delta$, by the abstract moment map condition ($\tilde{F}_\Delta$ is fixed by $\mathfrak{t}_\Delta$), and moreover change by elements of $B^\flat(\Lambda \cap \mathfrak{t}_\Delta^\perp)$ from component to component in $\tilde{F}_\Delta$. But $B^\flat(\mathfrak{t}_\Delta^\perp) = \ker(\mathrm{pr}_{\mathfrak{t}_\Delta^*})$, so the components $\mathrm{pr}_{\mathfrak{t}_\Delta^*} \Phi_{\tilde{F}_\Delta}$, $\mathrm{pr}_{\mathfrak{t}_\Delta}^* \circ \Psi_{\tilde{F}_\Delta}''$ are constant on $\tilde{F}_\Delta$.

Note also that $T_\Delta T_\ell$ is a union of finitely many connected components indexed by $T_\ell/(T_\Delta \cap T_\ell)$. Moreover $\mathfrak{F}_\Delta^t = \mathfrak{F}_\Delta^s$ whenever s, t lie in the same component, i.e. differ by an element of T_Δ, because all $F \in \mathfrak{F}_\Delta$ are fixed by T_Δ. So the range of the summation only depends on the connected component of $T_\Delta T_\ell$.

Proof. The argument is a mild generalization of that carried out in Section 5: we take the Fourier transform, reverse the order of summation, and express the finite sums that appear in terms of Verlinde sums. First note that under the Fourier transform, the equivariant Chern character of $\wedge \mathfrak{n}_-$ becomes the product of finite difference operators ∇_α, $\alpha \in \mathfrak{R}_-$, so we instead prove the slightly simpler analogue of (51) for $\overline{N}_\Delta$ and its distributional character $\overline{Q}_\Delta$.

Let q denote the right hand side of (51) without the factor $\mathrm{Ch}(\wedge \mathfrak{n}_-, -)$ and multiplied by a factor of $\#T_\ell$. We must prove $q = \overline{Q}_\Delta$, and we will do this by showing $\widehat{q} = \overline{N}_\Delta$. For $\lambda \in \Lambda^*$,

$$\widehat{q}(\lambda, k) = (\#T_\ell) \int_{v \in T_\Delta T_\ell} dv\, v^{-\lambda} \sum_{F \in \mathfrak{F}_\Delta^v} \int_F \mathrm{Ch}^v(\mathcal{L}_{\tilde{F}_\Delta})\mathcal{AS}^v(\nu_{\tilde{F},\Delta})\mathrm{Ch}^v(\mathrm{Sym}_{\tau_\Delta}(\nu_{\tilde{F},\Delta}^\perp)),$$

where as explained in Remark 6.7, the differential form $\mathrm{Ch}^v(\mathcal{L}_{\tilde{F}_\Delta})\mathcal{AS}^v(\nu_{\tilde{F},\Delta})$ is initially defined on $\tilde{F}_\Delta$, but it is $(\Lambda \cap \mathfrak{t}_\Delta^\perp)$-invariant, and so descends to F. For later reference we note in passing that pulling the integrand back by the multiplication map $T_\Delta \times T_\ell \to T_\Delta T_\ell$ and incorporating the factor of $(\#T_\ell)$, the measure we should use on $T_\Delta \times T_\ell$ is the product of normalized Haar measure on T_Δ and counting measure on T_ℓ.

We begin by simplifying the Atiyah-Singer integrand. Recall (see Appendix A)

$$\mathcal{AS}^v(\nu_{\tilde{F},\Delta}) = \frac{\operatorname{Td}(\tilde{F}_\Delta)}{\operatorname{Ch}^v(\mathsf{S}^*_{\nu_{\tilde{F},\Delta}})}$$

where

$$\mathsf{S}_{\nu_{\tilde{F},\Delta}} = \operatorname{Hom}_{\operatorname{Cl}(\nu^\perp_{\tilde{F},\Delta})}(\wedge_{-\tau_\Delta}\nu^\perp_{\tilde{F},\Delta}, \mathsf{S}_{\nu_{\tilde{F}}}|_{\tilde{F}_\Delta}).$$

One has

$$\nu_{\tilde{F},\Delta} = \nu'_{\tilde{F},\Delta} \oplus \nu''_{\tilde{F}_\Delta}, \qquad \nu'_{\tilde{F}} = \nu'_{\tilde{F},\Delta} \oplus \nu^\perp_{\tilde{F},\Delta}$$

where $\nu''_{\tilde{F}_\Delta} = \nu^{t_F}_{\tilde{F}_\Delta} = \nu''_{\tilde{F}}|_{\tilde{F}_\Delta}$. To keep the notation from becoming excessive, we sometimes omit the restriction from $\tilde{F}$ to $\tilde{F}_\Delta$ below. Equipping $\nu'_{\tilde{F},\Delta}$ with the v-polarized complex structure J_v, we have

$$\mathcal{AS}^v(\nu_{\tilde{F},\Delta}) = \frac{\operatorname{Td}(\tilde{F}_\Delta)}{\operatorname{Ch}^v(\mathsf{R}^*_{\nu''_{\tilde{F}}})\mathcal{D}^v_\mathbb{C}(\nu'_{\tilde{F},\Delta})},$$

where $\mathsf{R}_{\nu''_{\tilde{F}}}$ is the spinor module for $\nu''_{\tilde{F}}$ defined by

$$\mathsf{R}_{\nu''_{\tilde{F}}} = \operatorname{Hom}_{\operatorname{Cl}(\nu'_{\tilde{F},\Delta})}(\wedge_v\nu'_{\tilde{F},\Delta}, \mathsf{S}_{\nu_{\tilde{F},\Delta}}) = \operatorname{Hom}_{\operatorname{Cl}(\nu'_{\tilde{F}})}(\wedge_v\nu'_{\tilde{F},\Delta}\widehat{\otimes}\wedge_{-\tau_\Delta}\nu^\perp_{\tilde{F},\Delta}, \mathsf{S}_{\nu_{\tilde{F}}}).$$

On the other hand we had already defined a spinor module $\mathsf{S}_{\nu''_{\tilde{F}}}$ for $\nu''_{\tilde{F}}$ by

$$\mathsf{S}_{\nu''_{\tilde{F}}} = \operatorname{Hom}_{\operatorname{Cl}(\nu'_{\tilde{F}})}(\wedge_v\nu'_{\tilde{F}}, \mathsf{S}_{\nu_{\tilde{F}}}).$$

Then

$$\mathsf{R}_{\nu''_{\tilde{F}}} = \mathsf{S}_{\nu''_{\tilde{F}}}\widehat{\otimes}D$$

where D is the $\mathbb{Z}_2$-graded line bundle

$$D = \operatorname{Hom}_{\operatorname{Cl}(\nu_{\tilde{F}''})}(\mathsf{S}_{\nu''_{\tilde{F}}}, \mathsf{R}_{\nu''_{\tilde{F}}}) = \operatorname{Hom}_{\operatorname{Cl}(\nu^\perp_{\tilde{F},\Delta})}(\wedge_{-\tau_\Delta}\nu^\perp_{\tilde{F},\Delta}, \wedge_v\nu^\perp_{\tilde{F},\Delta}) = \det{}_v(\nu^{\perp+}_{\tilde{F},\Delta}).$$

Consequently

$$\frac{1}{\operatorname{Ch}^v(\mathsf{R}^*_{\nu''_{\tilde{F}}})} = \frac{\operatorname{Ch}^v(D)}{\operatorname{Ch}^v(\mathsf{S}^*_{\nu''_{\tilde{F}}})} \quad \Rightarrow \quad \mathcal{AS}^v(\nu_{\tilde{F},\Delta}) = \frac{\operatorname{Td}(\tilde{F})\operatorname{Ch}^v(D)}{\operatorname{Ch}^v(\mathsf{S}^*_{\nu''_{\tilde{F}}})\mathcal{D}^v_\mathbb{C}(\nu'_{F,\Delta})}.$$

Thus

$$\widehat{q}(\lambda,k) = (\#T_\ell)\int_{v\in T_\Delta T_\ell} \mathrm{d}v\, v^{-\lambda} \sum_{F\in\mathfrak{F}^v_\Delta} \int_F \frac{\operatorname{Td}(\tilde{F})\operatorname{Ch}^v(\mathcal{L}_{\tilde{F}_\Delta})}{\operatorname{Ch}^v(\mathsf{S}^*_{\nu''_{\tilde{F}}}|_{\tilde{F}_\Delta})\mathcal{D}^v_\mathbb{C}(\nu'_{F,\Delta})}\operatorname{Ch}^v(D\otimes\operatorname{Sym}(\nu^\perp_{\tilde{F},\Delta})).$$

Let $t_{F,a}$ be as before, and decompose $v = t_{F,a}u$ where $u \in T_{F,0} \cap (T_\Delta T_\ell)$. We have

$$\operatorname{Ch}^v(\mathcal{L}_{\tilde{F}_\Delta}) = u^{k\,\mathrm{pr}_{t_F^*}\,\Phi_{\tilde{F}_\Delta}}\operatorname{Ch}^{t_{F,a}}(\mathcal{L}_{\tilde{F}_\Delta}),$$

and

$$\operatorname{Ch}^v(\mathsf{S}^*_{\nu''_{\tilde{F}}}|_{\tilde{F}_\Delta}) = u^{-\mathrm{h}^\vee\,\mathrm{pr}_{t_F^*}\,\Psi''_{\tilde{F}_\Delta}}\operatorname{Ch}^{t_{F,a}}(\mathsf{S}^*_{\nu''_{\tilde{F}}}|_{\tilde{F}_\Delta}),$$

where $\Phi_{\tilde{F}_\Delta}$, $\Psi''_{\tilde{F}_\Delta}$ denote the restrictions of $\Phi_{\tilde{F}}$, $\Psi''_{\tilde{F}}$ to $\tilde{F}_\Delta \subseteq \tilde{F}$. Hence

$$\frac{\mathrm{Td}(\tilde{F})\mathrm{Ch}^v(\mathcal{L}_{\tilde{F}_\Delta})}{\mathrm{Ch}^v(\mathsf{S}^*_{\nu''_{\tilde{F}}}|_{\tilde{F}_\Delta})} = u^{\sigma_{\tilde{F}_\Delta, \ell}} Q_{F,a,k}|_{\tilde{F}_\Delta}.$$

Rearranging the sums like before we arrive at
(52)

$$\widehat{q}(\lambda, k) = (\#T_\ell) \sum_{F \in \mathfrak{F}_\Delta} \int_F \sum_a t_{F,a}^{-\lambda} Q_{F,a,k} \int'_{u \in T_{F,0} \cap (T_\Delta T_\ell)} \frac{u^{\sigma_{\tilde{F}_\Delta, \ell} - \lambda}}{\mathcal{D}^{t_{F,a}}_{\mathbb{C}}(\nu'_{F,\Delta}, u)} \mathrm{Ch}^{t_{F,a}}(D \otimes \mathrm{Sym}(\nu^\perp_{F,\Delta}), u),$$

and the prime next to the integral means to only integrate over components of $T_{F,0} \cap (T_\Delta T_\ell) = T_\Delta(T_{F,0} \cap T_\ell)$ such that $t_{F,a}u$ acts with no eigenvalue equal to 1 on the bundle $\nu'_{F,\Delta}$. The expression

(53)
$$\int'_{u \in T_{F,0} \cap (T_\Delta T_\ell)} \frac{u^{\sigma_{\tilde{F}_\Delta, \ell} - \lambda}}{\mathcal{D}^{t_{F,a}}_{\mathbb{C}}(\nu'_{F,\Delta}, u)} \mathrm{Ch}^{t_{F,a}}(D \otimes \mathrm{Sym}(\nu^\perp_{F,\Delta}), u),$$

can be regarded as the Fourier transform of a product of two $\Omega(F)$-valued distributions on $T_{F,0}$, and in particular, the integral results in a function on Λ^* which is pulled back from a function on Λ^*_F. The $\Omega(F)$-valued distributions on $T_{F,0}$ in question are

(54)
$$u \mapsto \mathrm{Ch}^{t_{F,a}}(D \otimes \mathrm{Sym}(\nu^\perp_{F,\Delta}), u)$$

and

(55)
$$u \mapsto \frac{u^{\sigma_{\tilde{F}_\Delta, \ell}} \delta'_{T_{F,0} \cap T_\Delta T_\ell}(u)}{\mathcal{D}^{t_{F,a}}_{\mathbb{C}}(\nu'_{F,\Delta}, u)},$$

where $\delta'_{T_{F,0} \cap T_\Delta T_\ell}$ denotes the delta distribution supported on the union of components of $T_{F,0} \cap (T_\Delta T_\ell) = T_\Delta(T_{F,0} \cap T_\ell)$ such that $t_{F,a}u$ acts with no eigenvalue equal to 1 on the bundle $\nu'_{F,\Delta}$. Hence (53) results in a convolution of the Fourier transforms of the factors. The Fourier transform of (54) is the partition function $P^{t_{F,a}}_{\nu^\perp_{F,\Delta}, \tau_\Delta}$, as one checks by noting that the latter is the Fourier transform of the generalized function of $u \in T_{F,0}$:

$$\lim_{\epsilon \to 0^+} \frac{1}{\mathrm{Ch}^{t_{F,a}}(\bigwedge_{-v} \nu^\perp_{F,\Delta}, u \exp(-\mathrm{i}\epsilon\tau_\Delta))},$$

and using the identity (49). For the Fourier transform of (55), we decompose u into a product st with $s \in T_\Delta$ and $t \in T_\ell$, and so re-write the Fourier transform as

(56)
$$\sum'_{t \in T_\ell \cap T_{F,0}} \frac{t^{\sigma_{\tilde{F}_\Delta, \ell} - \lambda}}{\mathcal{D}^{t_{F,a}}_{\mathbb{C}}(\nu'_{F,\Delta}, t)} \int_{s \in T_\Delta} s^{\sigma_{\tilde{F}_\Delta, \ell} - \lambda}.$$

We have used the fact that T_Δ acts trivially on $\nu^\perp_{F,\Delta}$ to omit s from the denominator. The measure on T_Δ is the normalized Haar measured, as we noted in passing earlier on, and so the integral over T_Δ results in the characteristic function of $\Lambda^*_F \cap (\mathrm{ann}(\mathfrak{t}_\Delta) + \sigma_{\tilde{F}_\Delta, \ell})$.

The sum over $T_\ell \cap T_{F,0}$ in (56) is the Verlinde sum $V^{t_{F,a}}_{\nu'_{F,\Delta}}$ translated by $\sigma_{\tilde{F}_\Delta,\ell}$. Since the weights of the $T_{F,0}$ action on $\nu'_{F,\Delta}$ are contained in $\mathrm{ann}(\mathfrak{t}_\Delta)$, the latter Verlinde sum is supported on a collection of translates of $\Lambda^*_F \cap \mathrm{ann}(\mathfrak{t}_\Delta)$; more precisely the sum over $T_\ell \cap T_{F,0}$ in (56) takes the form

$$(57) \qquad \sum_{\Delta'} V^{t_{F,a},\sigma_{\tilde{F}_0}}_{\nu'_{F,\Delta},\Delta'}(\mathrm{pr}_{\mathfrak{t}^*_F}(\lambda),\ell),$$

where the sum is over affine subspaces of the form $\Delta' = \Delta + \eta$, $\eta \in \Lambda$, and we are using the shifted Verlinde sum defined in (44). Multiplying by the characteristic function of $\Lambda^*_F \cap (\mathrm{ann}(\mathfrak{t}_\Delta) + \sigma_{\tilde{F}_\Delta,\ell})$ coming from the integral over $s \in T_\Delta$, the $\Delta' = \Delta$ term in (57) is picked out. The final result is

$$\widehat{q}(\lambda,k) = \sum_{F \in \mathfrak{F}_\Delta} \int_F \sum_a t^{-\lambda}_{F,a} Q_{F,a,k} V^{t_{F,a},\sigma_{\tilde{F}_0}}_{\nu'_{F,\Delta},\Delta} \star P^{t_{F,a}}_{\nu^\perp_{F,\Delta},\tau_\Delta}\big(\mathrm{pr}_{\mathfrak{t}^*_F}(\lambda),\ell\big).$$

This shows $\widehat{q} = \overline{N}_\Delta$, completing the proof. $\qquad\qquad\qquad\qquad\qquad\square$

As a corollary of Theorem 6.5 and Remark 6.3, we have the following alternative characterization of the contributions in the (46).

Corollary 6.8. *Let $N_\Delta(-,k)$ denote the Fourier transform of the measure*

$$(58) \qquad Q_\Delta(t,k) = \delta_{T_\Delta T_\ell}(t) \sum_{F \in \mathfrak{F}^t_\Delta} \int_F \mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_\Delta}) \mathcal{AS}^t(\nu_{\tilde{F},\Delta}) \mathrm{Ch}^t(\mathrm{Sym}_{\tau_\Delta}(\nu^\perp_{F,\Delta}) \otimes \wedge \mathfrak{n}_-).$$

The multiplicity function N admits a decomposition

$$(59) \qquad\qquad\qquad N = \sum_{\Delta \in \mathcal{S}} N^{\mathrm{qpol}}_\Delta$$

where N^{qpol}_Δ is the unique function such that (i) N^{qpol}_Δ is quasi-polynomial on all subsets $C_{\Delta,\delta}$, $\delta \in \Lambda^* \otimes \mathbb{Q}$, and (ii) there is a constant K and an open neighborhood $\mathfrak{b}$ of ϵ_Δ in $\mathfrak{t}$ such that $N^{\mathrm{qpol}}_\Delta(\lambda,k) = N_\Delta(\lambda,k)$ for $\lambda \in \ell \cdot \mathfrak{b}$, $k > K$.

In other words the contribution N^{qpol}_Δ is obtained by taking the 'quasi-polynomial germ at ϵ_Δ in the $\mathfrak{t}^\perp_\Delta$-directions' of the Fourier transform of (58). It will be shown in Corollary 6.13 below that the non-vanishing contributions in (59) are labelled exactly by the set $\mathfrak{B}_\epsilon$. Thus Corollary 6.8 is a norm-square localization formula for N.

Remark 6.9. The integral in (58) has the form of a fixed-point formula on the manifold X_Δ, for the index of a Dirac operator twisted by the infinite dimensional graded vector bundle $\mathrm{Sym}_{\tau_\Delta}(\nu^\perp_{F,\Delta})$. It is tempting to view this as the quantization of the non-compact space $\mathrm{tot}(\nu^\perp_{F,\Delta})$, and then to go a step further and interpret the passage to the 'quasi-polynomial germ' as the quantization of $\mathrm{tot}(\nu^\perp_{F,\Delta}|_U)$ where U is a suitable *open* neighborhood of $X_\Delta \cap \Phi^{-1}(\exp(\epsilon_\Delta))$ in X_Δ. We will not need this and so will not pursue this here. For the simpler analogous case of Duistermaat-Heckman measures of Hamiltonian G-spaces, an interpretation of the contributions in the norm-square localization formula along these lines was given by Harada-Karshon [11]. See also [17] for an analogue in the case of Hamiltonian LG-spaces.

6.3. **The Kirillov-Berline-Vergne formula.** Let D be a Dirac operator on a compact Rie-mannian G-manifold M acting on sections of a G-equivariant Clifford bundle E. The Kirillov-Berline-Vergne formula [6] for the equivariant index is

$$(60) \qquad \mathrm{index}(\mathsf{D})(g\exp(\xi)) = \int_{M^g} \mathcal{AS}^g(\nu, \xi)$$

where ν is the normal bundle to M^g in M, and the formula holds for $\xi \in \mathfrak{g}_g$ sufficiently small. Here $\mathcal{AS}^g(\nu, \xi)$ is an equivariant extension of the usual Atiyah-Segal-Singer integrand; see Appendix A for a brief introduction.

In this section we derive a formula of this type for the distributional character Q_Δ. For the reader's convenience, we recall equation (51) here:

$$(61) \qquad Q_\Delta(t, k) = \delta_{T_\Delta T_\ell}(t) \sum_{F \in \mathfrak{F}_\Delta^t} \int_F \mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_\Delta}) \mathcal{AS}^t(\nu_{\tilde{F},\Delta}) \mathrm{Ch}^t(\mathrm{Sym}_{\tau_\Delta}(\nu_{\tilde{F},\Delta}^\perp) \otimes \wedge\mathfrak{n}_-).$$

Recall that one subtlety of (61) is that $\mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_\Delta})$, $\mathcal{AS}^t(\nu_{\tilde{F},\Delta})$ are initially defined on the covering space $\tilde{F}_\Delta$, and then we argued that the product descends to F, when $t \in T_\Delta T_\ell$. T-equivariant extensions of these differential forms, as in (60), do not descend to F. For example, the equivariant extension of $\mathrm{Ch}^t(\mathcal{L}_{\tilde{F}_\Delta})$ will involve the moment map of $\mathcal{L}_{\tilde{F}_\Delta}$, which does not descend. This means that the Kirillov-Berline-Vergne-type formula *must* be formulated in terms of covering spaces.

This leads to a second issue. For $g = 1$, equation (60) involves an integral over the entire manifold. It is not initially clear what manifold should play this role in our setting. Because of the comments in the previous paragraph, it should contain all of the covering spaces $\tilde{F}$ as fixed-point submanifolds. A manifold that contains all the $\tilde{F}$ as submanifolds is the infinite dimensional Hamiltonian loop group space $\mathcal{M}$, but this obviously cannot be used to formulate an analogue of (60). We are thus led to consider a new manifold, mentioned briefly already in Remark 4.8. We give a fresh description here.

The map

$$T \times \mathfrak{t}^\perp \to G, \qquad (t, \xi) \mapsto t\exp(\xi)$$

restricts to a $N_G(T)$-equivariant diffeomorphism on a sufficiently small ball $\mathsf{B}_r(\mathfrak{t}^\perp) \subseteq \mathfrak{t}^\perp$ around the origin in $\mathfrak{t}^\perp$. Let U be the image of $T \times \mathsf{B}_r(\mathfrak{t}^\perp)$ under this map; U is a $N_G(T)$-invariant tubular neighborhood of T in G, with canonical maps $\pi_T \colon U \to T$ and $\pi_{\mathfrak{g}/\mathfrak{t}} \colon U \to \mathfrak{t}^\perp \simeq \mathfrak{g}/\mathfrak{t}$.

Definition 6.10. Define

$$X = \Phi^{-1}(U)$$

a $N_G(T)$-invariant open subset of the q-Hamiltonian space M. Let

$$\tilde{X} = \mathfrak{t} \times_T Y$$

be the fibre product, which is a Λ-covering space of X.

Note that $\tilde{X}$ has canonical maps

$$\Phi_{\tilde{X}} \colon \tilde{X} \to \mathfrak{t} \simeq \mathfrak{t}^*, \qquad \mu_{\mathfrak{g}/\mathfrak{t}} \colon \tilde{X} \to \mathfrak{g}/\mathfrak{t}.$$

It was explained in [21] that $\tilde{X}$ embeds $N_G(T) \ltimes \Lambda$-equivariantly into the Hamiltonian loop group space, compatible with the embedding of all the $\tilde{F}$. The spinor module S for p^*TM then pulls back to a spinor module for $\tilde{X}$.

For $\Delta \in \mathcal{S}$ define

$$X_\Delta = \exp(\Delta) \times_T X^{\mathfrak{t}_\Delta} \subseteq X, \qquad \tilde{X}_\Delta = \Delta \times_T X_\Delta \subseteq \tilde{X}.$$

Note that $\tilde{X}_\Delta$ is a $\Lambda \cap \mathfrak{t}_\Delta^\perp$-covering space of X_Δ. As X_Δ contains all the $F \in \mathfrak{F}_\Delta^t$ (see (61)), $\tilde{X}_\Delta$ contains the $\tilde{F}$, $F \in \mathfrak{F}_\Delta^t$ as submanifolds.

Let $\Lambda_\Delta = \Lambda \cap \mathfrak{t}_\Delta^\perp$. The manifold $\tilde{X}_\Delta$ has a $T \ltimes \widehat{\Lambda}_\Delta$-equivariant spinor module

$$\text{(62)} \qquad \mathsf{S}_\Delta = \mathrm{Hom}_{\mathrm{Cl}(\nu_{\tilde{X}_\Delta})} \big(\wedge_{-\tau_\Delta} \nu_{\tilde{X}_\Delta}, \mathsf{S}|_{\tilde{X}_\Delta} \big),$$

where we use the $-\tau_\Delta$-polarized complex structure on the normal bundle $\nu_{\tilde{X}_\Delta}$ to $\tilde{X}_\Delta$ in $\tilde{X}$. Using $T \ltimes \widehat{\Lambda}_\Delta$-invariant connections, the Atiyah-Singer integrand $\mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi)$ is a well-defined T-equivariant differential form on $\tilde{X}_\Delta^g$ (see Appendix A). Similarly one has a $T \ltimes \widehat{\Lambda}_\Delta$-invariant twisted Chern character form $\mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi)$. These differential forms transform at level $\mathrm{h}^\vee$, k respectively under the action of Λ_Δ, hence

$$\text{(63)} \qquad \eta^* \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi) \mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi) = \big(g \exp(\xi)\big)^{\ell\eta} \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi) \mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi),$$

for all $\eta \in \Lambda_\Delta$.

Applying the discussion from Section B to $V = \mathfrak{t}^\perp \simeq \mathfrak{g}/\mathfrak{t} \simeq \mathfrak{n}_-$, and taking the Thom form τ_{V^g} to have support contained in the ball $\mathsf{B}_r(V^g)$, results in a differential form $\mathrm{Ch}^g(\mathfrak{b}, \xi)$ whose pullback by $\mu_{\mathfrak{g}/\mathfrak{t}}$ to X_Δ^g has compact support. Pulling back further to $\tilde{X}_\Delta^g$, we obtain a closed $T \times \Lambda_\Delta$-equivariant differential form. To avoid excessive notation, we will denote this pullback by $\mathrm{Ch}^g(\mathfrak{b}, \xi)$ as well.

Theorem 6.11. *The expression*

$$\text{(64)} \qquad Q_{\Delta, g}(\xi, k) = \int_{\tilde{X}_\Delta^g} \mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi) \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi) \mathrm{Ch}^g(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta}), \xi)$$

defines a distribution on a neighbourhood of $0 \in \mathfrak{t}$, whose push-forward under the map

$$\exp_g \colon \xi \in \mathfrak{t} \mapsto g \exp(\xi) \in T$$

agrees with Q_Δ on a small neighbourhood of $g \in T$.

Proof. Choose a fundamental domain $\tilde{X}_{\Delta,0}^g \subseteq \tilde{X}_\Delta^g$ for the action of Λ_Δ. Since the differential form $\mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi)$ (resp. $\mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi)$) transforms at level k (resp. $\mathrm{h}^\vee$) under the action of Λ_Δ on $\tilde{X}_\Delta^g$, equation (64) can be written

$$\text{(65)} \quad Q_{\Delta, g}(\xi, k) = \sum_{\eta \in \Lambda_\Delta} (g \exp(\xi))^{\ell\eta} \int_{\tilde{X}_{\Delta,0}^g} \mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi) \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi) \mathrm{Ch}^g(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta}), \xi).$$

Let $g = \exp(\xi_g)$ for some $\xi_g \in \mathfrak{t}$. By the Poisson summation formula (cf. [20] for the version used here),

$$\text{(66)} \quad Q_{\Delta, g}(\xi, k) = \sum_V \delta_V(\xi + \xi_g) \int_{\tilde{X}_{\Delta,0}^g} \mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi) \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi) \mathrm{Ch}^g(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta}), \xi).$$

The sum is over $\mathfrak{t}_\Delta$ cosets $V = \mu + \mathfrak{t}_\Delta$, $\mu \in \ell^{-1}\Lambda^*$. The normalization of δ_V is induced by the Haar measure on $T_\Delta T_\ell = \exp(\ell^{-1}\Lambda^* + \mathfrak{t}_\Delta)$ having total integral 1. From (66) and using the same argument as Remark 6.6, it follows that $Q_{\Delta, g}$ defines a distribution on a sufficiently small

neighborhood of $0 \in \mathfrak{t}$, where the equivariant characteristic form $\mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi)$ is an analytic function of ξ.

The delta distribution in (66) forces $\xi + \xi_g$ to lie in $\mathfrak{t}_\Delta + \ell^{-1}\Lambda^* \subseteq \mathfrak{t}$. By equation (63), when $\xi + \xi_g$ is in the support of the delta distribution, the integrand in (66) descends to a smooth form on $X_\Delta^g = \tilde{X}_\Delta^g / \Lambda_\Delta$, and

$$(67) \quad Q_{\Delta,g}(\xi, k) = \sum_V \delta_V(\xi + \xi_g) \int_{X_\Delta^g} \mathrm{Ch}^g(\mathcal{L}_{\tilde{X}_\Delta^g}, \xi) \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi) \mathrm{Ch}^g(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta}), \xi).$$

The manifold X_Δ^g may be non-compact, but the Chern character of the Bott element has compact support, ensuring that the integrand is compactly supported. The integrand is closed for the equivariant differential $\mathrm{d}_\xi = \mathrm{d} + 2\pi i \iota(\xi_X)$, and thus localizes to the fixed-point set of the vector field generated by ξ on X_Δ^g. Let $t = g \exp(\xi) = \exp_g(\xi)$. For ξ sufficiently small

$$(X_\Delta^g)^\xi = X_\Delta^t.$$

Applying the abelian localization formula, the integral in (67) becomes

$$Q_{\Delta,g}(\xi, k) = \sum_V \delta_V(\xi + \xi_g) \int_{X_\Delta^t} \mathrm{Ch}^t(\mathcal{L}_{\tilde{X}_\Delta^t}) \frac{\iota_{\tilde{X}_\Delta^t}^* \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi)}{\mathrm{Eul}(\nu_{\tilde{X}_\Delta^g, \tilde{X}_\Delta^t}, \xi)} \mathrm{Ch}^t(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta})).$$

Equations (50), (62), (93) show that

$$\frac{\iota_{\tilde{X}_\Delta^t}^* \mathcal{AS}^g(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^g}, \xi)}{\mathrm{Eul}(\nu_{\tilde{X}_\Delta^g, \tilde{X}_\Delta^t}, \xi)} = \mathcal{AS}^t(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^t}),$$

hence

$$(68) \quad Q_{\Delta,g}(\xi, k) = \sum_V \delta_V(\xi + \xi_g) \int_{X_\Delta^t} \mathrm{Ch}^t(\mathcal{L}_{\tilde{X}_\Delta^t}) \mathcal{AS}^t(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^t}) \mathrm{Ch}^t(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta})),$$

where $t = g \exp(\xi) = \exp(\xi + \xi_g)$, with ξ assumed to be sufficiently small and chosen such that $t \in T_\Delta T_\ell$ (equivalently, $\xi + \xi_g$ lies in the support of the delta distributions). Let χ be a bump function with support contained in a neighborhood of $0 \in \mathfrak{t}$ where $\exp$ is a diffeomorphism and (68) holds. Then

$$(\exp_g)_*(\chi Q_{\Delta,g})$$

is a well-defined distribution on T, supported on $T_\Delta T_\ell$ (this corresponds to the image of the cosets V under the map $\exp_g$).

If the coset tT_Δ contains a regular element $h \in T$, then, since $X_\Delta \subseteq X^{T_\Delta}$,

$$X_\Delta^t = X_\Delta^h \subseteq \Phi^{-1}(G^h) = \Phi^{-1}(T).$$

It follows that X_Δ^t is compact, and the Bott element $\mathfrak{b}$ can be replaced by its pullback to $0 \in \mathfrak{g}/\mathfrak{t}$, namely $[\wedge \mathfrak{n}_-] \in \mathrm{K}_T^0(\mathrm{pt})$. In this case the connected components of X_Δ^t are the $F \in \mathfrak{F}_\Delta^t$, and $\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^t}|_{\tilde{F}} = \nu_{\tilde{F},\Delta}$. Comparing equations (61), (68) shows that the distributions $(\exp_g)_*(\chi Q_{\Delta,g})$ and $((\exp_g)_*\chi)Q_\Delta$ agree on a neighbourhood of $g \in T$.

We claim that on the other hand, if tT_Δ does not contain any regular points, then the contribution in (68) vanishes. Indeed if tT_Δ does not contain any regular points, then t must be fixed by some non-trivial reflection $w \in W$ that also fixes T_Δ. Choose a representative $n \in N_G(T)$ for w. Then n lies in $G^t \cap G^{T_\Delta}$. The latter subgroup is connected, as one may

deduce by observing that $G^t \cap G^{T_\Delta} = G^{tt'}$ where t' is any topological generator of T_Δ, and using the fact that G^g is connected for any $g \in G$ since G is simply connected. In particular, the linear transformation Ad_n on $\mathfrak{g}^t$ is orientation-preserving. Since Ad_n reverses orientation on $\mathfrak{t} \subseteq \mathfrak{g}^t$, Ad_n reverses orientation on $(\mathfrak{g}/\mathfrak{t})^t$ as well. Hence, under pull-back by Ad_n (cf. (94), remembering $\mathrm{Ad}_n(t) = t$),

$$\mathrm{Ad}_n^* \mathrm{Ch}^t(\mathfrak{b}) = -\mathrm{Ch}^t(\mathfrak{b}),$$

in the compactly-supported cohomology of $(\mathfrak{g}/\mathfrak{t})^t$.

A second consequence of the connectedness of $G^t \cap G^{T_\Delta}$ is that n acts by an orientation-preserving diffeomorphism on each component $F \subseteq X_\Delta^t$, and acts trivially on the compactly-supported cohomology of F. The map $\mu_{\mathfrak{g}/\mathfrak{t}}$ is $N_G(T)$-equivariant. Combining these observations,

$$\mu_{\mathfrak{g}/\mathfrak{t}}^* \mathrm{Ch}^t(\mathfrak{b}) = n^* \mu_{\mathfrak{g}/\mathfrak{t}}^* \mathrm{Ch}^t(\mathfrak{b}) = \mu_{\mathfrak{g}/\mathfrak{t}}^* \mathrm{Ad}_n^* \mathrm{Ch}^t(\mathfrak{b}) = -\mu_{\mathfrak{g}/\mathfrak{t}}^* \mathrm{Ch}^t(\mathfrak{b}) \quad \Rightarrow \quad \mu_{\mathfrak{g}/\mathfrak{t}}^* \mathrm{Ch}^t(\mathfrak{b}) = 0$$

in the compactly-supported de Rham cohomology of F.

The other differential forms on the right hand side of (68) are preserved by n^* (in cohomology, or even exactly if we use suitably invariant connections for the Chern-Weil representatives). This is immediate for $\mathcal{AS}^t(\nu_{\tilde{X}_\Delta, \tilde{X}_\Delta^t})$, $\mathrm{Ch}^t(\mathcal{L}_{\tilde{X}_\Delta^t})$ because of the $N_G(T)$-equivariance of the spinor module and line bundle $\mathcal{L}$. For $\mathrm{Ch}^t(\mathrm{Sym}_{\tau_\Delta}(\nu_{X_\Delta}))$ this holds because $\mathrm{Ad}_n(\tau_\Delta) = \tau_\Delta$ as $\tau_\Delta \in \mathfrak{t}_\Delta$. Thus, for such cosets tT_Δ, the right hand side of (68) vanishes. $\square$

6.4. **An asymptotic decay result.** Fix $\Delta \in \mathcal{S}$ throughout this section. Let $\Sigma \subseteq X_\Delta$ be a connected component, and let $\tilde{\Sigma} = \Delta \times_T \Sigma \subseteq \tilde{X}_\Delta$ the corresponding covering space. Let $\mathfrak{F}_{\Delta,\Sigma}$ (resp. $\mathfrak{F}_{\Delta,\Sigma}^t$) be the subset of $F \in \mathfrak{F}_\Delta$ (resp. $\mathfrak{F}_\Delta^t$) such that $F \subseteq \Sigma$. Define $N_{\Delta,\Sigma}^{\mathrm{qpol}}$, $N_{\Delta,\Sigma}$, $Q_{\Delta,\Sigma}$ by replacing $\mathfrak{F}_\Delta$ with $\mathfrak{F}_{\Delta,\Sigma}$. Define $Q_{\Delta,\Sigma,g}$ as in (64), except restricting the range of integration to $\tilde{\Sigma}^g \subseteq \tilde{X}_\Delta^g$.

The main result of this section is the following.

Theorem 6.12. *Assume the perturbation ϵ used in the decomposition formula is sufficiently small. Let β be the orthogonal projection of 0 onto Δ and let $\Sigma \subseteq X_\Delta$ be a connected component. The contribution $N_{\Delta,\Sigma}^{\mathrm{qpol}}$ is zero unless $\exp(\beta) \in \Phi(\Sigma)$.*

Following the strategy of Szenes and Vergne in the case of compact Hamiltonian G-spaces [38], below we deduce Theorem 6.12 from the Kirillov-Berline-Vergne formula (64), combined with a stationary phase argument as the level $k \to \infty$. A further corollary of Theorem 6.12 is the following, which shows that the non-vanishing contributions N_Δ^{qpol} in the formula of Corollary 6.8 are indexed by $\mathfrak{B}_\epsilon$.

Corollary 6.13. *For a sufficiently small, generic perturbation ϵ, the contribution N_Δ^{qpol} vanishes unless $\epsilon_\Delta = \mathrm{pr}_\Delta(\epsilon) \in \mathfrak{B}_\epsilon$.*

Proof. Suppose $N_\Delta^{\mathrm{qpol}} \neq 0$. By Theorem 6.12 there exists a component $\Sigma \subseteq X_\Delta$ such that $\exp(\beta) \in \Phi(\Sigma)$. For ϵ sufficiently small this implies $\exp(\epsilon_\Delta) \in \Phi(\Sigma)$ since $\beta \approx \epsilon_\Delta$. Thus $\epsilon_\Delta \in \Phi_\mathcal{M}(\tilde{X}_\Delta)$ and therefore $\tilde{X}_\Delta \cap \Phi_\mathcal{M}^{-1}(\epsilon_\Delta) \neq \emptyset$. Since $\tilde{X}_\Delta \subseteq \mathcal{M}^{\mathfrak{t}_\Delta} \subseteq \mathcal{M}^{\epsilon_\Delta - \epsilon}$, this shows that $\epsilon_\Delta \in \mathfrak{B}_\epsilon$. $\square$

We turn to the proof of Theorem 6.12. The function $N_{\Delta,\Sigma}^{\mathrm{qpol}}$ is quasi-polynomial on $C_{\Delta,\delta}$ for every $\delta \in \Lambda^* \otimes \mathbb{Q}$. By an *open cone* in $C_{\Delta,\delta}$, we mean an open subset of $C_{\Delta,\delta}$ which is a

translate of a set invariant under scalar multiplication by $\mathbb{R}_{>0}$. A quasi-polynomial function that decays on an open cone must be identically zero. Thus, to show $N_{\Delta,\Sigma}^{\mathrm{qpol}}$ vanishes, it suffices to show that it decays on an open cone inside each subset $C_{\Delta,\delta}$. By Proposition 6.2, there is an open neighborhood b of ϵ_Δ in $\mathfrak{t}^*$ such that $N_{\Delta,\Sigma}^{\mathrm{qpol}}$, $N_{\Delta,\Sigma}$ are equal on the set of (λ, ℓ) such that $k = (\ell - \mathrm{h}^\vee) > K$ and which are contained in the set

(69) $$C_{\Delta,\delta} \cap C_{\mathsf{b}} \subseteq \mathfrak{t}^* \times \mathbb{R}_{>0}.$$

As the intersection in (69) contains an open cone inside $C_{\Delta,\delta}$, Theorem 6.12 follows immediately from the next lemma.

Lemma 6.14. *Suppose $\epsilon \in \mathfrak{t}$ is sufficiently close to 0. Let β be the orthogonal projection of 0 onto Δ, and suppose $\exp(\beta) \notin \Phi(\Sigma)$. Then there is an open neighborhood b of ϵ_Δ in $\mathfrak{t}^*$ such that $N_{\Delta,\Sigma}$ decays on the set (69).*

Proof. We begin by passing to a 'truncation' of $N_{\Delta,\Sigma}$ in order to simplify the stationary phase analysis. Symmetric powers $\mathrm{Sym}_{\tau_\Delta}^n(\nu_\Sigma)$ in (51) with $n \gg 0$ do not contribute to the restriction of $N_{\Delta,\Sigma}$ to $C_{\Delta,\delta}$. This is because the τ_Δ-polarized weights of the T_Δ action on ν_Σ form a pointed cone, and thus the minimum value of $\langle \alpha, \tau_\Delta \rangle$, as α ranges over the weights of T_Δ on $\mathrm{Sym}_{\tau_\Delta}^n(\nu_\Sigma)$, increases linearly with n. For n sufficiently large, this minimum is much larger than the constant $\langle \tau_\Delta, \Delta + \delta \rangle$. Taking Fourier transforms, this implies that the support of the contribution of $\mathrm{Sym}_{\tau_\Delta}^n(\nu_\Sigma)$ to $N_{\Delta,\Sigma}$ does not intersect $C_{\Delta,\delta}$. Therefore let $Q_{\Delta,\Sigma}^{(n)}$ denote the distribution on T defined by equation (51), except replacing $\mathrm{Sym}_{\tau_\Delta}(\nu_\Sigma)$ with the finite dimensional subbundle $\mathrm{Sym}_{\tau_\Delta}^{(n)}(\nu_\Sigma)$ only including symmetric powers not exceeding n. Let $N_{\Delta,\Sigma}^{(n)}$ be the Fourier transform of $Q_{\Delta,\Sigma}^{(n)}$. It suffices to prove the statement for $N_{\Delta,\Sigma}^{(n)}$.

Choose a finite open cover $\{U_g | g \in I\}$ of T by neighbourhoods as in Theorem 6.11, and let χ_g be bump functions on $\mathfrak{t}$ such that $\{(\exp_g)_* \chi_g | g \in I\}$ is a partition of unity subordinate to the cover. By Theorem 6.11

(70) $$Q_{\Delta,\Sigma} = \sum_{g \in I} (\exp_g)_* (\chi_g Q_{\Delta,\Sigma,g}).$$

The Fourier transform of (70) is

$$N_{\Delta,\Sigma}(\lambda, k) = \sum_{g \in I} \int_{\xi \in \mathfrak{t}} \chi_g(\xi) (g \exp(\xi))^{-\lambda} Q_{\Delta,\Sigma,g}(\xi, k).$$

The same formula holds for $N_{\Delta,\Sigma}^{(n)}$, replacing $Q_{\Delta,\Sigma,g}$ with $Q_{\Delta,\Sigma,g}^{(n)}$, where $Q_{\Delta,\Sigma,g}^{(n)}$ is defined as in (64) except replacing $\mathrm{Sym}_{\tau_\Delta}(\nu_\Sigma)$ with $\mathrm{Sym}_{\tau_\Delta}^{(n)}(\nu_\Sigma)$.

The n-truncated version of equation (65) for $Q_{\Delta,\Sigma,g}^{(n)}$ reads

(71) $$Q_{\Delta,\Sigma,g}^{(n)}(\xi, k) = \sum_{\eta \in \Lambda_\Delta} (g \exp(\xi))^{\ell \eta} \int_{\tilde{\Sigma}_0^g} \mathrm{Ch}^g(\mathcal{L}_{\tilde{\Sigma}^g}, \xi) \mathcal{AS}^g(\nu_{\tilde{\Sigma}, \tilde{\Sigma}^g}, \xi) \mathrm{Ch}^g(\mathfrak{b} \otimes \mathrm{Sym}_{\tau_\Delta}^{(n)}(\nu_\Sigma), \xi),$$

where $\tilde{\Sigma}_0^g$ is a fundamental domain for the action of $\Lambda_\Delta = \Lambda \cap \mathfrak{t}_\Delta^\perp$ on $\tilde{\Sigma}^g$ by Deck transformations. It is convenient to choose $\tilde{\Sigma}_0^g$ to be of the form $\Phi^{-1}(\Delta_0) \cap \tilde{\Sigma}^g$, where Δ_0 is a (closed) fundamental domain for the action of Λ_Δ on Δ.

The product $e^{-\ell\langle\Phi_{\tilde{\Sigma}},\xi\rangle}\mathrm{Ch}^g(\mathcal{L}_{\tilde{\Sigma}^g},\xi)\mathcal{AS}^g(\nu_{\tilde{\Sigma},\tilde{\Sigma}^g},\xi)$ is quasi-polynomial in k: the dependence on k comes the g-twisted Chern class of the level k prequantum line bundle $\mathcal{L}$, and in particular the polynomial degree is at most $\dim(M)/2$. Define

$$P_{\Sigma}^g(\xi,k) = \chi_g(\xi)e^{-\ell\langle\Phi_{\tilde{\Sigma}},\xi\rangle}\mathrm{Ch}^g(\mathcal{L}_{\tilde{\Sigma}^g},\xi)\mathcal{AS}^g(\nu_{\tilde{\Sigma},\tilde{\Sigma}^g},\xi)\mathrm{Ch}^g(\mathfrak{b}\otimes\mathrm{Sym}_{\tau_{\Delta}}^{(n)}(\nu_{\Sigma}),\xi).$$

Then P_{Σ}^g is a family of differential forms on $\tilde{\Sigma}^g$ with quasi-polynomial dependence on k and smooth dependence on ξ. Note that this last property would not hold if the full symmetric algebra bundle were used, or if the cutoff function $\chi_g(\xi)$ were omitted.

By (71),

$$(72) \qquad N_{\Delta,\Sigma}^{(n)}(\lambda,k) = \sum_{g\in I}\sum_{\eta\in\Lambda_{\Delta}} g^{\ell\eta-\lambda}\int_{\tilde{\Sigma}_0^g}\int_{\xi\in\mathfrak{t}} e^{2\pi\mathrm{i}\ell\langle\Phi_{\tilde{\Sigma}}+\eta-\ell^{-1}\lambda,\xi\rangle}P_{\Sigma}^g(\xi,k).$$

Recall $X = \Phi^{-1}(U)$, where $U\subseteq G$ is a $N_G(T)$-invariant tubular neighborhood of T, identified with $T\times\mathsf{B}_r(\mathfrak{g}/\mathfrak{t})$ via the identification $\mathfrak{g}/\mathfrak{t} = \mathfrak{t}^{\perp}$ and the map

$$(t,\xi)\in T\times\mathsf{B}_r(\mathfrak{g}/\mathfrak{t})\mapsto t\exp(\xi)\in U.$$

As $\exp(\beta)\notin\Phi(\Sigma)$, it is possible to find an open neighborhood b' of $\beta\in\mathfrak{t}$ and r' with $0 < r' < r$ such that

$$\Phi(\Sigma)\cap\exp(\mathsf{b}')\exp(B_{r'}(\mathfrak{g}/\mathfrak{t})) = \emptyset.$$

Choosing $\mathrm{Ch}(\mathfrak{b})\in\Omega(\mathfrak{g}/\mathfrak{t})$ to have support contained in $B_{r'}(\mathfrak{g}/\mathfrak{t})$, it follows that the pullback of $\mathrm{Ch}(\mathfrak{b})$ to U vanishes on $\Phi(\Sigma)\cap\pi_T^{-1}(\exp(\mathsf{b}'))$. Pulling back to X and then to $\tilde{X}$, $\mathrm{Ch}(\mathfrak{b})$ vanishes on $\Phi_{\tilde{\Sigma}}^{-1}(\mathsf{b}')$.

By choosing ϵ sufficiently small, we may assume $\epsilon_{\Delta}\in\mathsf{b}'$. Let b be a relatively compact open neighborhood of ϵ_{Δ} such that $\overline{\mathsf{b}}\subseteq\mathsf{b}'$. The distance between $\overline{\mathsf{b}}$ and $\mathfrak{t}\setminus\mathsf{b}'$ is bounded below by a constant $\varepsilon > 0$.

Suppose $\lambda\in\ell\cdot\mathsf{b}$, equivalently $\ell^{-1}\lambda\in\mathsf{b}$. Since P_{Σ}^g vanishes on $\Phi_{\tilde{\Sigma}}^{-1}(\mathsf{b}')$, the lower bound

$$|\Phi_{\tilde{\Sigma}}(x) - \ell^{-1}\lambda| > \varepsilon,$$

holds for x in the support of P_{Σ}^g. As the set where P_{Σ}^g vanishes is invariant under Deck transformations, the lower bound

$$(73) \qquad |\Phi_{\tilde{\Sigma}}(x) + \eta - \ell^{-1}\lambda| > \varepsilon$$

holds on the support of P_{Σ}^g for any $\eta\in\Lambda_{\Delta}$. For $\eta\in\Lambda_{\Delta}$ outside of a closed ball D centred at 0 we will have a stronger lower bound, of the form

$$(74) \qquad |\Phi_{\tilde{\Sigma}}(x) + \eta - \ell^{-1}\lambda| > \tfrac{1}{2}|\eta|,$$

where $x\in\tilde{\Sigma}_0^g$; this holds because $|\Phi_{\tilde{\Sigma}}|$ is bounded on $\tilde{\Sigma}_0^g$.

The infinite sum over η in (72) can be split into two parts: a finite sum

$$(75) \qquad N_{D,\Sigma}(\lambda,k) = \sum_{g\in I}\sum_{\eta\in\Lambda_{\Delta}\cap D} g^{\ell\eta-\lambda}\int_{\tilde{\Sigma}_0^g}\int_{\xi\in\mathfrak{t}} e^{2\pi\mathrm{i}\ell\langle\Phi_{\tilde{\Sigma}}+\eta-\ell^{-1}\lambda,\xi\rangle}P_{\Sigma}^g(k,\xi),$$

and an infinite tail

$$(76) \qquad N_{D^{\mathrm{c}},\Sigma}(\lambda,k) = \sum_{g\in I}\sum_{\eta\in\Lambda_{\Delta}\setminus D} g^{\ell\eta-\lambda}\int_{\tilde{\Sigma}_0^g}\int_{\xi\in\mathfrak{t}} e^{2\pi\mathrm{i}\ell\langle\Phi_{\tilde{\Sigma}}+\eta-\ell^{-1}\lambda,\xi\rangle}P_{\Sigma}^g(k,\xi).$$

In equation (75) we use the weak lower bound (73). Since P_Σ^g is quasi-polynomial in k (with polynomial degree at most $\dim(M)/2$) and depends smoothly on ξ, at each point $x \in \tilde{\Sigma}_0^g$, the principle of stationary phase gives a bound for the integral over $\mathfrak{t}$ of the form $C_j(x)(\ell\varepsilon)^{-j+\dim(M)/2}$, where $j \in \mathbb{Z}_{>0}$ can be chosen, and $C_j(x)$ is a constant depending on x and j. As the support of the integrand is a compact subset of the fundamental domain $\tilde{\Sigma}_0^g$, the constant can be taken to be uniform in x, hence we obtain a bound of the form

$$|N_{D,\Sigma}(\lambda, k)| < c_j \ell^{-j+\dim(M)/2}.$$

Taking $j > \dim(M)/2$ shows that $N_{D,\Sigma}(\lambda, k)$ decays as $\ell \to \infty$.

In equation (76) we use the stronger bound (74). Applying the principle of stationary phase as in the previous paragraph, we find a bound of the form

$$|N_{D^c,\Sigma}(\lambda, k)| < c_j \ell^{-j+\dim(M)/2} \sum_{\eta \in \Lambda_\Delta \backslash D} |\eta|^{-j}.$$

For $j > \dim(M)/2$ sufficiently large, the sum converges, hence $N_{D^c,\Sigma}(\lambda, k)$ decays as $\ell \to \infty$. $\qquad\square$

7. EXAMPLES

In this section we describe two simple examples of the norm-square localization formula. We use some basic calculations of 1-dimensional Verlinde sums from [20]. Further details can be found in [16].

7.1. The 4-sphere.
The 4-sphere S^4 can be given the structure of a quasi-Hamiltonian $SU(2)$-space (cf. [27] for details of its construction), and is prequantizable at any level $k \in \mathbb{Z}_{>0}$. The fixed-point formula is discussed in detail in [27], where it is shown that

$$Q(S^4)(t, k) = \frac{1 - t^\ell}{(1 - t)(1 - t^{-1})}, \qquad \ell = k + \mathrm{h}^\vee = k + 2.$$

The finite subgroup $T_\ell = \{t \in U(1) | t^{2\ell} = 1\} \subseteq T = U(1)$ is the group of $(2\ell)^{th}$ roots of unity. The multiplicity function is

$$N = \nabla_\alpha \overline{N}, \qquad \overline{N}(\lambda, k) = \tfrac{1}{2\ell} V_{1,-1}(\lambda, \ell) - \tfrac{1}{2\ell} V(\lambda - \ell, \ell),$$

where $\lambda \in \Lambda^* = \mathbb{Z}$, $\alpha = -2$ is the negative root of $SU(2)$, and $V_{1,-1}$ is the Verlinde sum

$$V_{1,-1}(\lambda, \ell) = {\sum_{t \in T_\ell}}' \frac{t^{-\lambda}}{(1 - t)(1 - t^{-1})}.$$

The decomposition formula for $V_{1,-1}$ with $\gamma \in (0, 2)$ is

$$V_{1,-1}(\lambda, \ell) = \left(\tfrac{1}{3}\ell^2 - \ell\lambda + \tfrac{1}{2}\lambda^2 - \tfrac{1}{12}\right) - 2\ell \sum_{m>0} x_+(\lambda - 2m\ell) - 2\ell \sum_{m \le 0} x_-(\lambda - 2m\ell),$$

where

$$x_+(r) = \begin{cases} 0, & \text{if } r \le 0 \\ r, & \text{if } r > 0 \end{cases}, \qquad x_-(r) = x_+(-r).$$

See [16, 20] for similar computations. One uses a similar expression for $V_{1,-1}(\lambda - \ell, \ell)$, but instead chooses $\gamma \in (-2, 0)$, as explained in Section 5.5.

Let $\theta_\pm = -\nabla_\alpha x_\pm$, i.e. $\theta_\pm(\lambda) = x_\pm(\lambda + 2) - x_\pm(\lambda)$. Then the norm-square localization formula for S^4 is

(77) $$N(\lambda, k) = 1 + (-1)^j \sum_{j>0} \theta_+\left(\lambda - j\ell\right) + (-1)^j \sum_{j\leq 0} \theta_-\left(\lambda - j\ell\right).$$

The contribution from $\Delta = \mathfrak{t}$ is the constant polynomial 1, while the other terms are contributions from 0-dimensional affine subspaces.

Remark 7.1. The 4-sphere is a multiplicity-free q-Hamiltonian $SU(2)$ space with surjective moment map. There are only two other such examples [13]. A second example is the fusion product $C \circledast C$, where C is the conjugacy class corresponding to the midpoint of the alcove. The norm-square localization formula for this example is given in [16]; in this case the contribution from $\Delta = \mathfrak{t}$ is the quasi-polynomial $\delta_{2\mathbb{Z}}(\lambda)$.

7.2. **A multiplicity-free q-Hamiltonian $SU(3)$-space.** A multiplicity-free q-Hamiltonian (or Hamiltonian) G-space is uniquely determined by its generic stabilizer (up to conjugacy) and moment map image (cf. [41, 13]). There is an example due to C. Woodward of a multiplicity-free q-Hamiltonian $SU(3)$-space M with moment polytope an equilateral triangle inscribed in the alcove, and trivial generic stabilizer (cf. [41] where a similar multiplicity-free Hamiltonian $SU(3)$-space is discussed). This example was constructed and studied in [16], where it was shown that M is prequantizable at level k if and only if k is even.

Identify $\mathfrak{su}(3) = \mathfrak{su}(3)^*$ using the basic inner product. Let ϖ_1, ϖ_2 denote the fundamental weights of $SU(3)$. The alcove has vertices $0, \varpi_1, \varpi_2$. The moment polytope is the triangle with vertices $\frac{1}{2}\varpi_1, \frac{1}{2}\varpi_2, \frac{1}{2}(\varpi_1 + \varpi_2)$ inscribed in the alcove. There are 3 fixed-point sets corresponding to the 3 edges of the moment polytope; each is topologically a 2-torus fixed by a 1-dimensional subtorus of T. (An interesting feature of this example is that there are no T-fixed points, which would be impossible for a compact Hamiltonian G-space.) Let F be the fixed-point set with infinitesimal stabilizer $\mathfrak{t}_F$ equal to the linear span of $\rho = \varpi_1 + \varpi_2$ in $\mathfrak{t}$. The other two fixed-point sets are conjugates of F under the $N_G(T)$-action; thus it suffices to give a formula for the contribution N_F of F to the multiplicity function, since then the other two contributions N_{F_1}, N_{F_2} will be determined by the (shifted) Weyl group anti-symmetry:

(78) $$N_{F_i}(\lambda, \ell) = (-1)^{|w_i|} N_F(w_i(\lambda + \rho) - \rho, \ell), \qquad i = 1, 2$$

where w_i denotes the reflection corresponding to the simple root α_i of $SU(3)$.

The edge of the moment polytope orthogonal to $\mathfrak{t}_F$ is the line segment

$$\mathsf{S} = \left[\tfrac{1}{2}\varpi_1, \tfrac{1}{2}\varpi_2\right].$$

For $\Phi_{\tilde{F}_0}$ one can choose

$$\Phi_{\tilde{F}_0} = \mathrm{pr}_{\mathfrak{t}_F^*}\left(\tfrac{1}{2}\varpi_1\right) = \mathrm{pr}_{\mathfrak{t}_F^*}\left(\tfrac{1}{2}\varpi_2\right).$$

Identifying $\mathfrak{t}_F = \mathfrak{t}_F^*$ with the basic inner product, $\Phi_{\tilde{F}_0} = \frac{\rho}{4}$. The torus

$$T_F = \exp(\mathfrak{t}_F) = \mathfrak{t}_F/(\Lambda \cap \mathfrak{t}_F) = \mathbb{R}\rho/\mathbb{Z}\rho.$$

Since $\langle \rho, \rho \rangle = 2$ for the basic inner product, the weight lattice of T_F is identified with $\mathbb{Z} \cdot \frac{\rho}{2}$. This already implies M is not prequantizable when k is odd, since $k\frac{\rho}{4} \in \mathbb{Z} \cdot \frac{\rho}{2}$ iff k is even.

The normal bundle ν_F is trivial of complex rank 4, and the weights of the T_F action are the projection to $\mathfrak{t}_F^*$ of the positive roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$, as well as $-\varpi_1$ (or equivalently
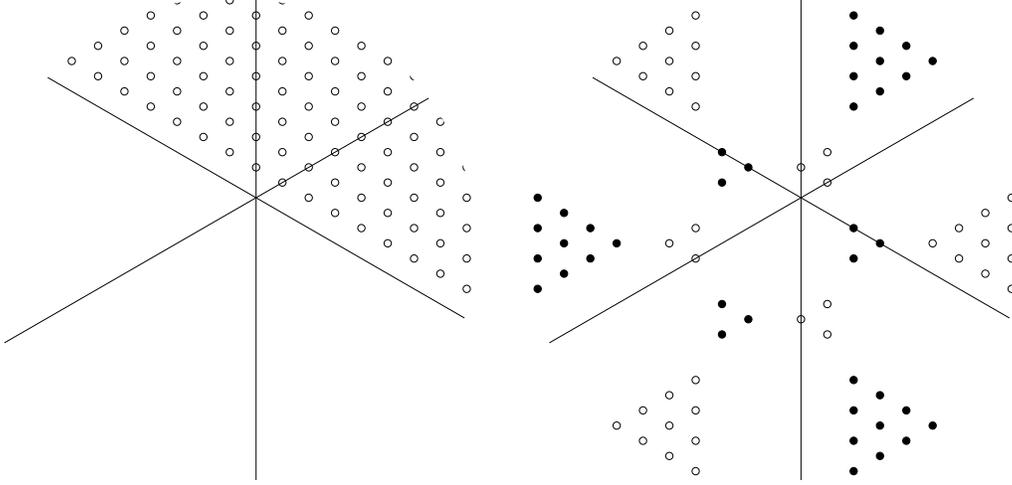
FIGURE 2. The left image shows a single contribution to the norm-square localization formula for a multiplicity-free q-Hamiltonian $SU(3)$-space (at level $k = 2$). The right image shows the sum of the first 6 contributions.

$-\varpi_2$). (One knows the positive roots appear since S intersects the interior of the alcove; the last weight and the fact that ν_F is trivial can be deduced from the construction [16].)

The evaluation of the fixed-point contribution for F involves an integral over F of the differential form $e^{k\omega}e^{c_1(\mathbb{Z})/2}$ (the A-hat class of a torus is 1). The latter integral is $3\ell = 3(k+\mathrm{h}^\vee)$ (the factor 3 can be read off from the moment polytope of F, see [16] for further details). On the other hand, $\#T_\ell = 3\ell^2$. Altogether, the fixed-point formula for N_F reduces to the following (translated) Verlinde sum,

$$(79) \qquad N_F(\lambda, k) = \frac{1}{\ell} \sideset{}{'}\sum_{t \in T_\ell \cap T_F} \frac{t^{\frac{k}{4}\varpi_1 - \lambda}}{1 - t^{\varpi_1}}.$$

The group $T_\ell \cap T_F \simeq \mathbb{Z}_\ell$, hence (79) is the pullback of a 1-dimensional Verlinde sum under the quotient map $\mathfrak{t}^* \to \mathfrak{t}_F^*$. The latter was evaluated in [20]. One finds

$$N_F(\lambda, k) = -\ell^{-1}\langle \lambda + \rho, \rho \rangle + \sum_{j \geq 0} h_+\left(\langle \lambda, \rho \rangle - \tfrac{k}{2} - j\ell\right) - \sum_{j < 0} h_-\left(\langle \lambda, \rho \rangle - \tfrac{k}{2} - j\ell + 1\right),$$

where $h_\pm$ are Heaviside functions

$$h_+(r) = \begin{cases} 1 & \text{if } r \geq 0 \\ 0 & \text{if } r < 0, \end{cases} \qquad h_-(r) = h_+(-r).$$

One obtains the multiplicity function by adding N_F together with N_{F_1}, N_{F_2} from (78). The part of the sum which is polynomial in λ is

$$\ell^{-1}\langle -(\lambda + \rho) + w_1(\lambda + \rho) + w_2(\lambda + \rho), \rho \rangle \equiv 0.$$

This is an example of the 'cancellations' observed in Section 6.4. Therefore, the norm-square localization formula is

$$N(\lambda, k) = \sum_{i=0,1,2} (-1)^{|w_i|} f\Big(\langle w_i(\lambda+\rho), \rho\rangle\Big), \quad f(r) = \sum_{j\geq 0} h_+\Big(r - \tfrac{k}{2} - 2 - j\ell\Big) - \sum_{j<0} h_-\Big(r - \tfrac{k}{2} - 1 - j\ell\Big),$$

where $w_0 = 1 \in W$. The sum of the first 6 non-trivial contributions is shown in Figure 2.

8. The quasi-polynomial multiplicity function

In this section we study the support of the non-trivial terms N_Δ^{qpol} in the norm-square localization formula (46) leading to a proof that $N(0, k)$ is a quasi-polynomial function of $k \geq 1$.

Recall from Proposition 5.7 that if $\Delta \neq \mathfrak{t}^*$ then $N_{\Delta,\Sigma}^{\mathrm{qpol}}(-, k)$ is supported in a half-space such that the in-pointing normal to boundary is τ_Δ. We now make this more precise.

Definition 8.1. For $\Sigma \subseteq X_\Delta$ a connected component, let $2\mathrm{h}^\vee \Psi_{\Delta,\Sigma}$ be the moment map for the action of T_Δ on the anti-canonical line bundle $\mathcal{Z}_{\tilde{\Sigma}} = \mathrm{Hom}_{\mathrm{Cl}(p^*TM)}(\mathsf{S}^*, \mathsf{S})|_{\tilde{\Sigma}}$; note that $\Psi_{\Delta,\Sigma} \in \mathfrak{t}_\Delta^*$ is constant, because $\Sigma \subseteq X^{T_\Delta}$ is connected, and the components of the covering space $\tilde{\Sigma} = \Delta \times_T \Sigma$ are related by Deck transformations lying in $\Lambda \cap \mathfrak{t}_\Delta^\perp$, which are annihilated by $\mathrm{pr}_{\mathfrak{t}_\Delta^*}$.

Definition 8.2. For $\Sigma \subseteq X_\Delta$ a connected component, let $\Theta_{\Delta,\Sigma} \in \mathfrak{t}_\Delta^*$ denote the sum of the complex weights for the action of T_Δ on the normal bundle ν_Σ to Σ in X, where ν_Σ is equipped with the τ_Δ-polarized complex structure.

Definition 8.3. Let $\Phi_\Delta = \mathrm{pr}_{\mathfrak{t}_\Delta^*}(\Phi_{\tilde{X}_\Delta})$, a constant in $\mathfrak{t}_\Delta^*$; equivalently $\{\Phi_\Delta\} = \mathrm{pr}_{\mathfrak{t}_\Delta^*}(\Delta)$. Note that $k\Phi_\Delta$ is the weight for the action of T_Δ on the line bundle $\mathcal{L}_{\tilde{X}_\Delta}$. (Unlike $\Psi_{\Delta,\Sigma}, \Theta_{\Delta,\Sigma}$, this does not depend on the connected component of X_Δ.)

Proposition 8.4. *Let $\Delta \neq \mathfrak{t}^*$ and let $\Sigma \subseteq X_\Delta$ be a connected component. Then $N_{\Delta,\Sigma}^{\mathrm{qpol}}(-, k)$ has support contained in the half-space described by the inequality $\langle -, \tau_\Delta\rangle \geq d_{\Delta,\Sigma}$ where*

$$(80) \qquad d_{\Delta,\Sigma} = k\langle \Phi_\Delta, \tau_\Delta\rangle + \mathrm{h}^\vee \langle \Psi_{\Delta,\Sigma}, \tau_\Delta\rangle + \frac{1}{2}\langle \Theta_{\Delta,\Sigma}, \tau_\Delta\rangle - \sum_{\{\alpha \in \mathfrak{R}_+ | \langle \alpha, \tau_\Delta\rangle > 0\}} \langle \alpha, \tau_\Delta\rangle.$$

Proof. We already did the requisite calculations in Section 6.2. The formula for $d_{\Delta,\Sigma}$ follows from the analogue of equation (51) for $Q_{\Delta,\Sigma}$, the function $N_{\Delta,\Sigma}^{\mathrm{qpol}}$ being related to $Q_{\Delta,\Sigma}$ by taking the Fourier transform, followed by replacing Verlinde sums with Verlinde quasi-polynomials (see equation (47)). Thus $d_{\Delta,\Sigma}$ equals the minimum of the function $\langle \tau_\Delta, \mu\rangle$ as μ ranges over the weights for action of T_Δ on the vector bundle

$$\mathcal{L}_{\tilde{F}_\Delta} \otimes \mathsf{S}_{\nu_{\tilde{F},\Delta}} \otimes \mathrm{Sym}_{\tau_\Delta}(\nu_{\tilde{F},\Delta}^\perp) \otimes \wedge \mathfrak{n}_-$$

over $\tilde{F}_\Delta$, for any $\tilde{F}_\Delta \subseteq \tilde{\Sigma}$ (the result is independent of the choice of $\tilde{F}_\Delta$, by the moment map property, and because $\Sigma \subseteq X^{T_\Delta}$ is connected). The line bundle $\mathcal{L}_{\tilde{F}_\Delta}$ contributes $k\Phi_\Delta$, while the factor $\wedge \mathfrak{n}_-$ contributes the (partial) sum over positive roots in (80). Since T_Δ fixes $\nu_{\tilde{F},\Delta}$, Schur's lemma implies T_Δ acts on $\mathsf{S}_{\nu_{\tilde{F},\Delta}}$ by some fixed weight. The weight for the action of

T_Δ on $\mathsf{S}_{\nu_{\tilde{F},\Delta}}$ is one half the weight for the action of T_Δ on its anti-canonical line bundle. By equation (50), the latter is

$$\det{}_{-\tau_\Delta}(\nu_{\tilde{F},\Delta}^\perp)^{-1} \otimes \mathcal{Z}_{\nu_{\tilde{F}_\Delta}} \tag{81}$$

where $\mathcal{Z}_{\nu_{\tilde{F}_\Delta}}$ is the canonical line bundle for $\mathsf{S}_{\nu_{\tilde{F}}}|_{\tilde{F}_\Delta}$. The weight for the action of T_Δ on (81) is $2\mathrm{h}^\vee \Psi_{\Delta,\Sigma} + \Theta_{\Delta,\Sigma}$, hence the result. $\square$

Theorem 8.5. *Let $k \geq 1$. Assume $0 \notin \Delta$, and $N_{\Delta,\Sigma}^{\mathrm{qpol}} \neq 0$. Then $d_{\Delta,\Sigma} > 0$.*

Proof. Let $\beta \in \mathfrak{t}_\Delta^*$ be the nearest point in Δ to the origin with respect to the inner product. As we are assuming that the perturbation ϵ is sufficiently small, $\beta \approx \tau_\Delta$. Since $0 \notin \Delta$, it suffices to prove $c_{\Delta,\Sigma} > 0$ where

$$c_{\Delta,\Sigma} := k\langle\Phi_\Delta, \beta\rangle + \mathrm{h}^\vee\langle\Psi_{\Delta,\Sigma}, \beta\rangle + \frac{1}{2}\langle\Theta_{\Delta,\Sigma}, \beta\rangle - \sum_{\{\alpha\in\mathfrak{R}_+ | \langle\alpha,\beta\rangle>0\}} \langle\alpha, \beta\rangle. \tag{82}$$

Each term in (82) other than the last is invariant under the Weyl group (with β replaced by $w\beta$, Δ by $w\Delta$ and Σ by $n \cdot \Sigma$, with $n \in N_G(T)$ being any choice of lift). On the other hand the last term is largest for $\beta \in \mathfrak{t}_+^*$. Thus, without loss of generality, we restrict to the 'worst' case when $\beta \in \mathfrak{t}_+^*$. The last term then becomes $2\langle\rho, \beta\rangle$. Note also that $\mathrm{pr}_{\mathfrak{t}_\Delta^*}(\beta) = \Phi_\Delta$, so that the first term is $k\|\beta\|^2$. Hence

$$c_{\Delta,\Sigma} = k\|\beta\|^2 + \mathrm{h}^\vee\langle\Psi_{\Delta,\Sigma}, \beta\rangle + \frac{1}{2}\langle\Theta_{\Delta,\Sigma}, \beta\rangle - 2\langle\rho, \beta\rangle.$$

Choose $w = (\bar{w}, \eta) \in W_{\mathrm{aff}}$ such that

$$\beta \in w\sigma$$

for a unique face $\sigma \subseteq \mathsf{a}$. Let $g = (\bar{g}, \eta) \in N_G(T) \ltimes \Lambda$ be a lift of w.

As $N_{\Delta,\Sigma}^{\mathrm{qpol}} \neq 0$, Theorem 6.12 implies $\exp(\beta) \in \Phi(\Sigma)$. Hence we may choose $x \in \Sigma \subseteq X^\beta$ with $\Phi(x) = \exp(\beta)$. Let $\hat{x} \in \tilde{\Sigma}$ be any lift of x. Let $2\mathrm{h}^\vee\Psi$ be a moment map for the action of T on $\mathcal{Z}$, the anti-canonical line bundle of S. Then Ψ transforms at level 1 with respect to the Λ-action, and thus

$$\Psi_{\Delta,\Sigma} = \mathrm{pr}_{\mathfrak{t}_\Delta^*}(\Psi(\hat{x})) = \mathrm{pr}_{\mathfrak{t}_\Delta^*}(\bar{w}\Psi(g^{-1}\hat{x}) + \eta).$$

By construction $g^{-1}\hat{x} \in \mathcal{Y}_\sigma^{\bar{w}^{-1}\beta}$, the $(\bar{w}^{-1}\beta)$-fixed-point subset of the cross-section $\mathcal{Y}_\sigma$ associated to the face $\sigma \subseteq \mathsf{a}$ of the alcove.

At this stage we need a fact about the anti-canonical line bundle $\mathcal{Z}$ that emerges from its construction: $\mathcal{Z}$ may be constructed by patching together the flow-outs, under the loop group action, of LG_σ-equivariant line bundles $\mathcal{K}_\sigma$ defined on the cross-sections $\mathcal{Y}_\sigma$, $\sigma \subseteq \mathsf{a}$ by

$$\mathcal{Z}_\sigma = \det{}_{\mathbb{C}}(T^*\mathcal{Y}_\sigma)^{-1} \otimes \mathbb{C}_{2(\rho-\rho'_\sigma)}$$

where one uses any compatible almost complex structure on the cross-section to define the complex determinant, and ρ'_σ is as in Section 2.2; see [31] for details. Moment maps depend on choices of connections, except along fixed-point subsets, where the corresponding components of the moment map are determined by the group action alone. It follows that

$$\mathrm{h}^\vee\langle\Psi(g^{-1}\hat{x}), \bar{w}^{-1}\beta\rangle = \frac{1}{2}\langle\Theta_\sigma, \bar{w}^{-1}\beta\rangle + \langle\rho - \rho'_\sigma, \bar{w}^{-1}\beta\rangle$$

or, equivalently,

$$\mathrm{h}^\vee \langle \bar{w}\Psi(g^{-1}\hat{x}), \beta \rangle = \frac{1}{2}\langle \bar{w}\Theta_\sigma, \beta \rangle + \langle \bar{w}(\rho - \rho'_\sigma), \beta \rangle$$

with $\Theta_\sigma \in \bar{w}^{-1}\mathfrak{t}^*_\Delta$ the sum of the complex weights for the action of $\bar{g}^{-1}T_\Delta\bar{g}$ on $T_{g^{-1}\hat{x}}\mathcal{Y}_\sigma$ (for any compatible almost complex structure).

By definition $\Theta_{\Delta,\Sigma}$ is the sum of the weights for the action of T_Δ on ν_Σ, where the latter is equipped with the τ_Δ-polarized complex structure. The tangent space is a direct sum

$$T_x X = T_x Y_{\bar{w}\sigma} \oplus \mathfrak{g}/\mathfrak{g}_{\bar{w}\sigma}$$

where $Y_{\bar{w}\sigma} = \Phi^{-1}(U_{\bar{w}\sigma})$ is the quasi-Hamiltonian cross-section and $\mathfrak{g}/\mathfrak{g}_{\bar{w}\sigma}$ is identified with G-orbit directions. Thus

$$\frac{1}{2}\langle \Theta_{\Delta,\Sigma}, \beta \rangle = \frac{1}{2}\langle \Theta^+_{\bar{w}\sigma}, \beta \rangle + \langle \rho - \rho_{w\sigma}, \beta \rangle$$

where $\Theta^+_{\bar{w}\sigma}$ is the sum of the τ_Δ-polarized weights for the subspace $T_x Y_{\bar{w}\sigma}$ (we have used the assumption $\beta \in \mathfrak{t}^*_+$, and also $\rho_{\bar{w}\sigma} = \rho_{w\sigma}$).

The combination

$$\frac{1}{2}\langle \bar{w}\Theta_\sigma, \beta \rangle + \frac{1}{2}\langle \Theta^+_{\bar{w}\sigma}, \beta \rangle \geq 0,$$

since any weight appearing in $\bar{w}\Theta_\sigma$ having negative pairing with $\beta \approx \tau_\Delta$ will cancel with the corresponding weight appearing in $\Theta^+_{\bar{w}\sigma}$. Dropping this term and the term $k\|\beta\|^2 > 0$, we obtain the lower bound

(83) $$c_{\Delta,\Sigma} > \langle \bar{w}(\rho - \rho'_\sigma) + \mathrm{h}^\vee\eta, \beta \rangle - \langle \rho + \rho_{w\sigma}, \beta \rangle.$$

The closure of $w\sigma$ is a compact, convex set. The right-hand-side of (83) is a linear function of β, and thus its minimum value on this set must occur at some vertex ξ of the face $w\sigma$. Hence

(84) $$c_{\Delta,\Sigma} > \langle \bar{w}(\rho - \rho'_\sigma) + \mathrm{h}^\vee\eta, \xi \rangle - \langle \rho + \rho_{w\sigma}, \xi \rangle.$$

Then $w^{-1}\xi$ is some vertex ν of a. Consider the sum

$$\bar{w}\rho'_\sigma + \rho_{w\sigma}.$$

$\rho_{w\sigma}$ is the half-sum of the positive roots $\mathfrak{R}_{\bar{w}\sigma,+} \subseteq \mathfrak{R}_+$ for the subalgebra $\mathfrak{g}_{\bar{w}\sigma}$, while $\bar{w}\rho'_\sigma$ is the half-sum of the (possibly different) set $\bar{w} \cdot \mathfrak{R}'_{\sigma,+}$ of positive roots for the same subalgebra. Those roots in the intersection $\mathfrak{R}_{\bar{w}\sigma,+} \cap \bar{w} \cdot \mathfrak{R}'_{\sigma,+}$ add, while the others cancel. Therefore the sum $\bar{w}\rho'_\sigma + \rho_{w\sigma}$ is a sum of a set of roots in $\mathfrak{R}_+$.

By a similar argument

$$\bar{w}\rho'_{\{\nu\}} + \rho_{\{\xi\}}$$

is a sum of a set of roots in $\mathfrak{R}_+$ (here $\{\nu\}$, $\{\xi\}$ are faces of the Stiefel diagram containing a single point). Since $\mathfrak{R}_{\bar{w}\sigma,+} \subseteq \mathfrak{R}_{\{\xi\},+}$ and $\mathfrak{R}'_{\sigma,+} \subseteq \mathfrak{R}'_{\{\nu\},+}$, the same positive roots will appear as in the sum $\bar{w}\rho'_\sigma + \rho_{w\sigma}$, and possibly some additional ones. Since $\xi \in \mathfrak{t}^*_+$ it follows that

$$\langle \bar{w}\rho'_{\{\nu\}} + \rho_{\{\xi\}}, \xi \rangle \geq \langle \bar{w}\rho'_\sigma + \rho_{w\sigma}, \xi \rangle.$$

Using this in (84) yields

(85) $$c_{\Delta,\Sigma} > \langle \bar{w}(\rho - \rho'_{\{\nu\}}) + \mathrm{h}^\vee\eta, \xi \rangle - \langle \rho + \rho_{\{\xi\}}, \xi \rangle.$$

By a special case of Lemma 2.5

$$\rho - \rho'_{\{\nu\}} = \mathrm{h}^\vee\nu$$

thus

$$\bar{w}(\rho - \rho'_{\{\nu\}}) + \mathrm{h}^\vee \eta = \mathrm{h}^\vee w \nu = \mathrm{h}^\vee \xi,$$

and (85) becomes

$$c_{\Delta, \Sigma} > \mathrm{h}^\vee \|\xi\|^2 - \langle \rho + \rho_\xi, \xi \rangle.$$

The result now follows from the Key Inequality (7). □

As a corollary, $N_{\Delta,\Sigma}^{\mathrm{qpol}}$ does not contribute to the multiplicity $N(0,k)$, and we obtain our main result:

Corollary 8.6. $N(0, k)$ *is a quasi-polynomial function of* k.

Proof. Theorem 8.5 implies

$$N(0, k) = \sum_{\{\Delta | 0 \in \Delta\}} \sum_{\Sigma \subseteq X_\Delta} N_{\Delta, \Sigma}^{\mathrm{qpol}}(0, k), \qquad \forall k \geq 1.$$

But if $0 \in \Delta$, then $N_{\Delta, \Sigma}^{\mathrm{qpol}}(0, k)$ is a quasi-polynomial function of k by Proposition 5.7. □

As explained in the introduction, Corollary 8.6 is closely related to the $[Q, R] = 0$ Theorem [3] for quasi-Hamiltonian G-spaces or Hamiltonian LG-spaces. The last step in a proof of the $[Q, R] = 0$ Theorem would be to apply the Kirillov-Berline-Vergne index formula of Section 6.3 and a stationary phase argument to deduce that $N(0, k)$ equals the quantization of the reduced space for large k, similar to [24, 18].

Appendix A. The Atiyah-Singer integrand for the Spin_c Dirac operator

Let M be a closed oriented Riemannian manifold of even dimension n. Let G be a compact Lie group with a given action on M by orientation-preserving isometries. Suppose that M is equipped with a G-equivariant Spin_c-structure: a G-equivariant principal $\mathrm{Spin}_c(n)$-bundle Q, together with a $G \times \mathrm{Spin}_c(n)$-equivariant bundle map $Q \to P$ to the oriented orthonormal frame bundle of M. Equivalently, in terms of the unique irreducible representation Δ of the Clifford algebra $\mathbb{C}l(n)$, M is equipped with a Hermitian vector bundle

$$\mathsf{S} = P \times_{\mathrm{Spin}_c(n)} \Delta$$

and an isomorphism $\mathbb{C}l(TM) \simeq \mathrm{End}(\mathsf{S})$. As a representation of $\mathrm{Spin}_c(n)$, Δ splits into a direct sum $\Delta^+ \oplus \Delta^-$, and accordingly $\mathsf{S} = \mathsf{S}^+ \oplus \mathsf{S}^-$ acquires a $\mathbb{Z}_2$-grading.

Fix $g \in G$, and let F be a connected component of M^g. The action of g defines sections g_ν, g_S of the group bundles $\mathrm{SO}(\nu)$, $\mathrm{Spin}_c(\nu)$ respectively. Since the G-action preserves the orientation, $n_0 = \dim(F)$ is even. Since the G-action preserves the Spin_c-structure, F is orientable (the proof is similar to [5, Proposition 6.14], combined with the observation that the variant of $\zeta(g)^{1/2}$ (see Definition A.4 below) defined on the oriented double cover of F is locally constant and reverses sign under the non-trivial deck transformation). Let $\nu = TF^\perp$ be the normal bundle to F, and let n_1 be its rank. Since $TM|_F$, TF are orientable, so is ν.

Let $x \in F$. A frame of ν_x is adapted to g if, relative to the frame, the matrix of g_x is block diagonal $\mathrm{diag}(-1_{n_\pi}, B_1, ..., B_r)$ where B_j is counterclockwise rotation by θ_j in $\mathbb{R}^2$, and $0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_r < \pi$. It is convenient to fix an orientation of ν, once and for all, such that any adapted frame is oriented (the latter condition only determines an orientation uniquely if $n_\pi = 0$). The orientations of $TM|_F$, ν determine an orientation of TF. An oriented

frame of $T_x M$ is adapted to g if its first n_0 elements form an oriented frame of $T_x F$ and its remaining n_1 elements form an adapted frame of ν_x. Thus along F the action of g determines a reduction of structure group $P_F \subseteq P|_F$, from $\mathrm{SO}(n)$ to $\mathrm{SO}(n_0) \times \mathrm{SO}(n_1)^g$, where $\mathrm{SO}(n_1)^g$ is the subgroup of $\mathrm{SO}(n_1)$ commuting with $\mathrm{diag}(-1_{n_\pi}, B_1, ..., B_r)$. The inverse image of P_F is a reduction of structure group $Q_F \subseteq Q|_F$ from $\mathrm{Spin}_c(n)$ to $\mathrm{Spin}_c(n_0) \times_{U(1)} \mathrm{Spin}_c(n_1)^g$. There is a unique element of $\mathrm{Spin}_c(n_1)$ determined by g (any choice of point in Q_F yields the same element), which we also denote by g when there is no risk of confusion. The Lie algebra

$$\mathrm{Lie}(\mathrm{Spin}_c(n_0) \times_{U(1)} \mathrm{Spin}_c(n_1)^g) = \mathfrak{so}(n_0) \oplus \mathfrak{spin}_c(n_1)^g = \mathfrak{so}(n_0) \oplus \mathfrak{so}(n_1)^g \oplus \mathfrak{u}(1)$$

splits into a direct sum, and we write elements of the Lie algebra as triples (A_0, A_1, a). For $j = 0, 1$, let Δ_j denote the irreducible $\mathbb{C}l(n_j)$-module. The map $\mathrm{Lie}(\mathrm{Spin}_c(n_0) \times_{U(1)} \mathrm{Spin}_c(n_1)) \to \mathfrak{spin}_c(n_1)$ makes Δ_1 into a representation of $\mathrm{Lie}(\mathrm{Spin}_c(n_0) \times_{U(1)} \mathrm{Spin}_c(n_1))$.

Definition A.1. The relative Chern character $\mathrm{Ch}_F^g(\mathsf{S}) \in H^\bullet(F)$ is the characteristic class of Q_F associated to the Ad-invariant analytic function

$$(86) \qquad\qquad (A_0, A_1, a) \mapsto \mathrm{sTr}_{\Delta_1}(g e^{\mathrm{i}(A_1+a)/2\pi})$$

on the Lie algebra of $\mathrm{Spin}_c(n_0) \times_{U(1)} \mathrm{Spin}_c(n_1)^g$.

The definition applies to any G-equivariant Spin_c-structure, and in particular $\mathrm{Ch}_F^g(\mathsf{S}^*)$ is defined, where $\mathsf{S}^* = \mathrm{Hom}(\mathsf{S}, \mathbb{C})$ is the dual spinor module. Since g acts without fixed vectors on ν, the class $\mathrm{Ch}_F^g(\mathsf{S}^*)$ is invertible.

Definition A.2. The Atiyah-Singer integrand is the characteristic class of Q_F given by

$$(87) \qquad\qquad \mathcal{AS}^g(\nu) = \frac{\widehat{\mathrm{A}}(F)}{\mathrm{Ch}_F^g(\mathsf{S}^*)}.$$

The fixed point contribution to the index of the Spin_c Dirac operator for S is the pairing of $\mathcal{AS}^g(\nu)$ with the fundamental class of F. Equation (87) for the fixed-point contribution may be derived from the general formula for the fixed-point contributions given by Atiyah and Singer. One of several other equivalent expressions is

$$(88) \qquad\qquad \mathcal{AS}^g(\nu) = \frac{\mathrm{Eul}(F)}{\mathrm{Ch}^g(\mathsf{S}^*)},$$

where $\mathrm{Eul}(F)$ is the Euler class of TF, and $\mathrm{Ch}^g(\mathsf{S}^*)$ is the ordinary Chern character of the $\mathbb{Z}_2$-graded complex vector bundle S^*; however note that in (88), the class $\mathrm{Ch}^g(\mathsf{S}^*)$ is not invertible, so the quotient only makes sense when interpreted in a suitable ring of characteristic classes.

This is an appropriate place to remark that, the definition of $\mathrm{Ch}_F^g(\mathsf{S}^*)$, and hence $\mathcal{AS}^g(\nu)$, does not depend on the ambient manifold M except through the g-equivariant Spin_c-structure on the Euclidean vector bundle $TF \oplus \nu$. This becomes relevant in our setting, in order to define $\mathcal{AS}^g(\nu)$ in a context where F is not canonically embedded into an ambient closed Spin_c manifold.

Since the Lie group $\mathrm{Spin}_c(n_0) \times_{U(1)} \mathrm{Spin}_c(n_1)$ does not split into a product, the spinor module S does not necessarily split into a graded tensor product along F. However one has

$$\mathrm{sTr}_{\Delta_1}(c) = (-2\mathrm{i})^{-n_0/2} \mathrm{sTr}_\Delta(e_1 \cdots e_{n_0} c),$$

for $c \in \mathrm{Cl}(n_1)$, where $e_1, \cdots, e_{n_0}$ is an oriented orthonormal frame of $\mathbb{R}^{n_0}$. This yields an alternative formula for (86) suited to the Chern-Weil construction. Let $\mathcal{V}_0 \in \Gamma(\det(TF))$ be the Riemannian volume form; it may be regarded as an element $\mathrm{Cl}(TM|_F)$ via the isomorphism $\wedge TM|_F \simeq \mathrm{Cl}(TM|_F)$. As F is totally geodesic, the Levi-Civita connection preserves the splitting $TM|_F = TF \oplus \nu$. Let $R = R_F \oplus R_\nu \in \Omega^2(F, \mathfrak{so}(TF) \oplus \mathfrak{so}(\nu))$ be the pullback of the Riemann curvature to F. Fix a connection on S compatible with the Levi-Civita connection, and let $R_\mathsf{S} \in \Omega^2(F, \mathfrak{spin}_c(TM|_F))$ be the pullback of its curvature to F. The image of R_S under the map $\mathfrak{spin}_c(TM|_F) \to \mathfrak{so}(TM|_F)$ is R.

Proposition A.3. *The differential form*

$$(-2\mathrm{i})^{-\dim(F)/2} \mathrm{sTr}_\mathsf{S}(\mathcal{V}_0 g_\mathsf{S}^{-1} e^{-\mathrm{i}(R_\mathsf{S} - R_F)/2\pi}) \in \Omega^\bullet(F)$$

represents $\mathrm{Ch}_F^g(\mathsf{S}^*)$.

We will use the same symbols $\mathrm{Ch}_F^g(\mathsf{S}^*)$, $\mathcal{AS}^g(\nu)$ to denote these Chern-Weil representatives; in particular we use this interpretation in more general contexts where F may be non-compact.

Let $\mathcal{Z} = \mathrm{Hom}_{\mathrm{Cl}}(\mathsf{S}^*, \mathsf{S})$ be the anti-canonical line bundle of the Spin_c structure. The element g acts on $\mathcal{Z}|_F$ by a phase factor $\zeta(g) \in U(1)$. The chosen orientation of ν determines a lift of $g_\nu \in \mathrm{SO}(\nu)$ to $\mathrm{Spin}(\nu)$, that we also denote g_ν, defined by the condition that the top-degree part of the image of g_ν under the isomorphism $\mathrm{Cl}(\nu) \simeq \wedge \nu$ is positive relative to the orientation of ν.

Definition A.4. The product $g_\mathsf{S} g_\nu^{-1} \in U(1)$ is a square-root of $\zeta(g)$ that we denote by $\zeta(g)^{1/2}$.

Remark A.5. Let $x \in F$ and fix a g-equivariant orthogonal complex structure J_x on $T_x M$ compatible with the orientation. There is an isomorphism of $\mathbb{C}l(T_x M)$-modules $\mathsf{S}_x \simeq \wedge_{J_x} T_x M \otimes D_x$ for a complex line D_x. Let $\kappa_{D_x}(g) \in U(1)$ be the phase factor for the induced action of g on D_x. Let $g_{J_x} \in U(T_x M, J_x)$ be the unitary transformation induced by g, and $g_{J_x}^{1/2} \in U(T_x M, J_x)$ the square root having eigenvalues $e^{\mathrm{i}\theta/2}$ with $\theta \in [0, 2\pi)$. Then $\zeta(g)^{1/2} = \kappa_{D_x}(g) \det_{\mathbb{C}}(g_{J_x}^{1/2})$.

By [5, Proposition 3.24] the function

$$A_1 \in \mathfrak{so}(n_1) \mapsto \det(1 - g e^{\mathrm{i}A_1/2\pi})$$

has an analytic square root $\det^{1/2}(1 - g e^{\mathrm{i}A_1/2\pi})$, where we fix the sign by demanding that $\det^{1/2}(1 - g) > 0$.

Definition A.6. Let $\mathcal{D}_\mathbb{R}^g(\nu) \in \Omega^\bullet(F)$ be the differential form

$$\mathcal{D}_\mathbb{R}^g(\nu) = \mathrm{i}^{\mathrm{rank}(\nu)/2} \det^{1/2}(1 - g_\nu e^{\mathrm{i}R_\nu/2\pi}).$$

Note that $\mathcal{D}_\mathbb{R}^g(\nu)$ is invertible since its component in $\Omega^0(F)$ is $\mathrm{i}^{n_1/2} \det^{1/2}(1 - g_\nu) \neq 0$. We use the same symbol $\mathcal{D}_\mathbb{R}^g(\nu)$ to denote its class in $H^\bullet(F)$.

Proposition A.7. *The Atiyah-Singer fixed-point contribution is*

$$(89) \qquad \mathcal{AS}^g(\nu) = \frac{\widehat{\mathrm{A}}(F) \zeta(g)^{1/2} e^{c_1(\mathcal{Z})/2}}{\mathcal{D}_\mathbb{R}^g(\nu)}.$$

This can be deduced from the formula (cf. [5, Proposition 3.23])

$$\det{}^{1/2}(1 - ge^{iA_1/2\pi}) = i^{-n_1/2}\mathrm{sTr}_{\Delta_1}(ge^{iA_1/2\pi}),$$

for the square-root. Note also that $\det(1 - u) = \det(1 - u^{-1})$ for any $u \in \mathrm{SO}(n_1, \mathbb{C})$. A similar calculation shows that

$$\mathrm{Ch}_F^g(\mathsf{S}^*)^2 = (-1)^{n_1/2} \det(1 - g_\nu e^{iR_\nu/2\pi})\mathrm{Ch}^g(\mathcal{Z}^{-1}),$$

hence $\mathrm{Ch}_F^g(\mathsf{S}^*)$ is a square-root of the right-hand-side.

Under various assumptions on F, the Atiyah-Singer fixed-point contribution admits further simplification.

Proposition A.8. *If F admits an almost complex structure J_F, and $\mathsf{S}_\nu = \mathrm{Hom}_{\mathrm{Cl}}(\wedge_{J_F} TF, \mathsf{S})$ is the induced $\mathbb{C}l(\nu)$-module, then*

$$(90) \qquad\qquad \mathcal{AS}^g(\nu) = \frac{\mathrm{Td}(F)}{\mathrm{Ch}^g(\mathsf{S}_\nu^*)}.$$

If furthermore $\mathsf{S}_\nu \simeq \wedge_{J_\nu}\nu \otimes D$ for a g-equivariant orthogonal complex structure J_ν on ν and line bundle D, then

$$(91) \qquad \mathcal{AS}^g(\nu) = \frac{\mathrm{Td}(F)\mathrm{Ch}^g(D)}{\mathcal{D}_{\mathbb{C}}^g(\nu)}, \qquad \mathcal{D}_{\mathbb{C}}^g(\nu) = \det{}_{\mathbb{C}}(1 - g_\nu^{-1}e^{-iR_\nu/2\pi}),$$

where g_ν, R_ν are regarded as endomorphisms of $\nu^{1,0}$ in the determinant.

The differential form $\mathcal{AS}^g(\nu)$ admits an *equivariant extension* $\mathcal{AS}^g(\nu, \xi)$, obtained by replacing curvatures with equivariant curvatures (cf. [5]) in the formulas, defined for $\xi \in \mathfrak{g}_g$ sufficiently small that the denominator $\mathrm{Ch}_F^g(\mathsf{S}^*, \xi)$ remains invertible:

$$(92) \qquad\qquad \mathcal{AS}^g(\nu, \xi) = \frac{\widehat{\mathrm{A}}(F)}{\mathrm{Ch}_F^g(\mathsf{S}^*, \xi)}.$$

Our convention for the equivariant curvature is such that $\mathcal{AS}^g(\nu, \xi)$ is closed for the differential $\mathrm{d}_\xi = \mathrm{d} + 2\pi i\iota(\xi_F)$. There are obvious analogues of (88), (89), (90).

Finally we briefly outline the proof of the Berline-Vergne-Kirillov index formula (cf. [5]). Let $\xi \in \mathfrak{g}_g$, let $\iota\colon F' = F^\xi \hookrightarrow F$ be the fixed-point set. Let $\nu_{F,F'}$ be the normal bundle to F' in F and $\nu' = \nu|_{F'} \oplus \nu_{F,F'}$. The action of ξ determines an orientation of $\nu_{F,F'}$. Let $g' = g\exp(\xi)$. For ξ sufficiently small, $\nu^{g'} = F^\xi = F'$. Using the equivariant analogue of (88) and multiplicativity of the Euler class,

$$(93) \qquad \frac{\iota^*\mathcal{AS}^g(\nu, \xi)}{\mathrm{Eul}(\nu_{F,F'}, \xi)} = \frac{\mathrm{Eul}(F')}{\iota^*\mathrm{Ch}^g(\mathsf{S}^*, \xi)} = \frac{\mathrm{Eul}(F')}{\mathrm{Ch}^{g'}(\mathsf{S}^*|_{F'})} = \mathcal{AS}^{g'}(\nu').$$

The Berline-Vergne-Kirillov index formula follows from this expression and the abelian localization formula in equivariant cohomology.

Appendix B. The Bott element and its Chern character.

Let V be a complex vector space with an effective T-action, and let $g \in T$. Viewing $\wedge V$ as a $\mathbb{Z}_2$-graded complex vector bundle over a point, its equivariant twisted Chern character is simply its ordinary character as a representation of T:

$$\mathrm{Ch}^g(\wedge V, \xi) = \det^V_{\mathbb{C}}(1 - g \exp(\xi)).$$

Let $g \in T$ and let V^g denote the subspace fixed by g. The T-equivariant Euler class of V^g (viewed as a vector bundle over a point) is the polynomial

$$\xi \mapsto \det^{V^g}_{\mathbb{C}}(\xi).$$

Let τ_{V^g} be a Thom form for V^g, i.e. a compactly supported closed differential form on V^g with integral 1. It has a T-equivariant extension $\tau_{V^g}(\xi)$ whose pullback to the origin in V^g is $\det^{V^g}_{\mathbb{C}}(\xi)$.

Let $\mathfrak{b} \in \mathrm{K}^0_T(V)$ denote the Bott element associated to the complex structure: the unique element of $\mathrm{K}^0_T(V)$ whose pullback to the origin is $\wedge V \in \mathrm{K}^0_T(\mathrm{pt}) \simeq R(T)$. The T-equivariant differential form

(94) $$\mathrm{Ch}^g(\mathfrak{b}, \xi) := \det^{V/V^g}_{\mathbb{C}}(1 - g \exp(\xi)) \det^{V^g}_{\mathbb{C}} \left(\frac{1 - \exp(\xi)}{\xi} \right) \tau_{V^g}(\xi).$$

on V^g represents its equivariant twisted Chern character. Indeed since pullback to the origin is injective for equivariant cohomology, this can be checked by pulling back (94) to the origin, where it becomes

$$\det^V_{\mathbb{C}}(1 - g \exp(\xi)) = \mathrm{Ch}^g(\wedge V, \xi).$$

References

1. A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.
2. A. Alekseev and E. Meinrenken, *Dirac structures and Dixmier-Douady bundles*, International Mathematics Research Notices **2012** (2012), no. 4, 904 – 956.
3. A. Alekseev, E. Meinrenken, and C. Woodward, *The Verlinde formulas as fixed point formulas*, J. Symplectic Geom. **1** (2001), no. 1, 1–46.
4. A. Alekseev, E. Meinrenken, and C. Woodward, *Duistermaat-Heckman measures and moduli spaces of flat bundles over surfaces*, Geom. and Funct. Anal. **12** (2002), 1–31.
5. N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators*, Grundlehren der mathematischen Wissenschaften, vol. 298, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
6. N. Berline and M. Vergne, *The equivariant index and Kirillov's character formula*, Amer. J. Math. **107** (1985), 1159–1190.
7. R. Bott, S. Tolman, and J. Weitsman, *Surjectivity for Hamiltonian loop group spaces*, Invent. Math. (2003), to appear.
8. J.-F. Boutot, *Singularités rationelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987), 65–68.
9. A. Boysal and M. Vergne, *Multiple Bernoulli series, an Euler-Maclaurin formula, and wall crossings*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 2, 821–858.
10. S. Chang, *Fixed point formula and loop group actions*, math.AG/9812148.
11. M. Harada and Y. Karshon, *Localization for equivariant cohomology with varying polarization*, Comm. Anal. Geom. **20** (2012), no. 5, 869–947.
12. F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton Univ. Press, Princeton, 1984.
13. F. Knop, *Multiplicity free quasi-hamiltonian manifolds*, Preprint (2016), arXiv:1612.03843.

14. B. Kostant, *On Macdonald's η-function formula, the Laplacian and generalized exponents*, Advances in Math. **20** (1976), no. 2, 179–212.

15. S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

16. Y. Loizides, *Norm-square localization for Hamiltonian LG-spaces*, Ph.D. thesis, 2017.

17. Y. Loizides, *Norm-square localization for Hamiltonian LG-spaces*, J. Geom. and Phys. **114** (2017), 420–449.

18. Y. Loizides, *Quasi-polynomials and the singular [Q,R]=0 theorem*, SIGMA, **15**(090) (2019).

19. Y. Loizides, *Geometric K-homology and the Freed-Hopkins-Teleman theorem*, J. Noncomm. Geom., to appear. arXiv:1804.05213.

20. Y. Loizides and E. Meinrenken, *The decomposition formula for Verlinde sums.*, Ann. Inst. Four., to appear.

21. Y. Loizides, E. Meinrenken, and Y. Song, *Spinor bundles for Hamiltonian loop group spaces*, J. Symplectic Geom., **18** (2020), no. 3, 889–937.

22. Y. Loizides and Y. Song, *Quantization of Hamiltonian loop group spaces*, Mathematische Annalen, **374** (2019), no. 1, 681–722.

23. Y. Loizides and Y. Song, *Norm-square localization and the quantization of Hamiltonian loop group spaces*, J. Func. Anal.,**278** (2020), no. 9.

24. E. Meinrenken, *On Riemann-Roch formulas for multiplicities*, J. Amer. Math. Soc. **9** (1996), 373–389.

25. _____, *Lie groups and Clifford algebras*, 2005, Lecture notes, University of Toronto. Available at: http://www.math.toronto.edu/mein/teaching/lectures.html.

26. E. Meinrenken, *Symplectic surgery and the Spin-c Dirac operator*, Adv. Math., **134** (1998), 240–277.

27. _____, *Quantization of q-Hamiltonian SU(2)-spaces*, Geometric Aspects of Geometry and Mechanics (in honor of the 65th birthday of Hans Duistermaat), Birkhäuser, 2011, pp. 257–293.

28. _____, *Twisted K-homology and group-valued moment maps*, International Mathematics Research Notices **2012 (20)** (2012), 4563–4618.

29. E. Meinrenken and R. Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), no. 4, 699–762.

30. E. Meinrenken and C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, J. Differential Geom. **50** (1999), 417–470.

31. _____, *Canonical bundles for Hamiltonian loop group manifolds*, Pacific J. Math. **198** (2001), no. 2, 477–487.

32. P.-E. Paradan, *Localization of the Riemann-Roch character*, J. Funct. Anal. **187** (2001), no. 2, 442–509.

33. A. Pressley and G. Segal, *Loop groups*, Oxford University Press, Oxford, 1988.

34. R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. (2) **134** (1991), 375–422.

35. A. Szenes, *Iterated residues and multiple Bernoulli polynomials*, Internat. Math. Res. Notices (1998), no. 18, 937–956.

36. _____, *Residue theorem for rational trigonometric sums and Verlinde's formula*, Duke Math. J. **118** (2003), no. 2, 189–227.

37. A. Szenes and O. Trapeznikova, *The parabolic Verlinde formula: iterated residues and wall-crossing*, Preprint, 2022. arXiv:2112.15149.

38. A. Szenes and M. Vergne, *[Q, R] = 0 and Kostant partition functions*, Enseign. Math. **63** (2017), 471–516.

39. V. Toledano Laredo, *Positive energy representations of the loop groups of non-simply connected groups*, Comm. Math. Phys. **207** (1999), 307–339.

40. E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nuclear Phys. B **300** (1988), 360–376.

41. C. Woodward, *The classification of transversal multiplicity-free group actions*, Ann. Global Anal. Geom. **14** (1996), 3–42.