LECTURES ON GROUP-VALUED MOMENT MAPS AND VERLINDE FORMULAS

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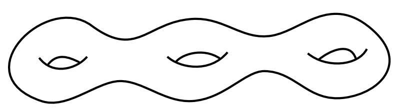
1. Introduction

These notes are based on lectures delivered at the 'Summer School on Quantization' at Notre Dame University, May 31-June 4, 2011. Some additional material is included from a lecture series at the IGA workshop in Adelaide, September 2011. The audience for the summer school were postdoctoral and graduate students, with a variety of backgrounds. I made an effort to keep the lectures at a moderate pace, and to present motivation and foundational material, without going into technical details. These notes, while more detailed than the actual lectures, are written with a similar audience in mind.

I thank David Li-Bland for his help in preparing this notes, and valuable comments.

2. Motivation: Moduli spaces of flat bundles

Suppose G is a compact, simply connected Lie group, and $\cdot$ an invariant inner product ('metric') on its Lie algebra $\mathfrak{g}$. Let Σ be a closed, connected, oriented 2-manifold of genus h

$\Sigma =$ 

Since G is assumed to be simply connected, any principal G-bundle over Σ is trivial. Let $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ be the infinite-dimensional affine space of connections on the trivial G-bundle over Σ. (We are treating infinite-dimensional manifolds in an informal manner; in any case we will soon pass to a finite-dimensional picture.) The group $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, G)$ acts on $\mathcal{A}(\Sigma)$ by *gauge transformations*,

$$g.A = \mathrm{Ad}_g(A) - g^*\theta^R.$$

(We denote by $\theta^R, \theta^L \in \Omega^1(G, \mathfrak{g})$ the right-invariant Maurer-Cartan form on G.) The *curvature*

$$\mathrm{curv}(A) = \mathrm{d}A + \tfrac{1}{2}[A, A] \in \Omega^2(\Sigma, \mathfrak{g})$$

transforms nicely under this action: $\mathrm{curv}(g.A) = \mathrm{Ad}_g \,\mathrm{curv}(A)$. In particular, the subset $\mathcal{A}_{\mathrm{flat}} = \{A \in \Omega^1(\Sigma, \mathfrak{g}) |\ \mathrm{curv}(A) = 0\}$ of *flat connections* is gauge invariant. Let

$$\mathcal{M}(\Sigma) = \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma)$$

be the *moduli space of flat connections* on the trivial G-bundle over Σ. As observed by Atiyah-Bott [11, 10], the space $\mathcal{M}(\Sigma)$ carries a natural symplectic structure, depending

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only on the choice of the metric $\cdot$ on $\mathfrak{g}$. Here the symplectic form is obtained by symplectic reduction, as follows. (We suggest the book [19] for background on symplectic reduction; this particular example is discussed on p. 158 of the book.) First, the affine space $\mathcal{A}(\Sigma)$ carries a symplectic form, given on tangent vectors $a, b \in T_A\Omega^1(\Sigma, \mathfrak{g}) = \Omega^1(\Sigma, \mathfrak{g})$ by

$$\omega_A(a, b) = \int_\Sigma a \cdot b.$$

The action of the gauge group $\mathcal{G}(\Sigma)$ preserves this 2-form, and is in fact *Hamiltonian*, with moment map the curvature $\mathrm{curv}\colon \mathcal{A}(\Sigma) \to \Omega^2(\Sigma, \mathfrak{g})$. That is,

$$\omega(\xi_{\mathcal{A}(\Sigma)}, \cdot) = -\mathrm{d} \int_\Sigma \mathrm{curv} \cdot \xi.$$

Here the integral on the right hand side is the function $A \mapsto \int_\Sigma \mathrm{curv}(A) \cdot \xi$, and 'd' is the exterior differential on the infinite-dimensional manifold $\mathcal{A}(\Sigma)$. The moduli space is hence recognized as a symplectic reduction

$$\mathcal{M}(\Sigma) = \mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma) = \mathrm{curv}^{-1}(0)/\mathcal{G}(\Sigma).$$

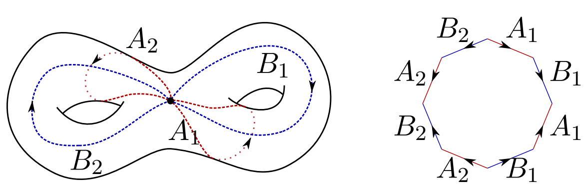
To see that $\mathcal{M}(\Sigma)$ is finite-dimensional, choose a base point x_0 on Σ, and let $\mathcal{G}(\Sigma, x_0) \subset \mathcal{G}(\Sigma)$ be the gauge transformations that are trivial at the base point. For any flat connection A on Σ, its holonomy along a based loop in Σ depends only on the homotopy class of that loop. It hence determines a group homomorphism $\kappa(A)\colon \pi_1(\Sigma; x_0) \to G$. Under the gauge action of $g \in \mathcal{G}(\Sigma)$, $\kappa(g.A) = \mathrm{Ad}_{g(x_0)}(\kappa(A))$. Conversely (using that G is simply connected), any homomorphism $\pi_1(\Sigma; x_0) \to G$ arises from a flat connection. Hence there is a canonical identification,

$$\mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma, x_0) \cong \mathrm{Hom}(\pi_1(\Sigma; x_0), G),$$

equivariant for the action of $\mathcal{G}(\Sigma)/\mathcal{G}(\Sigma, x_0) \cong G$. In particular,

$$\mathcal{M}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma; x_0), G)/G.$$

To be more explicit, we use a presentation of the fundamental group. This is done, as usual, by cutting the surface along A-cycles (winding around the handles) and B-cycles (going along the handles), as in the picture:



After cutting, the surface becomes a polygon with $4h$ sides, where h is the genus (number of handles) of the surface. Each handle gives rise to a word $A_iB_iA_i^{-1}B_i^{-1}$, and we obtain the relation $\prod_{i=1}^h A_iB_iA_i^{-1}B_i^{-1} = 1$ since the boundary of the polygon is contractible. Thus

$$\pi_1(\Sigma; x_0) = \big\langle A_1, B_1, \ldots, A_h, B_h \big|\ \prod_{i=1}^h A_iB_iA_i^{-1}B_i^{-1} \big\rangle$$

is a presentation of the fundamental group. Letting $a_i, b_i \in G$ be the holonomies of a connection along the paths A_i, B_i we obtain,

$$\mathrm{Hom}(\pi_1(\Sigma; x_0), G) = \Phi^{-1}(e)$$

where $\Phi\colon G^{2h} \to G$ is the map

$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^{h} a_i b_i a_i^{-1} b_i^{-1},$$

and finally

$$\mathcal{M}(\Sigma) = \Phi^{-1}(e)/G.$$

From this description, it is evident that $\mathcal{M}(\Sigma)$ is a compact space. If $h \geq 2$, then the subset $\Phi^{-1}(e)_{\mathrm{reg}}$ of points whose stabilizer in G equals the center $Z(G) \subset G$, is open and dense in $\Phi^{-1}(e)$. One may check that Φ has maximal rank at such points. It follows that $\Phi^{-1}(e)_{\mathrm{reg}}/G$ is a smooth symplectic manifold of dimension $(2h-2)\dim G$.

In the 1990s, the holonomy picture was used as a starting point for finite-dimensional constructions of the symplectic form on the moduli space, and an investigation of its cohomology. Important references include [31, 33, 34, 38, 40, 41, 65]. Jeffrey and Huebschmann [34, 38] developed an approach where the *logarithm* of the map Φ is viewed as a moment map, proving that $\mathcal{M}(\Sigma)$ can be written as a symplectic quotient of a finite-dimensional Hamiltonian G-space. Unfortunately, since the 'logarithm' is not globally defined, one cannot take this Hamiltonian space to be compact, and consequently many of the standard techniques of Hamiltonian geometry do not apply. One of the purposes of the theory of group-valued moment maps is to provide a more natural framework, in which the holonomy map $\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^{h} a_i b_i a_i^{-1} b_i^{-1}$ (rather than its logarithm) is directly viewed as a moment map.

3. Group-valued moment maps

Given a Lie group G, we denote by $\theta^L \in \Omega^1(G, \mathfrak{g})$ the *left-invariant Maurer-Cartan form* and by $\theta^R \in \Omega^1(G, \mathfrak{g})$ the *right-invariant Maurer-Cartan form.* In terms of a matrix representation of G, we have

$$\theta^L = g^{-1}\,\mathrm{d}g, \quad \theta^R = \mathrm{d}g g^{-1}.$$

Suppose $\mathfrak{g}$ carries an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form ('metric'), denoted by a dot '$\cdot$'. Thus $\mathrm{Ad}_g(\xi_1) \cdot \mathrm{Ad}_g(\xi_2) = \xi_1 \cdot \xi_2$ for all $\xi_1, \xi_2 \in \mathfrak{g}$. We denote by

$$\eta = \tfrac{1}{12}[\theta^L, \theta^L] \cdot \theta^L \in \Omega^3(G)$$

the *Cartan 3-form.* Since $\theta^R = \mathrm{Ad}_g\,\theta^L$ and since $\cdot$ is invariant, we may also write $\eta = \frac{1}{12}[\theta^R, \theta^R] \cdot \theta^R$. Thus η is a bi-invariant form on G, and hence it is closed: $\mathrm{d}\eta = 0$.

Definition 3.1 (Alekseev-Malkin-M [3]). A *q-Hamiltonian G-space* (M, ω, Φ) is a G-manifold M, with a G-invariant 2-form $\omega \in \Omega^2(M)$ and a G-equivariant map $\Phi \in C^\infty(M, G)$, called the *moment map*, satisfying

(i) $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi, \quad \xi \in \mathfrak{g}$
(ii) $\mathrm{d}\omega = -\Phi^*\eta,$
(iii) $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0.$

Here the G-equivariance of Φ is relative to the conjugation action on G.

Remark 3.2. In the original definition [3], an alternative version of condition (iii) was used, requiring

$$\text{(iii}')\qquad \ker(\omega_m)=\{\xi_M(m)|\ \operatorname{Ad}_{\Phi(m)}\xi=-\xi\}.$$

However, assuming conditions (i), (ii) one may show that (iii') is equivalent to (iii). This was observed by Bursztyn-Crainic [18] and Xu [67], independently.

Remark 3.3. In [3], the theory of group-valued moment maps was developed under the assumption that the metric $\cdot$ on $\mathfrak{g}$ is positive definite, which only happens if the adjoint group $G/Z(G)$ is compact. Using the more conceptual approach via Dirac geometry, initiated by [18], the main results all generalize to possibly non-compact groups (e.g. semi-simple Lie groups with $\cdot$ the Killing form on $\mathfrak{g}$), as well as to the holomorphic category. For details, see [2, 51].

Let us contrast the definition of q-Hamiltonian spaces with the usual definition of a Hamiltonian G-space. The latter is given by a G-manifold M with an invariant 2-form ω and an equivariant map $\Phi\colon M\to\mathfrak{g}^*$ satisfying the conditions,

(i) $\iota(\xi_M)\omega=-\langle \mathrm{d}\Phi,\xi\rangle$,
(ii) $\mathrm{d}\omega=0$,
(iii) $\ker(\omega)=0$.

Remark 3.4. Assuming (i),(ii), the condition $\ker(\omega)=0$ can be shown to be equivalent to a condition $\ker(\omega)\cap\ker(\mathrm{d}\Phi)=0$.

We will now discuss the main examples and basic properties of q-Hamiltonian spaces parallel to their Hamiltonian counterparts.

3.1. **Examples.**

3.1.1. *Coadjoint orbits, conjugacy classes.* The first examples of Hamiltonian G-spaces are the orbits $\mathcal{O}\subset\mathfrak{g}^*$ of the co-adjoint action

$$g.\mu=(\operatorname{Ad}_{g^{-1}})^*\mu,\quad g\in G,\ \mu\in\mathfrak{g}^*.$$

(The choice of an invariant metric on $\mathfrak{g}$ identifies the coadjoint and adjoint actions; hence we will denote the coadjoint action also by $\operatorname{Ad}_g\mu:=(\operatorname{Ad}_{g^{-1}})^*\mu$.) The moment map is the inclusion $\Phi\colon\mathcal{O}\hookrightarrow\mathfrak{g}^*$. The 2-form on the coadjoint orbit $\mathcal{O}$ is determined by the moment map condition, and is given at any point $\mu\in\mathcal{O}$ by the formula

$$\omega(\xi_{\mathcal{O}},\xi'_{\mathcal{O}})_\mu=\langle\mu,[\xi,\xi']\rangle,\quad \xi,\xi'\in\mathcal{O}.$$

Similarly, the first examples of q-Hamiltonian G-spaces are the orbits of the conjugation action on G. The moment map for a conjugacy class is the inclusion $\Phi\colon\mathcal{C}\hookrightarrow G$, and the 2-form is uniquely determined by the moment map condition:

$$\omega(\xi_{\mathcal{C}},\xi'_{\mathcal{C}})_a=\tfrac{1}{2}(\operatorname{Ad}_a-\operatorname{Ad}_{a^{-1}})\xi\cdot\xi'.$$

Since $\mathrm{d}\Phi$ is injective in this example, condition (iii) is automatic. Note that the 2-form ω may well-be degenerate: If elements of $\mathcal{C}$ square to central elements, the 2-form is even zero. Note also that conjugacy classes may be odd-dimensional (e.g. the conjugacy class $\mathcal{C}\cong S^1$ of O(2) consisting of reflections in the plane) or non-orientable (e.g. the conjugacy class

$\mathcal{C} \cong \mathbb{R}P(2)$ of SO(3) consisting of rotations by π). On the other hand, one can show that connected q-Hamiltonian G-spaces for connected, simply connected groups G are always even-dimensional and oriented (see [7, 4]).

3.1.2. *Cotangent bundles, the double.* The cotangent bundle T^*G, with the cotangent lift of the $G \times G$-action on G, $(g_1, g_2).a = g_1 a g_2^{-1}$, is an example of a Hamiltonian $G \times G$-space. Using left trivialization $T^*G \cong G \times \mathfrak{g}^*$ of the cotangent bundle, the cotangent action reads $(g_1, g_2).(a, \mu) = (g_1 a g_2^{-1}, \mathrm{Ad}_{g_2} \mu)$. The two components of the moment map are $\Phi_1(a, \mu) = \mathrm{Ad}_a(\mu),\ \Phi_2(a, \mu) = -\mu$.

Similarly, an example of a q-Hamiltonian $G \times G$-space is the *double* $D(G) \cong G \times G$, with action

$$(g_1, g_2).(a, b) = (g_1 a g_2^{-1},\ g_2 b g_1^{-1})$$

moment map components

$$\Phi_1(a, b) = ab, \quad \Phi_2(a, b) = a^{-1} b^{-1},$$

and 2-form

$$\omega = \tfrac{1}{2} a^*\theta^L \cdot b^*\theta^R + \tfrac{1}{2} a^*\theta^R \cdot b^*\theta^L$$

(here we view a, b as maps $D(G) \to G$). Replacing the variable b with $d = ba$ makes this look similar to the action on T^*G in left trivialization; for instance $\Phi_1 = \mathrm{Ad}_a(d),\ \Phi_2 = d^{-1}$.

One can also consider T^*G with the cotangent lift of the conjugation action, with corresponding moment map $(a, \mu) \mapsto \mathrm{Ad}_a \mu - \mu$. The q-Hamiltonian analogue is the double $\mathbf{D}(G) = G \times G$ [1] with the action $g.(a, b) = (\mathrm{Ad}_g(a), \mathrm{Ad}_g(b))$, with moment map the Lie group commutator

$$\Phi(a, b) = aba^{-1}b^{-1},$$

and with the 2-form

$$\omega = \tfrac{1}{2} a^*\theta^L \cdot b^*\theta^R + \tfrac{1}{2} a^*\theta^R \cdot b^*\theta^L + \tfrac{1}{2} (ab)^*\theta^L \cdot (a^{-1}b^{-1})^*\theta^R.$$

This is a special case of the *fusion* operation to be discussed below.

3.1.3. *Linear spaces, spheres.* The space $\mathbb{C}^n = \mathbb{R}^{2n}$, with its standard symplectic form, is a Hamiltonian $\mathrm{U}(n)$-space. Similarly, the even-dimensional sphere S^{2n} is a q-Hamiltonian $\mathrm{U}(n)$-space, where the action is defined by the embedding $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n) \subset \mathrm{SO}(2n+1)$. This example was found independently in [7], [36] for $n = 2$, and generalized to higher dimensions by Hurtubise-Jeffrey-Sjamaar [35]. There is a similar pair of examples, due to Eshmatov [24], of a Hamiltonian $\mathrm{Sp}(n)$-action on the quaternionic space $\mathbb{H}^n$, and a q-Hamiltonian $\mathrm{Sp}(n)$-action on quaternionic projective space $\mathbb{H}\mathrm{P}(n)$.

[1] We use the bold face notation to indicate that we consider the double as a G-space, rather than as a $G \times G$-space.

3.1.4. *Moduli spaces of surfaces with boundary.* Assume G simply connected. Let Σ be a compact, connected surface with a single boundary component. Fix a base point $x_0 \in \partial(\Sigma)$ on the boundary, and let

$$\mathcal{M}(\Sigma) = \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma; x_0)$$

be the moduli space of flat connections on Σ, up to gauge transformations that are trivial at x_1. The space $\mathcal{M}(\Sigma)$ carries a residual action of $\mathcal{G}(\Sigma)/\mathcal{G}(\Sigma; x_1) \cong G$, and the map taking the holonomy around $\partial\Sigma$ descends to a G-equivariant map $\Phi\colon \mathcal{M}(\Sigma) \to G$. A generalization of the Atiyah-Bott gauge theory construction discussed above gives 2-form ω on $\mathcal{M}(\Sigma)$, making $(\mathcal{M}(\Sigma), \omega, \Phi)$ into a q-Hamiltonian G-space. More generally, if Σ has r boundary components, fix one base point on each boundary component. Then the moduli space $\mathcal{M}(\Sigma)$ of flat connections modulo based gauge transformations is a q-Hamiltonian G^r-space. It turns out that the space associated to a cylinder (2-punctured sphere) is isomorphic to $D(G)$, while the space associated to a 1-punctured torus is isomorphic to $\mathbb{D}(G)$.

3.2. **Basic constructions: products.** Given two Hamiltonian G-spaces, their direct product, with the diagonal G-action and with the sum of moment maps and 2-forms, is again a Hamiltonian G-space. For q-Hamiltonian spaces, the product operation uses the product of the moment maps, but it is necessary to modify the sum of the 2-forms.

Proposition 3.5. [3] *Suppose* $(M_i, \omega_i, \Phi_i),\ i = 1, 2$ *are two q-Hamiltonian* G*-spaces. Then their* fusion product

$$(M_1 \times M_2, \omega_1 + \omega_2 + \tfrac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R, \Phi_1\Phi_2),$$

is again a q-Hamiltonian G*-space.*

Here the modification of the 2-form is required due to the following property of the 3-form η under group multiplication $\mathrm{Mult}\colon G \times G \to G$,

$$\mathrm{Mult}^*\eta = \mathrm{pr}_1^*\,\eta + \mathrm{pr}_2^*\,\eta - \tfrac{1}{2}\mathrm{d}\,\mathrm{pr}_1^*\,\theta^L \cdot \mathrm{pr}_2^*\,\theta^R,$$

where $\mathrm{pr}_1, \mathrm{pr}_2\colon G \times g \to G$ are the two projections. More generally, if $(M, \omega, (\Phi_1, \Phi_2))$ is a q-Hamiltonian $G \times G$-space, then we obtain a q-Hamiltonian G-space $(M_{fus}, \omega_{fus}, \Phi_{fus})$, where M_{fus} is M with the diagonal G-action, $\Phi_{fus} = \Phi_1\Phi_2$ and $\omega_{fus} = \omega + \frac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R$.

Remark 3.6. The fusion property finds a natural proof within the framework of *Dirac structures* [2, 18, 51] . Here, the axioms of a q-Hamiltonian are absorbed into a morphism of Manin pairs (strong Dirac morphism) from M into G, equipped with the so-called *Cartan-Dirac structure*. Since group multiplication in G is again a morphism of Manin pairs, the fusion operation becomes simply a composition of morphisms.

As an application, the space G^{2h}, with G acting diagonally by conjugation and with moment map $\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^h a_i b_i a_i^{-1} b_i^{-1}$ carries the structure of a q-Hamiltonian G-space as an h-fold fusion of the double $\mathbf{D}(G)$. The following nice way of looking at the 2-form was described by Pavol Ševera in [58]. For any manifold X, the space $C^\infty(X, G) \times \Omega^2(X)$ has a group structure

$$(q_1, \omega_1)(q_1, \omega_2) = (q_1 q_2, \omega_1 + \omega_2 + \tfrac{1}{2} q_1^*\theta^L \cdot q_2^*\theta^R).$$

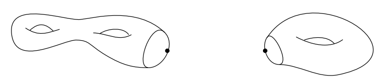
Take $X = G^{2h}$, with elements $x = (a_1, b_1, \ldots, a_h, b_h)$, and let $q_1, \ldots, q_{4h} \colon G^{2g} \to G$ be the maps

$$x \mapsto \ a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_h^{-1}, b_h^{-1}.$$

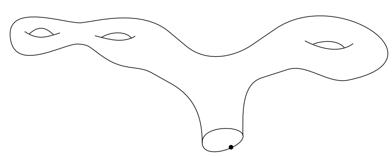
Then $(q_1, 0) \cdots (q_{4h}, 0) = (\Phi, \omega)$ defines the q-Hamiltonian 2-form $\omega \in \Omega^2(G^{2h})$ and the moment map Φ.

The name '*fusion*' corresponds to the fusion of surfaces, as in the following example. See [54] for a similar discussion for Hamiltonian loop group actions.

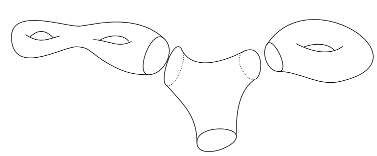
Example 3.7 (Fusion of moduli spaces). Suppose G is simply connected. For any compact, oriented surface Σ with boundary component, with a marked point on each boundary component, we denote by $\mathcal{M}(\Sigma)$ the moduli space of flat connections on Σ, up to gauge transformations that are trivial at the marked points. (See Section 3.1.4.) Suppose Σ has (at least) two boundary components. For instance, Σ could be a disjoint union of two surfaces Σ_1 and Σ_2 with one boundary component, as in the following picture.



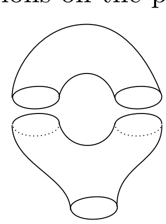
Then the fusion $\mathcal{M}(\Sigma)_{fus}$ is naturally identified with the moduli space $\mathcal{M}(\Sigma_{fus})$ of the surface Σ_{fus},



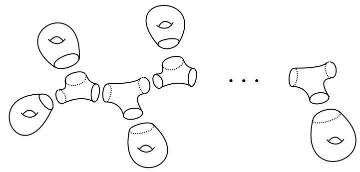
which is obtained by joining the two boundary components of Σ by a pair of pants: the two pant legs are attached to two boundaries.



For example, the moduli space of flat connections on the cylinder can be identified with the double $D(G) \cong G \times G$, a q-Hamiltonian $G \times G$ space. Fusing $D(G)$ with itself, we obtain the moduli space $\mathbf{D}(G)$ of flat connections on the punctured torus.



One can construct the punctured surface of genus h by joining h copies of the punctured torus together with $h - 1$ pairs of pants.



Thus, the moduli space of flat $\mathfrak{g}$-connections on the punctured surface of genus h is

$$\mathcal{M}(\Sigma_h) := \underbrace{\mathbf{D}(G) \times \cdots \times \mathbf{D}(G)}_{h} \cong G^{2h}.$$

3.3. **Reduction.** Symplectic reduction of q-Hamiltonian G-spaces works just the same as for ordinary Hamiltonian spaces. Suppose (M, ω, Φ) is a q-Hamiltonian G-space, such that the group unit e is a regular value of the moment map. Then it is automatic that the G-action on $\Phi^{-1}(e)$ has discrete stabilizers. If G is compact (or more generally if the action is proper), it follows that the quotient

$$M/\!\!/G = \Phi^{-1}(e)/G$$

is an orbifold. Furthermore, the pull-back of ω to $\Phi^{-1}(e)$ is *basic*, and the resulting 2-form on $M/\!\!/G$ is *symplectic* – even though ω itself was neither closed nor non-degenerate. This is possible because ω_m is non-degenerate for all $m \in \Phi^{-1}(e)$, and since its pull-back to $\Phi^{-1}(e)$ is closed. If e is a singular value of Φ the space $M/\!\!/G$ is a singular symplectic space in the sense of Sjamaar-Lerman [60].

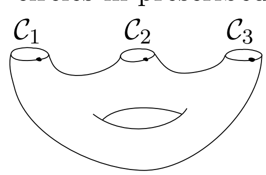
As an application of reduction, we obtain a symplectic structure on the moduli space of flat G-bundles, viewed as a symplectic quotient,

$$\mathcal{M}(\Sigma_h) = G^{2h}/\!\!/G.$$

Note that e is never a regular value of $\Phi\colon G^{2h} \to G$, since $\Phi^{-1}(e)$ contains the point $(e, \ldots, e)$ whose stabilizer is the entire group G. More generally, if $\mathcal{C}_1, \ldots, \mathcal{C}_r \subset G$ are conjugacy classes,

$$\mathcal{M}(\Sigma_h^r, \mathcal{C}_1, \ldots, \mathcal{C}_r) = (G^{2h} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r)/\!\!/G \tag{1}$$

is the moduli space of flat G bundles over a surface with r boundary components, with holonomies around boundary circles in prescribed conjugacy classes $\mathcal{C}_j$.



One of the main results in [3] asserts that, for G compact and simply connected, the symplectic structure obtained by q-Hamiltonian reduction coincides with that coming from the Atiyah-Bott construction.

3.4. **Convexity theorem.** We next describe some convexity results for q-Hamiltonian spaces. Here we assume that the group G is compact and simply connected.

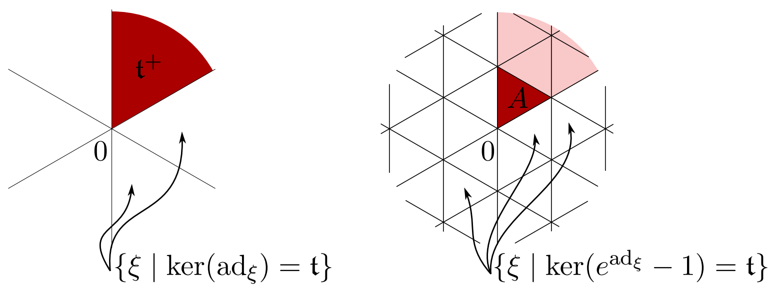
3.4.1. *Weyl chamber, alcove.* We fix a maximal torus T in G, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. An *open Weyl chamber* in $\mathfrak{t}$ is a connected component of the set

$$\{\xi \in \mathfrak{t}|\ \ker(\mathrm{ad}_\xi) = \mathfrak{t}\} = \{\xi \in \mathfrak{t}|\ G_\xi = T\},$$

while an *open alcove* is a connected component of the set

$$\{\xi \in \mathfrak{t}|\ \ker(e^{\mathrm{ad}_\xi} - 1) = \mathfrak{t}\} = \{\xi \in \mathfrak{t}|\ G_{\exp\xi} = T\},$$

here G_ζ resp. G_g are the stabilizers of $\zeta \in \mathfrak{g}$ resp. $g \in G$ under the adjoint action. Pick an open Weyl chamber, let $\mathfrak{t}_+$ be its closure, and let $\mathfrak{A}$ be the unique closed alcove with $0 \in \mathfrak{A} \subset \mathfrak{t}_+$.



Let $\mathfrak{t}^*_+$ be the image of $\mathfrak{t}_+$ under the isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ defined by the invariant metric on $\mathfrak{g}$. (This does not depend on the choice of metric.)

The fundamental Weyl chamber labels the set of coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, in the sense that every such orbit is of the form $G.\mu$ for a unique element $\mu \in \mathfrak{t}^*_+$. (See [16, Ch. IX, §2, Proposition 7].) Similarly, the fundamental Weyl alcove labels the conjugacy classes $\mathcal{C} \subset G$, in the sense that every conjugacy class is of the form $G.\exp\xi$ for a unique $\xi \in \mathfrak{A}$. (See [16, Ch. IX, §5, Corollary 2].)

3.4.2. *Convexity theorem.* The following result is known as the Hamiltonian *convexity theorem.*

Theorem 3.8 (Atiyah [9], Guillemin-Sternberg [28, 30], Kirwan [45])**.** *Let (M, ω, Φ) be a compact connected Hamiltonian G-space. Then*

(a) *the fibers of Φ are connected,*
(b) *the set*

$$\Delta(M) = \{\mu \in \mathfrak{t}^*_+|\ \mu \in \Phi(M)\}$$

is a convex polytope.

Similarly, the q-Hamiltonian convexity theorem states:

Theorem 3.9 (M-Woodward [54])**.** *Let (M, ω, Φ) be a compact connected q-Hamiltonian G-space. Then*

(a) *the fibers of Φ are connected,*
(b) *the set*

$$\Delta(M) = \{\xi \in \mathfrak{A}|\ \exp\xi \in \Phi(M)\}$$

is a convex polytope.

The result was phrased in [54] in terms of Hamiltonian loop group actions; the formulation in terms of q-Hamiltonian spaces follows using the equivalence theorem in [3].

3.4.3. *Eigenvalue problems.* The Hamiltonian convexity theorem has nice applications to eigenvalue problems. Let $\mathbb{R}^n_+$ be the set of $\lambda \in \mathbb{R}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. For a complex Hermitian $n \times n$ matrix A, let $\lambda(A) \in \mathbb{R}^n_+$ be its ordered tuple of eigenvalues. One then has:

Corollary 3.10 (Horn polytope). *Let $\mu, \mu' \in \mathbb{R}^n_+$ be given. Then the set $\gamma \in \mathbb{R}^n_+$ for which there exist Hermitian matrices A, A' with*

$$\lambda(A) = \mu,\ \ \lambda(A') = \mu', \quad \lambda(A + A') = \gamma,$$

is a convex polytope.

In short, the possible eigenvalues of a sum of Hermitian matrices with prescribed eigenvalues form a convex polytope. Corollary 3.10 follows by identifying Hermitian matrices with $\mathfrak{u}(n)^*$, the cone $\mathbb{R}^n_+$ with the positive Weyl chamber $\mathfrak{t}^*_+$, and matrices with prescribed eigenvalues with coadjoint orbits $\mathcal{O} \subset \mathfrak{u}(n)^*$. The Corollary is then an immediate consequence of the Hamiltonian convexity theorem applied to a product of coadjoint orbits $\mathcal{O} \times \mathcal{O}'$. A description of the faces of this polytope in terms of explicit eigenvalue inequalities was known as the *Horn conjecture*, this was solved by Klyachko [46] in 1994. For more general compact groups, the inequalities for the moment polytopes of products of coadjoint orbits in general were determined by Berenstein-Sjamaar [13].

The q-Hamiltonian convexity theorem has applications to *multiplicative* eigenvalue problems. The eigenvalues of any special unitary matrix $A \in \mathrm{SU}(n)$ may be written in the form $e^{2\pi\sqrt{-1}\lambda_1(A)}, \ldots, e^{2\pi\sqrt{-1}\lambda_n(A)}$, for a unique $\lambda(A) \in \mathbb{R}^n$ with

$$\sum_{i=1}^n \lambda_i(A) = 0, \quad \lambda_1(A) \geq \cdots \geq \lambda_n(A) \geq \lambda_1(A) - 1.$$

Corollary 3.11 (M-Woodward). *Given $\mu, \mu' \in \mathbb{R}^n$, the set*

$$\{\gamma \in \mathbb{R}^n |\ \exists A, A' \in \mathrm{SU}(n) \colon \lambda(A) = \mu,\ \ \lambda(A') = \mu', \quad \lambda(AA') = \gamma\},$$

is a convex polytope.

In short, the possible eigenvalues of a product of special unitary matrices with prescribed eigenvalues forms a convex polytope. The corollary is obtained by applying Theorem 3.9 to a fusion product of two conjugacy classes, $\mathcal{C} \times \mathcal{C}'$. The problem of determining the faces of this polytope was solved by Agnihotri-Woodward [1]. The moment polytope for products of conjugacy classes in a general compact simply connected Lie group was determined by Teleman-Woodward [61].

3.4.4. *Connectivity of the fibers.* Let us also note the following consequences of the first part of Theorem 3.9, concerning connectivity of the fibers of the moment map.

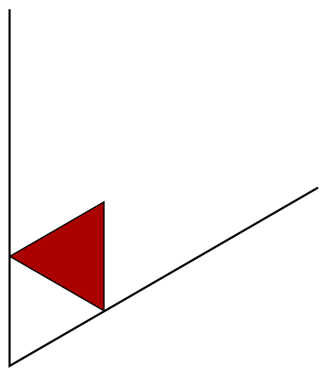
Corollary 3.12. *Let G be a compact, simply connected Lie group, and (M, ω, Φ) a compact connected q-Hamiltonian G-space. Then the symplectic quotient $M/\!/G$ is connected.*

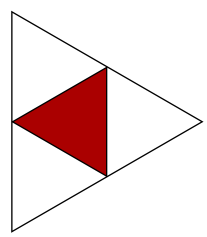
In particular, the moduli spaces (1) are connected.

Corollary 3.13. *For any compact, simply connected Lie group G, the fibers of the commutator map $G \times G \to G,\ (a,b) \mapsto aba^{-1}b^{-1}$ are connected.*

This follows by applying Theorem 3.9 to the double $\mathbf{D}(G)$. Note that this result is not easy to prove 'by hand'.

3.4.5. *Multiplicity-free spaces.* An interesting class of Hamiltonian G-spaces are the *multiplicity-free spaces*. These are spaces such that the map $M/G \to \Delta(M)$ is a homeomorphism; equivalently, the symplectic quotients are 0-dimensional. In case G is a torus, Delzant [20] proved that multiplicity-free spaces are determined by their moment polytopes. This result was extended by Woodward [66] to 'reflective' multiplicity-free spaces for non-abelian groups. The classification of multiplicity-free spaces in general is more involved, and was completed only recently by F. Knop [47] following Losev's proof of the 'Knop conjecture'. The definition of multiplicity-free spaces carries over verbatim to the q-Hamiltonian setting. For instance, the q-Hamiltonian $\mathrm{SU}(n)$-space S^{2n} and the q-Hamiltonian $\mathrm{Sp}(n)$-space $\mathbb{H}\mathrm{P}(n)$ are multiplicity free. The following picture shows the moment polytopes for a reflective multiplicity free Hamiltonian SU(3)-space (left) and a reflective multiplicity free q-Hamiltonian SU(3)-space (right). These examples are due to Chris Woodward.





3.5. **Volume forms.** The *Liouville form* of a symplectic manifold (M, ω) is the volume form defined as $\Gamma = \frac{1}{n!}\omega^n$, or equivalently as the top degree part of the exponential of ω,

$$\Gamma = (\exp \omega)_{[top]}.$$

In local Darboux coordinates $q_1, p_1, \ldots, q_n, p_n$, one has $\omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$, and the Liouville form is $\mathrm{d}q_1 \wedge \mathrm{d}p_1 \cdots \wedge \mathrm{d}q_n \wedge \mathrm{d}p_n$. Given a compact Hamiltonian G-space (M, ω, Φ), one defines the *Duistermaat-Heckman measure* [23] to be the push-forward on $\mathfrak{g}^*$ of the associated measure, $\mathfrak{m} = \Phi_*|\Gamma|$. It has interesting properties, and may be calculated using localization techniques.

For a q-Hamiltonian G-space (M, ω, Φ), we saw that the 2-form ω may be degenerate or even zero. Assuming that G is compact and simply connected, it turns out that there is nevertheless a distinguished volume form on M. In particular, M carries a canonical orientation. The construction involves a certain differential form on G.

Proposition 3.14. *For any compact, simply connected Lie group G, the function $g \mapsto \det(\frac{\mathrm{Ad}_g + 1}{2})$ admits a smooth global square root, equal to 1 at $g = e$. Furthermore, there is* *a well-defined smooth differential form $\psi \in \Omega(G)$, given on the set where* $\det(\mathrm{Ad}_g + 1) \neq 0$ *by*

$$\psi = \det{}^{1/2}\big(\tfrac{1+\mathrm{Ad}_g}{2}\big)\ \exp\big(\tfrac{1}{4}\tfrac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\theta^L \cdot \theta^L\big).$$

Note that the set where $\det(\mathrm{Ad}_g + 1) \neq 0$ is open and dense in G. Note that the 2-form inside the exponential becomes singular on the subset where $\det(\mathrm{Ad}_g + 1) = 0$, but the scalar factor in front of the exponential has zeroes there. The Proposition says that the zeroes compensate the singularities, so that the form extends smoothly across the set $\det(\mathrm{Ad}_g + 1) = 0$.

Theorem 3.15. [7] *Suppose G is compact and simply connected. For any q-Hamiltonian G-space (M, ω, Φ), the top degree part of the form* $\exp(\omega)\Phi^*\psi$ *is a G-invariant volume form,*

$$\Gamma = (e^\omega \Phi^* \psi)_{[top]}.$$

In particular, M is even-dimensional and carries a canonical orientation. A conceptual explanation of the volume form is given in [2, 51], where the differential form ψ is identified as a *pure spinor*, and the Theorem is interpreted as the non-degeneracy of a pairing between two pure spinors. As shown in [7], the push-forward $\mathfrak{m} = \Phi_*|\Gamma| \in \mathcal{E}'(G)$ plays the role of a *Duistermaat-Heckman measure*, with properties similar to the Hamiltonian Duistermaat-Heckman measure. In particular, it encodes volumes of symplectic quotients, and for G compact and simply connected it can be computed by localization [5].

3.6. **Kirwan surjectivity.** There are many other aspects of the Hamiltonian theory that carry over the q-Hamiltonian setting, with suitable modifications. One result of central importance for Hamiltonian spaces is the *Kirwan surjectivity theorem.* We assume that G is compact. For any G-manifold M, let $H^\bullet_G(M)$ be its equivariant cohomology ring with coefficients in $\mathbb{R}$. It may be realized as the cohomology of the *Cartan complex* $(\Omega^\bullet_G(M), \mathrm{d}_G)$ where

$$\Omega^k_G(M) = \bigoplus_{2i+j=k} (S^i\mathfrak{g}^* \otimes \Omega^j(M))^G.$$

Viewing elements of $\Omega_G(M)$ as G-equivariant polynomial maps $\beta\colon \mathfrak{g} \to \Omega(M)$, the differential is given by

$$(\mathrm{d}_G\beta)(\xi) = \mathrm{d}\beta(\xi) - \iota(\xi_M)\beta(\xi), \quad \xi \in \mathfrak{g}.$$

Example 3.16.
(a) If (M, ω, Φ) is a Hamiltonian G-space, then $\omega^G = \omega + \Phi \in \Omega^2_G(M)$ is an example of a closed equivariant 2-form.
(b) If G carries an invariant metric $\cdot$, then

$$\eta^G(\xi) = \eta + \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi$$

defines a closed equivariant 3-form $\eta^G \in \Omega^3_G(M)$. Conditions (i),(ii) in the definition of a q-Hamiltonian G-space may be combined into a single condition $\mathrm{d}_G\omega = -\Phi^*\eta^G$.

Theorem 3.17 (Kirwan [44])**.** *Let (M, ω, Φ) be a Hamiltonian G-space, with 0 a regular value of the moment map Φ. Then the pull-back map*

$$H_G(M) \to H_G(\Phi^{-1}(0)) \cong H(M/\!\!/G)$$

is a surjective ring homomorphism.

Thus, all cohomology classes on the symplectic quotient are obtained from equivariant cohomology classes on the unreduced space. For instance, the class $[\omega_G]$ descends to the class of the symplectic form on $M/\!\!/G$. This result is the starting point for the calculation of intersection pairings on $M/\!\!/G$ using localization on M, see e.g. [39], [62].

For a q-Hamiltonian G-space, the map $H_G(M) \to H_G(\Phi^{-1}(e)) = H(M/\!\!/G)$ need not be surjective, in general. There are in fact examples where $H^2_G(M) = 0$, so that the class of the symplectic form on $M/\!\!/G$ need not lie in the image of this map. It turns out that the correct version of the surjectivity theorem involves the topology of the group G. We assume that G is compact and simply connected. As is well-known, the inclusion of bi-invariant differential forms $(\wedge\mathfrak{g}^*)^G \cong \Omega(G)^{G\times G} \hookrightarrow \Omega(G)$ induces an isomorphism in cohomology. Since the de Rham differential restricts to zero on bi-invariant differential forms, it follows that

$$H(G) = (\wedge\mathfrak{g}^*)^G.$$

On the other hand, it is known that the invariants $(\wedge\mathfrak{g}^*)^G$ are an exterior algebra over a graded subspace $P^\bullet \subset (\wedge^\bullet\mathfrak{g}^*)^G$ of *primitive elements.*

$$(\wedge\mathfrak{g}^*)^G = \wedge P.$$

Here $\dim P = l$ equals the rank of G, and all homogeneous elements in P are of odd degree. Let $\eta_1, \ldots, \eta_l \in \Omega^{2d_i-1}(G)$ be a homogeneous basis of P, where η_1 is the Cartan 3-form. For instance, if $G = \mathrm{SU}(n+1)$, the generators of $(\wedge\mathfrak{g}^*)^G$ are of degree $3, 5, 7, \ldots, 2n+1$. It turns out that each of the η_i admits an extension $\eta_i^G \in \Omega_G^{2d_i-1}(G)$ to an equivariantly closed form. These may be constructed using an equivariant version of the Bott-Shulman complex [37] (see also [50]). In particular, $\eta_1^G = \eta^G$.

Suppose now that (M, ω, Φ) is a q-Hamiltonian G-space. Define a new complex,

$$\widetilde{\Omega}_G(M) = \Omega_G(M)[u_1, \ldots, u_l],$$

where $[u_1, \ldots, u_l]$ denotes the graded ring of polynomials in given variables u_i of degree $2d_i - 2$, and with the differential

$$\widetilde{\mathrm{d}}_G = \mathrm{d}_G + \sum_{i=1}^{l} \Phi^*\eta_i^G \tfrac{\partial}{\partial u_i}.$$

(Here $\Phi^*\eta_i^G$ acts by exterior multiplication, raising the degree by $2d_i - 1$, while the differentiation $\frac{\partial}{\partial u_i}$ lowers the degree by $2d_i - 2$. We hence see that $\widetilde{\mathrm{d}}_G$ raises the degree by 1, as required.) The cohomology of this complex is denoted $\widetilde{H}^\bullet_G(M)$. Let

$$\widetilde{\Omega}^\bullet_G(M) \to \widetilde{\Omega}^\bullet_G(\Phi^{-1}(e)) \to \Omega^\bullet_G(\Phi^{-1}(e)) \tag{2}$$

be the cochain map, given by pull-back followed by the augmentation map for $[u_1, \ldots, u_l]$ (setting these variables equal to zero). For instance, the element

$$\omega + u_1$$

is a cocycle (since $\mathrm{d}_G\omega = -\Phi^*\eta^G$), and its image under the map (2) is simply the pull-back of ω to $\Phi^{-1}(e)$ (a closed, basic form).

Theorem 3.18 (Kirwan surjectivity for q-Hamiltonian G-spaces)**.** *Suppose* (M,ω,Φ) *is a q-Hamiltonian G-space, where G is compact and simply connected, and suppose e is a regular value of* Φ. *Then the map*

$$\widetilde{H}^\bullet_G(M) \to H^\bullet_G(\Phi^{-1}(e)) = H^\bullet(M/\!\!/G)$$

is a surjective ring homomorphism.

The surjectivity result was originally proved by Bott, Tolman and Weitsman [15] in terms of Hamiltonian loop group actions. In unpublished work with A. Alekseev, we obtained the reformulation above, using a 'small model' for the equivariant cohomology of the loop group space. As an application, one obtains generators for the cohomology rings of moduli spaces, see e.g. [50].

4. Quantization of Hamiltonian G-spaces

Our aim in these lectures is to explain the quantization of q-Hamiltonian G-spaces. In this Section, we set the stage by reviewing aspects of the quantization of ordinary Hamiltonian G-spaces. The term 'quantization' will be used in a wide sense. Ideally, the quantization of a symplectic manifold should be Hilbert space, and a Hamiltonian G-action (thought of as classical symmetries) should be quantized to define a representation of G by unitary operators on the Hilbert space (thought of as quantum symmetries). The method of geometric quantization produces such G-representations, but requires further data and additional assumptions. Rather than dealing with concrete Hilbert spaces, we will be content with *isomorphism classes* of G-representations. That is, we will take the quantization of a Hamiltonian G-space to be a certain element of the representation ring of G.

4.1. **Background in representation theory.** In this Section, we take G to be compact and connected. For any G-representation $\pi\colon G \to \mathrm{Aut}(V)$, let $\chi_V \in C^\infty(G)$ be its character, $\chi_V(g) = \mathrm{tr}(\pi(g))$. Characters have the properties

$$\chi_{V\oplus W} = \chi_V + \chi_W, \quad \chi_{V\otimes W} = \chi_V \chi_W, \quad \chi_{V^*} = \chi_V^*,$$

hence they form a subring $R(G) \subset C^\infty(G)$ of the ring of complex-valued functions, invariant under the involution $*$. As an additive group, $R(G)$ is the $\mathbb{Z}$-module spanned by the characters of irreducible representations, also called the *irreducible characters.*

Fix a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$, and let $P \subset \mathfrak{t}^*$ be the *(real) weight lattice*. Thus $\mu \in \mathfrak{t}^*$ lies in P if and only if the Lie algebra homomorphism

$$\mathfrak{t} \to \mathfrak{u}(1),\ \xi \mapsto 2\pi\sqrt{-1}\langle \mu, \xi\rangle$$

exponentiates to a group homomorphism $e_\mu\colon T \to \mathrm{U}(1)$. For any G-representation $\pi\colon G \to \mathrm{Aut}(V)$, we define the *weight spaces* $V_\mu = \{v \in V |\ \forall t \in T\colon \pi(t)v = e_\mu(t)v\}$, $\ \mu \in P$, and the set of weights

$$P(V) = \{\mu \in P |\ V_\mu \neq 0\}.$$

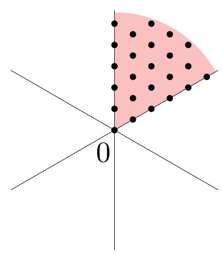
Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a choice of fundamental Weyl chamber. It is known that if V is irreducible, then there is a unique weight $\mu \in P(V)$ such that $\mu + \epsilon$ has maximal length, for any $\epsilon \in \mathrm{int}(\mathfrak{t}^*_+)$. This element $\mu \in P(V) \cap \mathfrak{t}^*_+$ is called the *highest weight* of V. By *H. Weyl's* *theorem*, this sets up a 1-1 correspondence between the set of irreducible representations and the set

$$P_+ = P \cap \mathfrak{t}^*_+$$

of *dominant weights* of G. Thus, as a $\mathbb{Z}$-module we have

$$R(G) = \mathbb{Z}[P_+],$$

with basis the irreducible characters χ_μ indexed by dominant weights $\mu \in P_+$. In the figure below, the shaded area is the fundamental Weyl chamber for the group SU(3), while the dominant weights are indicated as dots.



4.2. **Quantization of Hamiltonian G-spaces.** Suppose now that (M, ω, Φ) is a Hamiltonian G-space, with moment map $\Phi\colon M \to \mathfrak{g}^*$.

Definition 4.1. A pre-quantum line bundle $L \to M$ is a G-equivariant Hermitian line bundle with connection ∇, such that

(a) $\mathrm{curv}(\nabla) = \omega$,

(b) The $\mathfrak{g}$-action on L is given by Kostant's formula

$$\xi_L = \mathrm{Lift}_\nabla(\xi_M) + \langle \Phi, \xi \rangle \partial_\theta$$

where $\partial_\theta \in \mathfrak{X}(L)$ generates the S^1-action on L.

Remarks 4.2. (a) The existence of a pre-quantum line bundle is equivalent to the integrality of the 2-form ω.

(b) If G is simply connected, the existence of the pre-quantum lift of the G-action from M to L is automatic. Indeed, the formula for ξ_L defines a Lie algebra action of $\mathfrak{g}$ on L by infinitesimal Hermitian automorphisms, and this Lie algebra action integrates to a Lie group action.

(c) If a G-equivariant pre-quantum line bundle exists, then the choice of L is unique up to a *flat* G-equivariant line bundle.

Given an equivariant pre-quantization, we obtain an element $\mathcal{Q}(M)$ of the representation ring, as follows. Let $J\colon TM \to TM$ be a G-invariant compatible almost complex structure, i.e. $g(v, w) = \omega(Jv, w)$ is a Riemannian metric. (In other words, every tangent space admits an isomorphism $T_mM \to \mathbb{C}^n = \mathbb{R}^{2n}$ taking ω_m to the standard symplectic structure $\sum_{i=1}^n e_{2i-1} \wedge e_{2i}$ and J_m to the standard complex structure $e_{2i-1} \mapsto e_{2i},\ \ e_{2i} \mapsto -e_{2i-1}$.) The space of G-invariant compatible almost complex structures is well-known to be contractible; hence the particular choice of J is unimportant for what follows. Let $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ be the decomposition into $\pm i$ eigenbundles of J. Then $\wedge T^{0,1}M$ is a spinor module over the Clifford bundle $\mathbb{C}\mathrm{l}(TM)$, where the Clifford action of $T^{0,1}M$ is by exterior multiplication and that of $T^{1,0}M$ is by contraction. (See Section 7 below.) Tensoring with L one obtains a new spinor module,

$$\mathsf{S} = \wedge T^{0,1}M \otimes L$$

Let $\partial\!\!\!/ \colon \Gamma(\mathsf{S}) \to \Gamma(\mathsf{S})$ be the associated Dirac operator, given by the covariant derivative $\nabla\colon \Gamma(\mathsf{S}) \to \Gamma(T^*M \otimes \mathsf{S})$ followed by the Clifford action of $T^*M \cong TM \subset \mathbb{C}\mathrm{l}(TM)$ on S. Then $\partial\!\!\!/$ is a G-equivariant elliptic operator, and hence it has a G-equivariant index. Let $\mathsf{S}^+,\ \mathsf{S}^-$ be the even, odd part of the spinor bundle.

Definition 4.3. The *quantization* $\mathcal{Q}(M) \in R(G)$ of the pre-quantized Hamiltonian G-space (M, ω, Φ) is the G-index

$$\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/) = \chi_{\ker(\partial\!\!\!/|_{\mathsf{S}^+})} - \chi_{\ker(\partial\!\!\!/|_{\mathsf{S}^-})}.$$

For any given L, the construction of $\partial\!\!\!/$ involves a few choices such as the choice of J and of connections; however, the stability property of indices guarantees that $\mathcal{Q}(M)$ is independent of those choice. (In fact, it turns out that for G connected, even the choice of L does not affect $\mathcal{Q}(M)$. This is immediate from the equivariant index formula of Berline and Vergne [14], cf. [49].) The basic properties of the quantization are as follows:

(a) $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
(b) $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
(c) $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
(d) *Borel-Weil-Bott* (weak version): $G.\mu,\ \mu \in \mathfrak{t}^*_+$ is pre-quantized if and only if $\mu \in P_+$. In this case,

$$\mathcal{Q}(G.\mu) = \chi_\mu.$$

Property (d) is a weak version of the Borel-Weil-Bott theorem: the strong version uses *Kähler quantization*, and realizes the irreducible representation corresponding to μ as a space of holomorphic sections of the pre-quantum line bundle.

Let $R(G) \to \mathbb{Z},\ \chi \mapsto \chi^G$ be the group homomorphism defined on basis elements by $\chi^G_\mu = \delta_{\mu,0}$. That is, χ^G is the coefficient of the basis element χ_0 in χ. The map $\chi \mapsto \chi^G$ may be regarded as the 'quantum counterpart' to symplectic reduction (taking the coefficient of $\mu = 0$ corresponds to setting the moment map value equal to 0). The following fact was conjectured by Guillemin-Sternberg in the 1980s. (In [29], Guillemin and Sternberg gave a full proof of a similar statement for Kähler quantization.)

Theorem 4.4 (Quantization commutes with reduction)**.**

$$\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G).$$

Remark 4.5. The right hand side of this result requires some explanation. If 0 is a regular value of the moment map, and G acts freely on the zero level set $\Phi^{-1}(0)$, then $M/\!\!/G$ is a symplectic manifold with pre-quantum line bundle $L/\!\!/G = L|_{\Phi^{-1}(0)}/G$. In this case, the right hand side is defined as the (non-equivariant) index of the corresponding Spin_c-Dirac operator. If the action on the zero level set is only locally free, then $L/\!\!/G$ becomes an orbifold line bundle over the orbifold $M/\!\!/G$, and the index has to be interpreted accordingly (using Kawasaki's index theorem for orbifolds). In the most general case, one can define the right hand side by a partial desingularization of $M/\!\!/G$, reducing to the orbifold case. In this generality, the result was proved in [53].

Example 4.6. Let $N_{\mu_1\mu_2\mu_3}$ for $\mu_1, \mu_2, \mu_3 \in P_+$ be the tensor coefficients, defined by

$$\chi_{\mu_1}\chi_{\mu_2} = \sum_{\mu_3 \in P_+} N_{\mu_1\mu_2\mu_3}\chi^*_{\mu_3}.$$

Equivalently, $N_{\mu_1\mu_2\mu_3} = (\chi_{\mu_1}\chi_{\mu_2}\chi_{\mu_3})^G$. Let $\mathcal{O}_i$ be the coadjoint orbits of $\mu_i \in P_+$. Then

$$N_{\mu_1\mu_2\mu_3} = \mathcal{Q}(\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 /\!\!/ G),$$

realizing the tensor coefficients as an index.

Given a pre-quantized Hamiltonian G-space (M, ω, Φ), let $N(\mu) \in \mathbb{Z}$ be the multiplicity of χ_μ in the quantization $\mathcal{Q}(M)$,

$$\mathcal{Q}(M) = \sum_{\mu \in P_+} N(\mu)\chi_\mu.$$

Thus $N(0) = \mathcal{Q}(M)^G$. For any $\mu \in \mathfrak{t}^*_+$, let $M/\!\!/_\mu G = \Phi^{-1}(\mu)/G_\mu$ be the symplectic quotient at level $\mu \in \mathfrak{g}^*$. The *shifting trick* expresses $M/\!\!/_\mu G$ as a reduction at 0:

$$M/\!\!/_\mu G = (M \times (G.\mu)^*)/\!\!/ G;$$

here $(G.\mu)^*$ denotes the coadjoint orbit $G.\mu$ with minus the standard symplectic structure and minus the inclusion as a moment map. Suppose $\mu \in P_+ \subset \mathfrak{g}^*$. Since

$$\mathcal{Q}((G.\mu)^*) = \mathcal{Q}(G.\mu)^* = \chi^*_\mu,$$

Theorem 4.4 shows that the multiplicity of 0 in $\mathcal{Q}(M \times (G.\mu)^*)$ equals the multiplicity $N(\mu)$ of μ in $\mathcal{Q}(M)$. Thus

$$N(\mu) = \mathcal{Q}(M/\!\!/_\mu G).$$

4.3. **Localization.** In most cases, the 'quantization commutes with reduction' theorem is not very practical for the calculation of weight multiplicities in $\mathcal{Q}(M)$. Instead, the result is often used in the opposite direction: One obtains the indices of symplectic quotients $\mathcal{Q}(M/\!\!/_\mu G)$ from the knowledge of $\mathcal{Q}(M)$. The main technique for the computation of $\mathcal{Q}(M)$ is *localization.*

The Atiyah-Segal-Singer equivariant index theorem for elliptic operators, specialized to the case of Spin_c-Dirac operators, gives the formula

$$\mathcal{Q}(M)(g) = \sum_{F \subset M^g} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}|_F, g)^{1/2}}{D_\mathbb{R}(\nu_F, g)}$$

where the sum is over fixed point manifolds $F \subset M^g$ for the action of g. Here $\mathcal{L}$ is the 'Spin_c-line bundle' $\mathcal{L} = L^2 \otimes K^{-1}$, with K the canonical bundle, and ν_F is the normal bundle to F. The terms $\widehat{A}(F), \mathrm{Ch}(\mathcal{L}|_F, g)^{1/2}$, and $D_\mathbb{R}(\nu_F, g)$ are certain characteristic classes of TF, $\mathcal{L}|_F$, ν_F. (For details, see e.g. [52, Section 5.3].)

Remark 4.7. The fixed point formula can also be written in 'Riemann-Roch form',

$$\mathcal{Q}(M)(g) = \sum_{F \subset M^g} \int_F \frac{\mathrm{Td}(F)\,\mathrm{Ch}(L|_F, g)}{D_\mathbb{C}(\nu_F, g)},$$

which is often easier to use for computations. However, the 'Spin_c-form' will be more convenient for our discussion.

Remark 4.8. If one tries develop a similar quantization procedure for q-Hamiltonian G-spaces (M, ω, Φ), one is faced with several obstacles. First, the 2-form ω need not be closed, hence it cannot be the curvature form of a line bundle. Secondly, since ω can be degenerate, there is no obvious notion of 'compatible complex structure'. (In fact, there are examples of conjugacy classes $\mathcal{C}$ of compact, simply connected Lie groups not admitting any Spin_c-structure.) Hence, there is no suitable Dirac operator in sight. In the following sections we will explain how to get around these problems.

5. The level k fusion ring

From the correspondence with Hamiltonian loop group spaces, we expect that the result of the quantization procedure of q-Hamiltonian spaces should be an element not of the representation ring but of the *fusion ring* of G, at suitable level.

For the remainder of these lecture notes, we will assume that G *is compact, simply connected and* simple. We fix a maximal torus T and a fundamental Weyl chamber $\mathfrak{t}^*_+$. Recall that $P_+ = P \cap \mathfrak{t}^*_+$ are the dominant weights. Let $\theta \in P_+$ be the highest root, i.e. the highest weight of the adjoint representation of G on $\mathfrak{g}^{\mathbb{C}}$. The fundamental alcove has the following description

$$\mathfrak{A} = \{\xi \in \mathfrak{t}_+ |\ \langle \theta, \xi \rangle \leq 1\}.$$

We denote by $\rho \in P_+$ the unique shortest weight in $P_+ \cap \mathrm{int}(\mathfrak{t}^*_+)$; equivalently 2ρ is the sum of the positive roots of G. The *basic inner product* $\cdot$ on $\mathfrak{g}$ is the unique invariant inner product such that $\theta \cdot \theta = 2$ (using the identification $\mathfrak{g} \cong \mathfrak{g}^*$ given by the inner product). We will use the basic inner product to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. The *dual Coxeter number* of G is the positive integer defined by

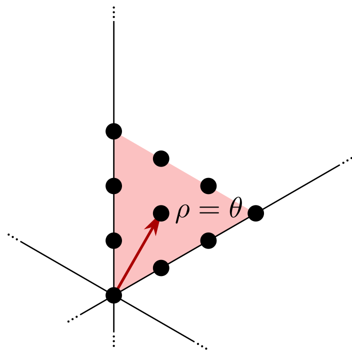
$$\mathsf{h}^\vee = 1 + \theta \cdot \rho.$$

For $G = \mathrm{SU}(N)$ one has $\mathsf{h}^\vee = N$.

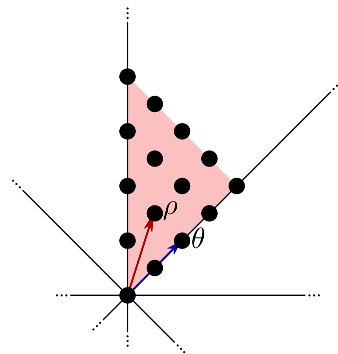
Definition 5.1. Let $k \in \{1, 2, \ldots\}$. The *level k weights* of G are the elements of

$$P_k = P \cap k\mathfrak{A}.$$

The following pictures show the set of level k weights, as well as the weights ρ, θ, in two examples. The shaded area is $k\mathfrak{A}$.



$$G = \mathrm{SU}(3), \quad k = 3$$



$$G = \mathrm{Spin}(5), \quad k = 4$$

For $\lambda \in P_k$ define the special element

$$t_\lambda = \exp(\tfrac{\lambda+\rho}{k+\mathsf{h}^\vee}) \in T.$$

Definition 5.2. The *level k fusion ring* (Verlinde algebra) is the quotient

$$R_k(G) = R(G)/I_k(G)$$

by the *level k fusion ideal*, $I_k(G) = \{\chi \in R(G)|\ \chi(t_\lambda) = 0\ \ \forall\ \lambda \in P_k\}$.

The fusion ring $R_k(G)$ plays an important role in conformal field theory (see e.g. [25]). It is also known as the *level k Verlinde algebra*, after the physicist Erik Verlinde [63].

Remark 5.3. $R_k(G)$ is also the fusion ring of level k projective representations of the loop group LG. However, we will not need this interpretation here.

Some basic properties of the level k fusion ring are as follows:

(a) the unit and involution of $R(G)$ descend to a unit and involution of $R_k(G)$,

(b) $R_k(G)$ has *finite* $\mathbb{Z}$-basis the images τ_μ of $\chi_\mu, \mu \in P_k$. Thus

$$R_k(G) = \mathbb{Z}[P_k].$$

(c) $R_k(G)$ has a trace,

$$R_k(G) \to \mathbb{Z},\ \tau \mapsto \tau^G$$

where $\tau_\mu^G = \delta_{\mu,0}$.

(d) The integers

$$N^{(k)}_{\mu_1\mu_2\mu_3} = (\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3})^G, \quad \mu_i \in P_k$$

are called the *level k fusion coefficients*. They encode the multiplication in $R_k(G)$:

$$\tau_{\mu_1}\tau_{\mu_2} = \sum_{\mu_3 \in P_k} N^{(k)}_{\mu_1\mu_2\mu_3}\tau^*_{\mu_3}.$$

If $\mu_1, \mu_2, \mu_3 \in P_+$ are fixed, the fusion coefficients become independent of k for sufficiently large k, and coincide with the tensor coefficients:

$$N^{(k)}_{\mu_1\mu_2\mu_3} = N_{\mu_1\mu_2\mu_3}, \qquad k >> 0.$$

Example 5.4. For $G = \mathrm{SU}(2)$, it is not difficult to determine the level k fusion ring 'by hand'. Identify $\mathfrak{t} \cong \mathbb{R}$ in such a way that $P_+ = \{0, 1, \ldots\}$. Here $m \geq 0$ is realized as the dominant weight for the m-th symmetric power of the defining representation, $S^m\mathbb{C}^2$. We have $\rho = 1$, $\theta = 2$, and the alcove is the interval $[0, 1] \subset \mathbb{R}$. Hence $P_k = \{0, 1, \ldots, k\}$. The product in $R(\mathrm{SU}(2))$ is given by the well-known formula

$$\chi_l \chi_m = \chi_{l+m} + \chi_{l+m-2} + \ldots + \chi_{|l-m|}.$$

Equivalently, the tensor coefficients are given by

$$N_{m_1 m_2 m_3} = 1$$

if $m_1 + m_2 + m_3$ is even with $m_i \leq \frac{1}{2}(m_1 + m_2 + m_3)$ for $i = 1, 2, 3$, and are zero in all other cases. One finds that the level k fusion ideal is $I_k(\mathrm{SU}(2)) = \langle \chi_{k+1} \rangle$, and the quotient map $R(G) \to R_k(G)$ is 'signed reflection' across indices $k + 1,\ 2k + 3,\ 3k + 5, \ldots$.

To illustrate, if $k = 5$ we find $\tau_3\tau_4 = \tau_3 + \tau_1$ since

$$\chi_3\chi_4 = \chi_7 + \chi_5 + \chi_3 + \chi_1,$$

and because $\chi_7 \mapsto -\tau_5,\ \chi_5 \mapsto \tau_5$ under the quotient map. For $m_1, m_2, m_3 \in \{0, 1, \ldots, k\}$, the $\mathrm{SU}(2)$ fusion coefficients at level k are given by

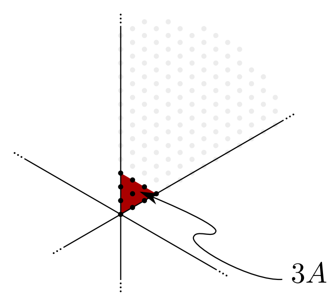
$$N^{(k)}_{m_1 m_2 m_3} = 1$$

provided $m_1 + m_2 + m_3$ is even with

$$m_i \leq \tfrac{1}{2}(m_1 + m_2 + m_3) \leq k$$

for $i = 1, 2, 3$, and are zero in all other cases.

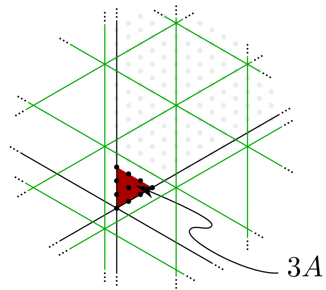
For a general compact simple Lie group G, the quotient map $R(G) \to R_k(G)$ is a 'signed reflection' for a shifted Stiefel diagram. We illustrate the quotient map for $G = \mathrm{SU}(3)$ and level $k = 3$. Consider the set P_k of level k weights



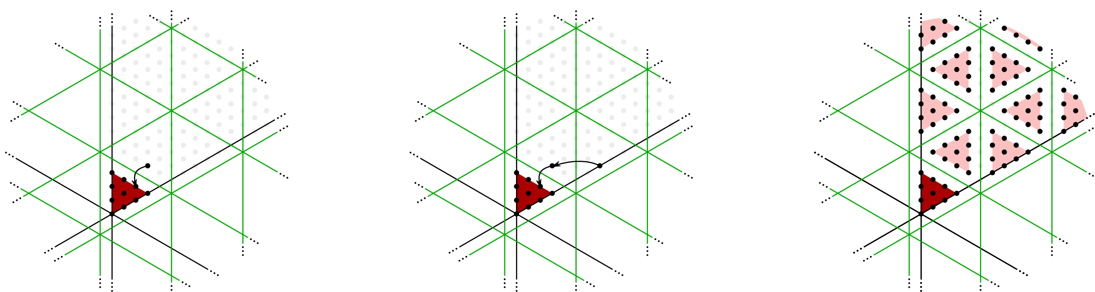
One can show that the weights P_k, shifted by ρ, are exactly the weights in the interior the shifted alcove at level $k + \mathsf{h}^\vee$:

$$P_k + \rho = P \cap (k + \mathsf{h}^\vee)\,\mathrm{int}(\mathfrak{A}).$$

The affine reflections across the faces of the shifted alcove $(k + \mathsf{h}^\vee)\mathfrak{A} - \rho$ alcove generate the *ρ-shifted level $k + \mathsf{h}^\vee$ Stiefel diagram*, shown in the following picture.



The shifted affine Weyl group is the group of transformations of $\mathfrak{t}$, generated by reflections across these affine hyperplanes:



The last picture shows the weights that can be reflected into P_k. If $\mu \in P_+$ lies on the walls of the shifted Stiefel diagram, then χ_μ lies in the kernel of the quotient map $R_k(G)$. Otherwise, the quotient map takes χ_μ to $\pm\tau_\nu$, where $\nu \in P_k$ is the unique level k weight related to μ by a sequence of affine reflections, and where the sign (plus or minus) is given by the parity (even or odd) of the required number of reflections.

Remark 5.5. It was shown by Gepner [27] and Bouwknegt-Ridout [17] that for $G = \mathrm{SU}(N)$, the level k fusion ideal has the description

$$I_k(G) = \langle \chi_{(k+1)\varpi_1}, \dots, \chi_{(k+N-1)\varpi_1} \rangle,$$

where ϖ_1 (the first fundamental weight) is the highest weight of the defining representation of $\mathrm{SU}(N)$ on $\mathbb{C}^N$. There is a similar presentation of the fusion ideal for the symplectic group $\mathrm{Sp}(r)$. Explicit presentations of the fusion rings for the other simple groups, with small numbers of generators, were obtained by C. Douglas in [22].

By definition of the ideal $I_k(G)$, the evaluation maps

$$R(G) \to \mathbb{C},\ \chi \mapsto \chi(t_\lambda)$$

for $\lambda \in P_k$ vanishes on $I_k(G)$, hence they descend to the fusion ring:

$$R_k(G) \to \mathbb{C}, \quad \tau \mapsto \tau(t_\lambda).$$

After complexification, we obtain a new additive basis $\tilde{\tau}_\mu,\ \mu \in P_k$ of $R_k(G) \otimes \mathbb{C}$, characterized by the property

$$\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}.$$

In the new basis, the product is diagonalized: $\tilde{\tau}_\mu \tilde{\tau}_\nu = \delta_{\mu,\nu}\tilde{\tau}_\nu$. The two bases are related by the S-*matrix*

$$S \in \mathrm{End}(\mathbb{C}[P_k]).$$

The S-matrix is the unique *unitary* matrix with properties

$$S_{\mu,\nu} = S_{\nu,\mu}, \quad S_{0,\nu} > 0$$

for $\mu, \nu \in P_k$, and such that

$$\tau_\mu = \sum_{\nu \in P_k} S_{0,\nu}^{-1}\, S_{\mu,\nu}^* \tilde{\tau}_\nu;$$

In terms of the S-matrix, the fusion coefficients take on the form,

$$N^{(k)}_{\mu_1\mu_2\mu_3} = \sum_{\nu \in P_k} \frac{S_{\mu_1,\nu} S_{\mu_2,\nu} S_{\mu_3,\nu}}{S_{0,\nu}}.$$

6. Pre-quantization of q-Hamiltonian spaces

While the 2-form ω for a q-Hamiltonian G-space (M, ω, Φ) is not closed, in general, the pair $(\omega, -\eta)$ defines a *relative cocycle* for the map Φ. To explain in more detail, we recall the cone construction from homological algebra.

6.1. **Relative cohomology.**

Definition 6.1. Let $F^\bullet \colon S^\bullet \to R^\bullet$ be a cochain map between cochain complexes. The *algebraic mapping cone* is the cochain complex

$$\mathrm{cone}^k(F) = R^{k-1} \oplus S^k, \quad \mathrm{d}(x, y) = (f(y) - \mathrm{d}x, \mathrm{d}y).$$

Its cohomology is denoted $H^\bullet(F)$, and is called the relative cohomology of the cochain map $F^\bullet$.

The short exact sequence of cochain complexes $0 \to R^{k-1} \to \mathrm{cone}^k(F) \to S^k \to 0$ gives rise to a long exact sequence of cohomology groups,

$$\cdots \to H^{k-1}(R) \to H^k(F) \to H^k(S) \to H^k(R) \to \cdots .$$

The connecting homomorphism $H^\bullet(S) \to H^\bullet(R)$ is just the map induced by F.

Given a smooth map $\Phi \colon M \to N$ between manifolds, we define $\Omega^\bullet(\Phi) = \mathrm{cone}^\bullet(\Phi^*)$ to be the algebraic mapping cone for the pull-back of differential forms, $\Phi^* \colon \Omega^\bullet(N) \to \Omega^\bullet(M)$. Its cohomology $H^\bullet(\Phi) := H^\bullet(\Phi^*)$ is called the relative de Rham cohomology of the map Φ. The usual isomorphism with the singular cohomology with real coefficients carries over the the relative setting, and there is a *coefficient homomorphism*

$$H^\bullet(\Phi, \mathbb{Z}) \to H^\bullet(\Phi) = H^\bullet(\Phi, \mathbb{R}).$$

6.2. **Definition of pre-quantization.** For a q-Hamiltonian G-space, we have $\mathrm{d}\omega = -\Phi^*\eta$ and $\mathrm{d}\eta = 0$. Hence

$$(\omega, -\eta) \in \Omega^3(\Phi)$$

is a cocycle. (In fact, working with equivariant forms the pair $(\omega, -\eta_G)$ is an equivariant relative cocycle in $\Omega^3_G(\Phi)$, using the algebraic mapping cone for the Cartan complexes.) Suppose G simple, simply connected, $\cdot$ the basic inner product.

Definition 6.2. [48, 52] Let (M, ω, Φ) be a q-Hamiltonian G-space, $\Phi\colon M \to G$. A *level k pre-quantization* of (M, ω, Φ) is an integral lift of

$$k[(\omega, -\eta)] \in H^3(\Phi, \mathbb{R}).$$

There is an equivariant version of this condition, but for simply connected compact groups G the equivariance is automatic. Indeed, in this case the natural map $H^\bullet_G(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z})$ for a G-space X is an isomorphism in degrees ≤ 2, while for any G-map Φ the map $H^\bullet_G(\Phi, \mathbb{Z}) \to H^\bullet(\Phi, \mathbb{Z})$ is an isomorphism in degrees ≤ 3. Cf. Krepski [48, Section 3].

Remark 6.3. The geometric interpretation of the pre-quantization condition involves 'gerbes'. Loosely speaking, the pre-quantization of the condition $\mathrm{d}(k\omega) = -k\Phi^*\eta$ is given by a gerbe over G, with 3-curvature form $k\eta$, together with a trivialization of the pull-back of this gerbe to M, with $k\omega$ the curvature form of the trivialization. See Shahbazi [59] for further details.

6.3. **Basic properties, examples.** One has the following criterion for the integrality of the relative form $k(\omega, -\eta) \in \Omega^3(\Phi)$. For any manifold M, let $C_\bullet(M)$ be the chain complex of *smooth* singular chains on M (i.e. $C_k(M)$ consists of $\mathbb{Z}$-linear combinations of smooth maps $\Delta^k \to M$, where Δ^k is the k-simplex). Recall that a closed differential form $\alpha \in \Omega^k(M)$ is integral (i.e. its class $[\alpha] \in H^k(M, \mathbb{R})$ lies in the image of $H^k(M, \mathbb{Z})$) if and only if $\int_\Sigma \alpha \in \mathbb{Z}$ for all k-cycles $\Sigma \in Z_k(M)$. This criterion extends to the relative case, so that we have:

Proposition 6.4. *A q-Hamiltonian G-space (M, ω, Φ) is pre-quantizable at level k if and only if for all $\Sigma \in Z_2(M)$, and any $X \in C_3(G)$ with $\Phi(\Sigma) = \partial X$,*

$$k(\int_\Sigma \omega + \int_X \eta) \in \mathbb{Z}.$$

Note that for given Σ, it suffices to check for *any* X, due to the integrality of η. In particular, the criterion is satisfied if the second homology group $H_2(M, \mathbb{Z})$ is zero. Indeed, in this case we can take $X = \Phi(Y)$ with $Y \in C_3(M)$, $\partial Y = \Sigma$, and the criterion holds true by Stokes' theorem.

Example 6.5. The double $\mathbf{D}(G) = G \times G$, $\Phi(a, b) = aba^{-1}b^{-1}$ is pre-quantizable for all $k \in \mathbb{N}$, since $H_2(D(G), \mathbb{Z}) = 0$.

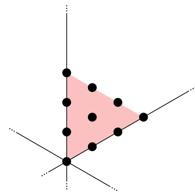
Example 6.6. The q-Hamiltonian $\mathrm{SU}(n)$-space $M = S^{2n}$ and the q-Hamiltonian $\mathrm{Sp}(n)$-space $M = \mathbb{H}\mathrm{P}(n)$ are pre-quantized at all levels $k \in \mathbb{N}$, since $H_2(M, \mathbb{Z}) = 0$ in these examples.

Recall that conjugacy classes $\mathcal{C} \subset G$ are parametrized by points in the alcove, where $\xi \in \mathfrak{A}$ corresponds to the conjugacy class $\mathcal{C} = G.\exp\xi$. We have:

Example 6.7. The level k pre-quantized conjugacy classes $\mathcal{C} \subset G$ are those indexed by

$$\xi \in \tfrac{1}{k} P_k \subset A.$$

The following picture shows the pre-quantized conjugacy classes for SU(3) at level $k = 3$.



$$\begin{aligned} G &= \mathrm{SU}(3) \\ k &= 3 \end{aligned}$$

In all these examples, the torsion subgroup of $H^2(M, \mathbb{Z})$ is trivial, hence the pre-quantization is unique.

Pre-quantizations are well-behaved with respects to products: If M_1, M_2 are level k-pre-quantized q-Hamiltonian G-spaces, then their fusion product $M_1 \times M_2$ inherits a level k pre-quantization. In particular, the q-Hamiltonian G-space $D(G)^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ has a level k pre-quantization, provided the conjugacy classes $\mathcal{C}_j$ have level k pre-quantizations.

Furthermore, if M is a level k pre-quantized q-Hamiltonian G-space, then the symplectic quotient $M/\!\!/G$ inherits a pre-quantization at level k, i.e. for the k-th multiple of the symplectic form. (If the symplectic quotient is singular, this statement should be interpreted as in [53].)

7. Twisted Spin$_c$-structures on q-Hamiltonian spaces

Besides the notion of pre-quantization, a key ingredient in the quantization of Hamiltonian G-spaces is the existence of a canonical Spin$_c$-structure (defined by a compatible almost complex structure). For q-Hamiltonian G-spaces, there need not be a Spin$_c$-structure in general, but it turns out that there is a canonical *twisted* Spin$_c$-structure.

7.1. Spin$_c$-structures.

We will use the following viewpoint toward Spin$_c$-structures. Given a Euclidean vector space V, let $\mathbb{C}\mathrm{l}(V)$ denote its complex Clifford algebra. Thus $\mathbb{C}\mathrm{l}(V)$ is the complex unital algebra with generators $v \in V$ and relations $v_1 v_2 + v_2 v_1 = 2\langle v_1, v_2 \rangle$. Using a basis $e_1, \ldots, e_n \in V$ to identify $V \cong \mathbb{R}^n$, the Clifford algebra has basis the products $e_I = e_{i_1} \ldots e_{i_k}$ for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$, with the convention $e_\emptyset = 1$. Thus, $\mathbb{C}\mathrm{l}(V) = \wedge(V) \otimes \mathbb{C}$ as a vector space. The Clifford algebra carries a $\mathbb{Z}_2$-grading, where the even (resp. odd) part is spanned by products of an even (resp. odd) number of elements of V.

Definition 7.1. Suppose $\dim V$ is even. A *spinor module* over $\mathbb{C}\mathrm{l}(V)$ is a $\mathbb{Z}_2$-graded Hermitian vector space S, together with an isomorphism $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\mathsf{S})$ preserving $\mathbb{Z}_2$-gradings and involutions $*$.

A concrete spinor module is obtained by the choice of an orthogonal complex structure $J \in \mathrm{End}(V)$: Let $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ be the decomposition into $\pm i$ eigenspaces of J, the space $\wedge V^{0,1}$ is a spinor module, with Clifford action of generators $v = v^{1,0} + v^{0,1}$ given by $\iota(v^{1,0}) + \epsilon(v^{0,1})$ (here ι denotes contraction, ϵ is exterior multiplication). One has the following fact:

Proposition 7.2. *For any two spinor modules* S, S' *over* $\mathbb{C}\mathrm{l}(V)$*, the space*

$$\mathrm{Hom}_{\mathbb{C}\mathrm{l}(V)}(\mathsf{S}, \mathsf{S}')$$

of linear maps $\mathsf{S} \to \mathsf{S}'$ *intertwining the Clifford actions is 1-dimensional.*

These definitions generalize to Euclidean vector bundles $V \to M$ in an obvious way. In particular, we define a spinor module over $\mathbb{C}\mathrm{l}(V)$ to be a $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathsf{S} \to M$, together with an even isomorphism of $*$-algebra bundles $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\mathsf{S})$. Given an orthogonal complex structure J on V, the bundle $\wedge V^{0,1} \to M$ is such a spinor module.

Definition 7.3. A Spin_c-structure on an even rank Euclidean vector bundle $V \to M$ is a spinor module S over $\mathbb{C}\mathrm{l}(V)$. A Spin_c-structure on an even-dimensional Riemannian manifold is a Spin_c-structure on TM.

Remarks 7.4.

(a) A Spin_c-structure on a Euclidean vector bundle V of rank n can also be defined as an orientation on V together with a lift of the structure group from $\mathrm{SO}(n)$ to the group $\mathrm{Spin}_c(n)$. The two definitions are equivalent [55].

(b) For Euclidean vector bundle of odd rank, one can define a Spin_c-structure on V to be a Spin_c-structure on $V \oplus \mathbb{R}$.

(c) There are two topological obstructions to the existence of a Spin_c-structure on V. The first obstruction $w^1(V) \in H^1(M, \mathbb{Z}_2)$ is simply the obstruction to orientability of V. The second obstruction $W^3(V) \in H^3(M, \mathbb{Z})$ is the *third integral Stiefel-Whitney class*, given as the image of $w^2(V) \to H^2(M, \mathbb{Z}_2)$ under the Bockstein homomorphism. Note that $W^3(V)$ is 2-torsion, which is consistent with the fact that $V \oplus V = V \otimes \mathbb{C}$ carries a Spin_c-structure.

(d) If S is spinor module, then so is the graded $\mathsf{S} \otimes L$ for any $\mathbb{Z}_2$-graded Hermitian line bundle L. (A $\mathbb{Z}_2$-grading on a complex line bundle $L \to M$ is just the assignment of an even or odd parity over each component of M.) Proposition 7.2 generalizes to the fact that any two Spin_c-structures S, S' on V differ by a $\mathbb{Z}_2$-graded Hermitian line bundle:

$$\mathsf{S}' = \mathsf{S} \otimes L; \quad L = \mathrm{Hom}_{\mathbb{C}\mathrm{l}(V)}(\mathsf{S}, \mathsf{S}').$$

7.2. **Dixmier-Douady theory.** Given a separable Hilbert space H, we denote by $\mathbb{K}(\mathsf{H})$ the $*$-algebra of *compact operators* on H, i.e. the norm closure of the algebra of operators of finite rank. One may think of $\mathbb{K}(\mathsf{H})$ as an appropriate notion of infinite matrices. The action of the unitary group $\mathrm{U}(\mathsf{H})$ by conjugation on $\mathbb{K}(\mathsf{H})$ descends to the projective unitary group $\mathrm{PU}(\mathsf{H}) = \mathrm{U}(\mathcal{H})/\mathrm{U}(1)$, and in fact it is known that

$$\mathrm{Aut}(\mathbb{K}(\mathsf{H})) = \mathrm{PU}(\mathsf{H}),$$

where $\mathrm{PU}(\mathsf{H})$ carries the strong topology.[2]

Definition 7.5. A ($\mathbb{Z}_2$-graded) *Dixmier-Douady bundle* over M is a ($\mathbb{Z}_2$-graded) bundle of $*$-algebras $\mathcal{A} \to M$, with typical fiber $\mathbb{K}(\mathsf{H})$ for some ($\mathbb{Z}_2$-graded) Hilbert space H. A *Morita trivialization* of the Dixmier-Douady bundle $\mathcal{A}$ is a bundle of ($\mathbb{Z}_2$-graded) Hilbert spaces $\mathcal{E} \to M$ with an even isomorphism of $*$-algebra bundles $\mathcal{A} \to \mathbb{K}(\mathcal{E})$.

[2] In the following discussion, some subtleties are being ignored. See [52], and references given there, for a more careful treatment.

Example 7.6. For an even rank Euclidean vector bundle $V \to M$, the Clifford bundle $\mathbb{C}\mathrm{l}(V)$ is a $\mathbb{Z}_2$-graded Dixmier-Douady bundle. A $\mathbb{Z}_2$-graded Morita trivialization of $\mathbb{C}\mathrm{l}(V)$ is the same thing as a spinor module S over $\mathbb{C}\mathrm{l}(V)$, i.e. it is a Spin_c-structure on V.

Generalizing this example, one finds that for any two Morita trivializations $\mathcal{E}, \mathcal{E}'$ of a $\mathbb{Z}_2$-graded DD-bundle $\mathcal{A} \to M$, the bundle $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$ is a $\mathbb{Z}_2$-graded Hermitian line bundle, and conversely any two Morita trivializations differ by such a line bundle

$$\mathcal{E}' = \mathcal{E} \otimes L; \qquad L = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}').$$

Given a DD-bundle $\mathcal{A} \to M$, there is an obstruction $\mathrm{DD}(\mathcal{A}) \in H^3(M, \mathbb{Z})$ to the existence of a Morita trivialization $\mathcal{E}$, called the *Dixmier-Douady class*. In the $\mathbb{Z}_2$-graded setting, there is an additional obstruction in $H^1(M, \mathbb{Z}_2)$ to introducing a compatible $\mathbb{Z}_2$-grading on $\mathcal{E}$.

Remark 7.7. One viewpoint towards the DD-class is as follows. Consider the principal $\mathrm{PU}(\mathsf{H})$-bundle $P \to M$ associated to $\mathcal{A}$. Choose a trivializing open cover U_α of M, so that P is described by transition functions $\chi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{PU}(\mathsf{H})$. Over triple overlaps, $\chi_{\alpha\beta}\chi_{\beta\gamma}\chi_{\alpha\gamma} = 1$. Lift to $\mathrm{U}(\mathsf{H})$-valued functions $\widetilde{\chi}_{\alpha\beta}$. Then $\psi_{\alpha\beta\gamma} = \widetilde{\chi}_{\beta\gamma}\widetilde{\chi}_{\alpha\gamma}^{-1}\widetilde{\chi}_{\alpha\beta}$ is a $\mathrm{U}(1)$-valued function on triple overlaps. On quadruple overlaps one has, by definition of ψ,

$$\psi_{\beta\gamma\delta}\psi_{\alpha\gamma\delta}^{-1}\psi_{\alpha\beta\delta}\psi_{\alpha\beta\gamma}^{-1} = 1,$$

which means that ψ is a Čech cocycle, defining a class in $H^2(M, \underline{\mathrm{U}(1)}) = H^3(M, \mathbb{Z})$. For a detailed discussion of Dixmier-Douady theory, see [56].

More generally, if $\mathsf{H}_1, \mathsf{H}_2$ are two Hilbert spaces, we have the Banach space $\mathbb{K}(\mathsf{H}_1, \mathsf{H}_2)$ of compact operators from H_1 to H_2, again defined as the norm closure of finite rank operators. It is a bimodule:

$$\mathbb{K}(\mathsf{H}_2) \circlearrowright \mathbb{K}(\mathsf{H}_1, \mathsf{H}_2) \circlearrowleft \mathbb{K}(\mathsf{H}_1).$$

If H_i carry $\mathbb{Z}_2$-gradings, then this bimodule structure is compatible with $\mathbb{Z}_2$-gradings.

Definition 7.8. Suppose $\mathcal{A}_i \to M_i,\ i = 1, 2$ are two ($\mathbb{Z}_2$-graded) Dixmier-Douady bundles, with typical fiber $\mathbb{K}(\mathsf{H}_i)$. A ($\mathbb{Z}_2$-graded) *Morita morphism* $(\Phi, \mathcal{E}) \colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ is a map $\Phi \colon M_1 \to M_2$ together with a ($\mathbb{Z}_2$-graded) Banach bundle $\mathcal{E} \to M_1$ of bimodules

$$\Phi^*\mathcal{A}_2 \circlearrowright \mathcal{E} \circlearrowleft \mathcal{A}_1$$

locally modeled on $\mathbb{K}(\mathsf{H}_2) \circlearrowright \mathbb{K}(\mathsf{H}_1, \mathsf{H}_2) \circlearrowleft \mathbb{K}(\mathsf{H}_1)$.

The composition of two Morita morphisms $(\Phi, \mathcal{E}) \colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ and $(\Phi', \mathcal{E}') \colon \mathcal{A}_2 \dashrightarrow \mathcal{A}_3$ has underlying map the composition $\Phi' \circ \Phi$, and bimodule a completion of the tensor product $\Phi^*\mathcal{E}' \otimes_{\Phi^*\mathcal{A}_2} \mathcal{E}$. A Morita morphism is invertible if Φ is; the inverse is defined using an 'opposite' bimodule.

A Morita trivialization of $\mathcal{A} \to M$ is equivalent to a Morita morphism $(p, \mathcal{E}) \colon \mathcal{A} \dashrightarrow \mathbb{C}$, where $\mathbb{C} \to \mathrm{pt}$ is the trivial DD bundle. Again, Morita morphisms may be twisted by line bundles, and any two Morita morphisms $\mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ differ by a Hermitian line bundle:

$$L = \mathrm{Hom}_{\Phi^*\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}') \quad \longleftrightarrow \quad \mathcal{E}' = \mathcal{E} \otimes L.$$

The Dixmier-Douady theorem states that DD-bundles $\mathcal{A} \to M$ are classified, up to Morita isomorphisms inducing the identity map on the base, by $H^3(M, \mathbb{Z})$. The result extends to G-equivariant DD-bundles, see [12].

7.3. **The Dixmier-Douady bundle $\mathcal{A}_G^{\mathrm{Spin}}$.** It is known that

$$H^2(\mathrm{SO}(n),\mathbb{Z})=0,\ \ H^3(\mathrm{SO}(n),\mathbb{Z})=\mathbb{Z},\ \ H^1(\mathrm{SO}(n),\mathbb{Z}_2)=\mathbb{Z}_2$$

for $n=3$ and all $n\geq 5$. If V is a Euclidean vector space of dimension $\dim V\geq 5$, we denote by $\mathcal{A}_{\mathrm{SO}(V)}\to\mathrm{SO}(V)$ the $\mathrm{SO}(V)$-equivariant $\mathbb{Z}_2$-graded DD-bundle whose characteristic classes in $H^3(\mathrm{SO}(V),\mathbb{Z})$ and $H^1(\mathrm{SO}(V),\mathbb{Z}_2)$ represent the generators. Since $H^2(\mathrm{SO}(V),\mathbb{Z})=0$, the particular choice of this DD-bundle does not matter. If $V\subset V'$ is a subspace of a larger Euclidean vector space, then $\mathcal{A}_{\mathrm{SO}(V)}$ is canonically Morita isomorphic to the pull-back of $\mathcal{A}_{\mathrm{SO}(V')}$ under the inclusion $\mathrm{SO}(V)\hookrightarrow\mathrm{SO}(V')$. Consequently, we may extend the definition to $\dim V<5$ by taking $\mathcal{A}_{\mathrm{SO}(V)}$ to be the pull-back of $\mathcal{A}_{\mathrm{SO}(V')}$, where $\dim V'\geq 5$. (E.g., take $V'=V\oplus\mathbb{R}^5$). An explicit construction of this bundle may be found in Atiyah-Segal [12], see also [4] for a discussion of their result.

Given a compact, connected Lie group G, with an invariant inner product on $\mathfrak{g}$, we let

$$\mathcal{A}_G^{\mathrm{Spin}}\to G$$

be the pull-back of $\mathcal{A}^{\mathrm{SO}(\mathfrak{g})}$ under the adjoint representation $G\to\mathrm{SO}(\mathfrak{g})$. (The notation is motivated by a relationship with the spin representation of the loop group.) A nice property of $\mathcal{A}_G^{\mathrm{Spin}}$ is that it is *multiplicative*: there is a Morita morphism $\mathcal{A}_G^{\mathrm{Spin}}\times\mathcal{A}_G^{\mathrm{Spin}}\dashrightarrow\mathcal{A}_G^{\mathrm{Spin}}$ covering group multiplication on G, and with an associativity property.

Remark 7.9. If G is compact, simple and simply connected, so that $H^3_G(G,\mathbb{Z})=H^3(G,\mathbb{Z})=\mathbb{Z}$, it is known that $\mathrm{DD}(\mathcal{A}_G^{\mathrm{Spin}})$ represents the $\mathsf{h}^\vee$-th multiple of the generator of $H^3(G,\mathbb{Z})$, where $\mathsf{h}^\vee$ is the dual Coxeter number.

7.4. **Twisted Spin_c-structure.**

Theorem 7.10 (Alekseev-M [4])**.** *Let G be a compact Lie group, with a positive definite invariant metric $\cdot$ on its Lie algebra. For any q-Hamiltonian G-space (M,ω,Φ), there is a distinguished G-equivariant $\mathbb{Z}_2$-graded Morita morphism*

$$(\Phi,\mathcal{E}^{\mathrm{Spin}})\colon\mathbb{C}\mathrm{l}(TM)\dashrightarrow\mathcal{A}_G^{\mathrm{Spin}}.$$

Keeping in mind that a Morita trivialization $\mathbb{C}\mathrm{l}(TM)\dashrightarrow\mathbb{C}$ is a Spin_c-structure, we think of this morphism as a *twisted* Spin_c*-structure* (following the terminology from [64]).

Remark 7.11. The restriction of $\mathcal{A}_G^{\mathrm{Spin}}$ to the group unit e is G-equivariantly Morita trivial, and the Morita trivialization is essentially unique (since there are no non-trivial G-equivariant line bundles over pt.) By composing the twisted Spin_c-structure with this Morita trivialization, it follows that the restriction $TM|_{\Phi^{-1}(e)}$ inherits an *ordinary* Spin_c-structure. It turns out [4] that this is equal to the Spin_c-structure defined by the non-degenerate 2-form given by the restriction of ω to $TM|_{\Phi^{-1}(e)}$, hence it induces the correct Spin_c-structure on $M/\!\!/G$.

In the Hamiltonian setting, the next step is to twist the Spin_c-structure coming from the almost complex structure by the pre-quantum line bundle L. Similarly, for q-Hamiltonian spaces we can twist by the pre-quantization. To simplify the discussion, we will return to the assumption that G is simple and simply connected. Let $\mathcal{A}_G^{(k)}\to G$ be any G-DD bundle over G whose Dixmier-Douady class is k times the generator of $H^3_G(G,\mathbb{Z})=\mathbb{Z}$. For example, $\mathcal{A}_G^{\mathrm{Spin}}$ may be used as $\mathcal{A}_G^{(\mathsf{h}^\vee)}$. A level k pre-quantization of (M,ω,Φ) determines a Morita morphism,

$$(\Phi,\mathcal{E}^{\mathrm{Preq}})\colon M\times\mathbb{C}\dashrightarrow\mathcal{A}_G^{(k)}.$$

(Classes in $H^3(\Phi,\mathbb{Z})$ may be realized in terms of DD bundles over the target, together with Morita trivializations of the pull-back under Φ.) Tensoring the two Morita morphisms, we obtain a G-equivariant Morita morphism,

$$(\Phi,\mathcal{E}^{\mathrm{Spin}}\otimes\mathcal{E}^{\mathrm{Preq}})\colon \mathbb{C}\mathrm{l}(TM)\dashrightarrow\mathcal{A}_G^{(k+\mathsf{h}^\vee)}. \tag{3}$$

In the following section we will use this Morita morphism to obtain a push-forward in twisted K-homology.

8. Quantization of q-Hamiltonian G-spaces

8.1. **Twisted K-homology.** Recall that a C^*-algebra is a Banach algebra with a conjugate linear involution $*$, isomorphic to a norm closed subalgebra of the algebra $\mathbb{B}(\mathsf{H})$ of bounded operators on a Hilbert space, with $*$ induced by the adjoint. For instance, $\mathbb{K}(\mathsf{H})$ is a C^*-algebra. If $\mathcal{A}\to X$ is a G-equivariant Dixmier-Douady bundle, the space

$$\mathsf{A}=\Gamma_0(X,\mathcal{A})$$

of sections vanishing at infinity (i.e. the closure of the space of sections of compact support) is a G-equivariant C^*-algebra.

Definition 8.1 (Donovan-Karoubi [21], Rosenberg [57]). The *twisted G-equivariant K-homology* of X with coefficients in $\mathcal{A}$ is defined as

$$K_\bullet^G(X,\mathcal{A}):=K_G^\bullet(\Gamma_0(X,\mathcal{A})),$$

the equivariant K-homology of the G-C^*-algebra $\Gamma_0(X,\mathcal{A})$.

Remark 8.2. Here we working with Kasparov's definition of the K-homology of G-C^*-algebras [42, 43]. Let us very briefly sketch Kasparov's approach; an excellent reference for this material is the book [32] by Higson and Roe. Let A be a $\mathbb{Z}_2$-graded C^* algebra. A *Fredholm module over* A is a $\mathbb{Z}_2$-graded Hilbert space H with a $*$-representation $\pi\colon\mathsf{A}\to\mathbb{B}(H)$, together with an odd element $F\in\mathbb{B}(H)$, s.t. $\forall a\in\mathsf{A}$

(a) $[\pi(a),F]\in\mathbb{K}(H)$,
(b) $(F^2+I)\pi(a)\in\mathbb{K}(H)$.

Kasparov defines the K-homology group $K^0(\mathsf{A})$ as the set of all Fredholm modules over A, modulo a suitable notion of 'homotopy'. (For $\mathsf{A}=C(X)$ the continuous functions on a compact Hausdorff space, a definition along similar lines had been proposed by Atiyah [8].) One puts $K^1(\mathsf{A})=K^0(\mathsf{A}\otimes\mathbb{C}\mathrm{l}(\mathbb{R}))$. It is a contravariant functor in C^*-algebras, hence $K_\bullet(X)=K^\bullet(C(X))$ is a covariant functor in spaces X. The definition has a straightforward extension to G-C^*-algebras, defining groups $K_G^\bullet(\mathsf{A})$.

The twisted K-homology groups are functorial with respect to Morita morphisms of Dixmier-Douady-bundles.

Example 8.3. There is a canonical ring isomorphism $K_0^G(\mathrm{pt})=R(G)$, where the ring structure on the left hand side is given by push-forward under the map $\mathrm{pt}\times\mathrm{pt}\to\mathrm{pt}$.

Example 8.4. Suppose D is an equivariant skew-adjoint odd elliptic differential operator acting on a $\mathbb{Z}_2$-graded Hermitian vector bundle $V = V^+ \oplus V^- \to M$ over a compact manifold M. It has an equivariant index $\mathrm{index}_G(D) := \chi_{\ker D|_{V^+}} - \chi_{\ker D|_{V^-}}$. The pair

$$H = \Gamma_{L^2}(M, V),\ F = \tfrac{D}{\sqrt{1+D^*D}}$$

with the natural action of $C(M)$ defines a K-homology class $[D] \in K_0^G(M)$. The index is a push-forward under the map $p\colon M \to \mathrm{pt}$ to a point:

$$p_*[D] = \mathrm{index}_G(D).$$

Example 8.5. Let M be a compact Riemannian G-manifold of even dimension. Then there is a *fundamental class*

$$[M] \in K_0^G(M, \mathbb{C}\mathrm{l}(TM)),$$

represented by the de Rham Dirac operator on $\Gamma(M, \wedge T^*M)$. A Spin_c-structure on M defines a Morita trivialization of $\mathbb{C}\mathrm{l}(TM)$ and a Spin_c-Dirac operator $\partial\!\!\!/_M$. The class $[\partial\!\!\!/_M]$ is the image of $[M]$ under the resulting isomorphism $K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \to K_0^G(M)$. Thus $\mathbb{C}\mathrm{l}(TM)$ plays the role of an 'orientation bundle' in K-theory. Compare with singular homology: Any compact manifold, regardless of orientation, has a fundamental class in the homology group $H_{\dim M}(M, o_M)$ with coefficients in the orientation bundle $o_M = \det(TM)$. An orientation on M trivializes the bundle o_M, and identifies the fundamental class as an element of $H_{\dim M}(M)$. Recall also that there is an isomorphism $H_{\dim M}(M, o_M) \cong H^0(M, \mathbb{Z})$, taking $[M]$ to 1. Similarly, for an even-dimensional Riemannian manifold there is an isomorphism

$$K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \cong K_G^0(M) \tag{4}$$

with equivariant K-theory, taking $[M]$ to the element $1 \in K_G^0(M)$.

Example 8.6. Let G be compact, simply connected, and simple. Denote by $\mathcal{A}_G^{(l)} \to G$ a G-Dixmier-Douady bundle at level $l \in \mathbb{Z} \cong H^3(G, \mathbb{Z})$. $K_0^G(G, \mathcal{A}_G^{(l)})$ has a ring structure defined by $(\mathrm{Mult}_G)_*$. (Note that $\mathrm{Mult}_G^* \mathcal{A}^{(l)}$ is Morita isomorphic to $\mathrm{pr}_1^* \mathcal{A}_G^{(l)} \otimes \mathrm{pr}_2^* \mathcal{A}_G^{(l)}$ since the two bundles have the same Dixmier-Douady class; the specific choice of Morita isomorphism is unimportant since $H_G^2(G \times G, \mathbb{Z}) = 0$.) The theorem of *Freed-Hopkins-Teleman* [26] shows that for all non-negative integers $k \geq 0$, there is a canonical isomorphism of rings

$$K_0^G(G, \mathcal{A}_G^{(k+\mathsf{h}^\vee)}) \cong R_k(G) \tag{5}$$

where $R_k(G)$ is the level k fusion ring (Verlinde ring).

8.2. **Quantization as a push-forward.** Suppose G is a compact, simple, simply connected Lie group, and (M, ω, Φ) is a level k pre-quantized q-Hamiltonian G-space. The Morita morphism (3) defines a push-forward in twisted K-homology,

$$\Phi_*\colon K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \to K_0^G(G, \mathcal{A}_G^{(k+\mathsf{h}^\vee)}).$$

Using the isomorphism (4) and the Freed-Hopkins-Teleman result (5), we have constructed an $R(G)$-module homomorphism

$$\Phi_*\colon K_0^G(M) \to R_k(G).$$

Definition 8.7. [52] The *quantization* of a level k pre-quantized q-Hamiltonian G-space (M,ω,Φ) is the element

$$\mathcal{Q}(M)=\Phi_*(1)\in R_k(G).$$

As shown in [52], the quantization of q-Hamiltonian spaces has properties parallel to those for Hamiltonian spaces:

(a) $\mathcal{Q}(M_1\cup M_2)=\mathcal{Q}(M_1)+\mathcal{Q}(M_2)$,
(b) $\mathcal{Q}(M_1\times M_2)=\mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
(c) $\mathcal{Q}(M^*)=\mathcal{Q}(M)^*$,
(d) Let $\mathcal{C}$ be the conjugacy class of $\exp(\frac{1}{k}\mu)$, $\mu\in P_k$. Then

$$\mathcal{Q}(\mathcal{C})=\tau_\mu.$$

Recall the trace $R_k(G)\to\mathbb{Z},\ \tau\mapsto\tau^G$ where $\tau_\mu^G=\delta_{\mu,0}$.

Theorem 8.8 (Quantization commutes with reduction)**.** *Let (M,ω,Φ) be a level k pre-quantized q-Hamiltonian G-space. Then*

$$\mathcal{Q}(M)^G=\mathcal{Q}(M/\!\!/G).$$

Similar to Example 4.6 we have:

Example 8.9. Let $\mathcal{C}_i$ be the conjugacy classes of $\exp(\frac{1}{k}\mu_i),\ \mu_i\in P_k$. Then

$$\mathcal{Q}(\mathcal{C}_1\times\mathcal{C}_2\times\mathcal{C}_3/\!\!/G)=(\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3})^G=N^{(k)}_{\mu_1\mu_2\mu_3}.$$

Example 8.10. The double $\mathbf{D}(G)=G\times G$, $\Phi(a,b)=aba^{-1}b^{-1}$ has level k quantization

$$\mathcal{Q}(D(G))=\sum_{\mu\in P_k}\tau_\mu\tau_\mu^*.$$

Remark 8.11. The Hamiltonian analogue of the double is the *non-compact* Hamiltonian G-space T^*G, with the cotangent lift of the conjugation action. Any reasonable quantization scheme for non-compact spaces gives

$$\mathcal{Q}(T^*G)=\sum_{\mu\in P_+}\chi_\mu\chi_\mu^*,$$

the character for conjugation action on $L^2(G)$, defined as an element of a completion of $R(G)$.

Since $\mathcal{Q}(M_1\times M_2)=\mathcal{Q}(M_1)\mathcal{Q}(M_2)$, we also get the quantization of iterated fusions of copies of $D(G)$ and of level k prequantized conjugacy classes $\mathcal{C}_j$. To work out the product, it is convenient to re-write these results in terms of the basis $\tilde{\tau}_\mu$ of $R_k(G)\otimes\mathbb{C}$, where $\tilde{\tau}_\mu(t_\lambda)=\delta_{\lambda,\mu}$:

$$\mathcal{Q}\big(G.\exp(\tfrac{1}{k}\mu)\big)=\tau_\mu=\sum_{\nu\in P_k}\frac{S^*_{\mu,\nu}}{S_{0,\nu}}\tilde{\tau}_\nu.$$

$$\mathcal{Q}(D(G))=\sum_{\nu\in P_k}\frac{1}{S^2_{0,\nu}}\tilde{\tau}_\nu$$

Using $\mathcal{Q}(M_1\times M_2)=\mathcal{Q}(M_1)\mathcal{Q}(M_2)$ this gives

Proposition 8.12. *Let $\mu_1, \ldots, \mu_r \in P_k$, and $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. Then the level k quantization of $D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ is given by the formula,*

$$\mathcal{Q}\Big(D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r\Big) = \sum_{\nu \in P_k} \frac{S^*_{\mu_1,\nu} \cdots S^*_{\mu_r,\nu}}{S^{2g+r}_{0,\nu}} \tilde{\tau}_\nu$$

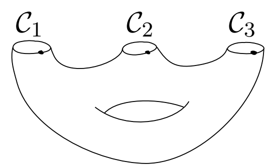
Hence, using the q-Hamiltonian 'quantization commutes with reduction' theorem, we obtain,

Theorem 8.13 (Symplectic Verlinde formulas)**.** *Let $\mu_1, \ldots, \mu_r \in P_k$, and $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. The level k quantization of the moduli space*

$$\mathcal{M}(\Sigma_g^r, \mathcal{C}_1, \ldots, \mathcal{C}_r) = (D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r) /\!\!/ G$$

is given by the formula

$$\mathcal{Q}\Big(\mathcal{M}(\Sigma_g^r, \mathcal{C}_1, \ldots, \mathcal{C}_r)\Big) = \sum_{\nu \in P_k} S_{\mu_1,\nu} \cdots S_{\mu_r,\nu}\ S_{0,\nu}^{-(2g+r-2)}$$



Remark 8.14. The choice of a complex structure on Σ, compatible with the orientation, defines a Kähler structure on the moduli space, and its pre-quantization is given by a holomorphic line bundle (using an appropriate interpretation in case the moduli space is singular). The more common setting for the Verlinde formulas is as the dimension of the space of holomorphic sections of the pre-quantum line bundle (i.e. the Kähler quantization). Provided the higher cohomology groups vanish, this dimension equals the index computed above.

8.3. **Localization.** As in the case of Hamiltonian spaces, the main technique for actually calculating the quantization of q-Hamiltonian spaces is by localization. Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space. Since the pull-back of the Cartan 3-form $\eta \in \Omega^3(G)$ to the maximal torus $T \subset G$ vanishes, the map in cohomology $H^3(G, \mathbb{R}) \to H^3(T, \mathbb{R})$ is the zero map. Due to the absence of torsion, this is also true with integer coefficients, proving that $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$ is Morita trivial. In fact, by considering the pull-back of the generator of $H^3_G(G, \mathbb{Z}) = \mathbb{Z}$ to a class in $H^3_T(T, \mathbb{Z})$ (see [52, Section 5.1]), one finds that it is $T_{k+\mathsf{h}^\vee}$-equivariantly Morita trivial, where $T_{k+\mathsf{h}^\vee} \subset T$ is the finite subgroup generated by the elements $t_\lambda, \lambda \in P_k$. (Note that while the conjugation action of $T_{k+\mathsf{h}^\vee}$ on T is trivial, there is still a non-trivial action on $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$.) Let us choose any such Morita trivialization, with the additional property that the resulting Morita trivialization of $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_e$ is G-equivariant. Even with this additional normalization the choice is not quite canonical: One may still twist by a line bunde over T with a trivial T-action.

Suppose now that $t \in T$ is a regular element (i.e. $G_t = T$), and let (M, ω, Φ) be a q-Hamiltonian G-space. If $F \subset M^t$ is a component of the fixed point set, then $\Phi(F) \subseteq T$, by

T-equivariance of the moment map. By composition, we obtain a $T_{k+\mathsf{h}^\vee}$-equivariant Morita morphism

$$\mathbb{C}\mathrm{l}(TM|_F) \dashrightarrow \mathcal{A}^{(k+\mathsf{h}^\vee)}|_T \dashrightarrow \mathbb{C},$$

or equivalently a $T_{k+\mathsf{h}^\vee}$-equivariant Spin_c-structure on $TM|_F$. Thus, even though M itself does not carry a global Spin_c-structure, one does have Spin_c-structures along the fixed point manifolds. Consequently, the fixed point contributions from the equivariant index theorem for Spin_c-Dirac operators are well-defined, even though there is no globally defined operator.

We specialize to the case $t = t_\lambda,\ \lambda \in P_k$. Recall again (Section 5) that the evaluation of elements $\tau \in R_k(G)$ at the points t_λ is well-defined, and τ can be recovered from the values $\tau(t_\lambda)$.

Theorem 8.15. *Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space. For $\lambda \in P_k$,*

$$\mathcal{Q}(M)(t_\lambda) = \sum_{F \subset M^{t_\lambda}} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}_F, t_\lambda)^{1/2}}{D_{\mathbb{R}}(\nu_F, t_\lambda)}$$

where $\mathcal{L}_F$ is the Spin_c*-line bundle for $TM|_F$.*

As a typical application, consider the case $M = D(G)^h = G^{2h}$. Since the G-action on M is just conjugation, and t_λ is regular, the fixed point set is simple $F = T^{2h} \subset G^{2h}$, with $\Phi(T^{2h}) = \{e\}$ and with a trivial normal bundle $\nu_F = (\mathfrak{g}/\mathfrak{t})^{2h}$. Since the geometry is so simple, the evaluation of the fixed points contributions poses no problems. See [6] or [52] for the calculation.

Remark 8.16. In Alekseev-M-Woodward [6], the quantization of a q-Hamiltonian space was essentially *defined* in terms of the localization formula. (The result was phrased in terms of loop group actions). However, it was unclear in [6] what the 'equivariant object' might be of which the right hand side of this formula are the localization contributions.

The 'quantization commutes with reduction' theorem for q-Hamiltonian spaces, Theorem 8.8, was proved in [6] using the definition in terms of fixed point data. The proof is somewhat complicated, and it would be of great interest to obtain a cleaner proof.

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