

# TOWARDS THE SMOOTHNESS OF OPTIMAL MAPS ON RIEMANNIAN SUBMERSIONS AND RIEMANNIAN PRODUCTS (OF ROUND SPHERES IN PARTICULAR)

YOUNG-HEON KIM AND ROBERT J. MCCANN

ABSTRACT. The variant **A3w** of Ma, Trudinger and Wang's condition for regularity of optimal transportation maps is implied by the nonnegativity of a pseudo-Riemannian curvature — which we call *cross-curvature* — induced by the transportation cost. For the Riemannian distance squared cost, it is shown that (1) cross-curvature nonnegativity is preserved for products of two manifolds; (2) both **A3w** and cross-curvature nonnegativity are inherited by Riemannian submersions, as is domain convexity for the exponential maps; and (3) the  $n$ -dimensional round sphere satisfies cross-curvature nonnegativity. From these results, a large new class of Riemannian manifolds satisfying cross-curvature nonnegativity (thus **A3w**) is obtained, including many whose sectional curvature is far from constant. All known obstructions to the regularity of optimal maps are absent from these manifolds, making them a class for which it is natural to conjecture that regularity holds. This conjecture is confirmed for complex projective space  $\mathbf{CP}^n$  and other Riemannian submersions of the round sphere which have no purely focal points; in the presence of purely focal points we obtain only continuity of optimal maps between positively bounded source and target densities.

## 1. INTRODUCTION

This paper addresses questions in optimal transportation theory and Riemannian geometry. For a general introduction to these subjects we refer to the books by Villani [V1] [V2] for optimal transport theory and the book by Cheeger and Ebin [CE] for Riemannian geometry.

### 1.1. Background: optimal transport and pseudo-Riemannian geometry.

In optimal transportation theory one is interested in phenomena which occur when moving mass distributions so as to minimize the transportation cost. Mathematically, there are source and target domains,  $M, \bar{M}$ , two differential manifolds equipped with a lower semi-continuous cost function  $c : M \times \bar{M} \rightarrow \mathbf{R} \cup \{\infty\}$ . Given two positive Borel probability measures  $\rho, \bar{\rho}$  on  $M, \bar{M}$ , respectively, one wants to understand the optimal map  $F : M \rightarrow \bar{M}$ , which minimizes the average

---

*Date:* November 18, 2009.

*2000 Mathematics Subject Classification.* 49N60, 35Jxx, 58E17, 90B06.

This research was supported in part by Natural Sciences and Engineering Research Council of Canada Grant 217006-03 and United States National Science Foundation Grant DMS-0354729.

©2008 by the authors.

cost

$$\int_M c(x, F(x)) d\rho(x)$$

among all such maps pushing-forward  $\rho$  to  $\bar{\rho}$ , denoted  $F_{\#}\rho = \bar{\rho}$  and meaning that

$$\rho(F^{-1}(E)) = \bar{\rho}(E) \quad \forall \text{ Borel } E \subset \bar{M}.$$

Particular attention has been devoted to the case  $M = \bar{M}$ , a complete Riemannian manifold with the cost  $c = \frac{1}{2}\text{dist}^2$ , where  $\text{dist}$  denotes the Riemannian distance function. Existence and uniqueness of optimal maps in this case is well known due to the work of Brenier [Br] and McCann [M] (and Sturm [S] and Figalli [F] for noncompact manifolds), under the condition that  $\rho$  doesn't charge lower dimensional submanifolds (see also [CNM]). Under suitable hypotheses, regularity ( $C^0/C^\alpha/C^\infty$ ) of such optimal maps is known for Euclidean space  $M = \mathbf{R}^n$  by results of Delanoë [D1], Caffarelli [Ca1] [Ca2], and Urbas [U], and for flat [Co] and near flat [D2] manifolds by Cordero-Erausquin and Delanoë. Beyond the flat case, Loeper [L2] deduced regularity on the round sphere, by combining his own breakthroughs with pioneering results of Ma, Trudinger and Wang [MTW][TW1][TW2] concerning regularity of optimal maps for general cost functions. Loeper's result is simplified in the work of present authors [KM] where we give an elementary and direct proof of a crucial maximum principle deduced by Loeper from Trudinger and Wang's theory. We referred to Loeper's maximum principle with the acronym **DASM** (see Theorem 2.7), and extended it to the manifold case. In the course of deriving new results, our method was employed and further developed by Figalli and Villani [FV], Figalli and Rifford [FR], Loeper and Villani [LV], and Villani [V2] [V3].

Our contributions in [KM] are based on a pseudo-Riemannian geometric structure  $h$  on  $M \times \bar{M}$  induced by the transportation cost. Namely when  $\dim M = \dim \bar{M}$  this pseudo-metric  $h$  is defined on  $N \subset M \times \bar{M}$  with  $c \in C^4(N)$  as the following symmetric bilinear form on  $TM \oplus T\bar{M}$ :

$$(1.1) \quad h := \begin{pmatrix} 0 & -\frac{1}{2}\bar{D}Dc \\ -\frac{1}{2}D\bar{D}c & 0 \end{pmatrix}$$

where  $D$  and  $\bar{D}$  denote the differentials along  $M$  and  $\bar{M}$ , respectively. For non-degeneracy of  $h$  we assume that  $D\bar{D}c$  and its adjoint  $\bar{D}Dc$  are non-degenerate (Ma, Trudinger and Wang's condition **A2**: see Section 2). This is automatically true for the Riemannian distance squared cost, away from the cut-locus, since  $\bar{D}Dc$  is a matrix of independent Jacobi fields. This pseudo-metric  $h$  geometrizes the regularity theory of optimal maps, by recasting the variant **A3w** [TW1] of Ma, Trudinger and Wang's [MTW] key cost hypothesis for regularity as the non-negativity of certain pseudo-Riemannian sectional curvatures of  $h$ . To explain this condition more precisely, we need

**Definition 1.1 (cross-curvature).** Let  $(x, \bar{x}) \in N \subset M \times \bar{M}$ . For each  $p \oplus \bar{p} \in T_{(x, \bar{x})}N = T_x M \oplus T_{\bar{x}} \bar{M}$ , the cross-curvature of  $p$  and  $\bar{p}$  is defined as

$$(1.2) \quad \text{cross}_{(x, \bar{x})}^{(N, h)}(p, \bar{p}) = R_h((p \oplus 0) \wedge (0 \oplus \bar{p}), (p \oplus 0) \wedge (0 \oplus \bar{p}))$$

where  $R_h$  is the Riemann curvature operator of the pseudo-metric  $h$ . We drop the superscript  $(N, h)$  and the subscript  $(x, \bar{x})$  when no ambiguity can occur.

Ma, Trudinger and Wang's **A3w** condition [TW1] and its strict version **A3s** [MTW] then assert

$$(1.3) \quad \mathbf{A3w}: \quad \text{cross}(p, \bar{p}) \geq 0 \text{ for all } p \oplus \bar{p} \text{ with } h(p \oplus \bar{p}, p \oplus \bar{p}) = 0;$$

$$(1.4) \quad \mathbf{A3s}: \quad \text{in addition, } \text{cross}(p, \bar{p}) = 0 \text{ in (1.3) implies } p = 0 \text{ or } \bar{p} = 0.$$

**A3w** and **A3s** are also called *weak regularity* and *strict regularity*, respectively. If  $(N, h)$  satisfies the inequality  $\text{cross}(p, \bar{p}) \geq 0$  for all  $p \oplus \bar{p}$  (whether or not  $h(p \oplus \bar{p}, p \oplus \bar{p})$  vanishes), then  $(N, h)$  is said to be *non-negatively cross-curved*. Loeper [L1] showed that **A3w** is necessary and **A3s** sufficient for continuity of optimal maps between suitable measures: without **A3w** there are discontinuous optimal maps between smooth measures  $\rho, \bar{\rho}$  on nice domains. Trudinger and Wang [TW1] had already shown the sufficiency of **A3w** for continuity (indeed smooth differentiability) of optimal mappings, under much stronger smoothness and convexity restrictions on  $\rho$  and  $\bar{\rho}$ . These restrictions on  $\rho, \bar{\rho}$  are relaxed in two dimensions by Figalli and Loeper [FL], where they showed a continuity result of optimal maps under **A3w**. It still remains an open question to show such continuity result in higher dimensions. However, in a separate work with Figalli [FKM1], we show non-negative cross-curvature allows these smoothness and convexity restrictions to be relaxed without sacrificing continuity of optimal maps and without assuming **A3s**, thus obtaining a continuity theory for smooth cost functions which extends Caffarelli's  $c(x, y) = \frac{1}{2}|x - y|^2$  result [Ca1] to new costs.

For the Riemannian distance squared cost  $c(x, \bar{x}) = \text{dist}^2(x, \bar{x})/2$  on a Riemannian manifold  $M = \bar{M}$ , an isometric copy of  $M$  is embedded totally geodesically as the diagonal of  $M \times M$  with respect to the pseudo-Riemannian metric  $h$ . Along this diagonal,

$$\text{cross}_{(x, x)}(p, \bar{p}) = \frac{4}{3}R_M(p \wedge \bar{p}, p \wedge \bar{p})$$

where  $R_M$  denotes the curvature operator of  $M$  (see [KM] for details). This provides some geometric intuition motivating Loeper's result [L1] that **A3w** implies nonnegative sectional curvature of the Riemannian metric. (However, the nonnegative/positive curvature does not imply **A3w**, as shown by a counterexample in [K].) Loeper also verified **A3s** for the standard round sphere [L2] and used it to obtain  $C^{1/\max\{5, 4n-1\}}$  and  $C^\infty$  regularity results for optimal maps in this spherical setting. (This Hölder exponent has since been improved to its sharp value by Liu [L].)

**1.2. Main results.** Throughout this paper, if not specified, each Riemannian manifold  $M$  is assumed to be complete and to be equipped with the cost function  $c = \frac{1}{2}\text{dist}^2$  and this cost induces the pseudo metric  $h$  on  $N = M \times M \setminus \text{cut-locus}$ . A Riemannian manifold  $M$  is said to be **A3w** / **A3s** / non-negatively cross-curved if  $(N, h)$  satisfies the corresponding cross-curvature condition.

Our main results provide methods of generating new examples of Riemannian manifolds which satisfy non-negative cross-curvature and thus **A3w**. As announced in [KM], we show:

**Theorem 1.2. (Products and submersions, particularly of round spheres)**

- 1)  $S^n$  with its standard round metric is non-negatively cross-curved.
- 2) Let  $\pi : \tilde{M} \rightarrow M$  be a Riemannian submersion (see Definition 4.1). If  $\tilde{M}$  is **A3w** / **A3s** / non-negatively cross-curved then so is  $M$ .
- 3) For product  $M_+ \times M_-$  of Riemannian manifolds, if each factor  $M_\pm$  is non-negatively cross-curved, then the resulting manifold  $M_+ \times M_-$  is non-negatively cross-curved, thus **A3w** holds (but never **A3s**).
- 4) Moreover, if either of the factors above fail to be non-negatively cross-curved then the product  $M_+ \times M_-$  fails to be **A3w**.

*Proof.* The proof of assertion 1) is a calculation given in Section 6. Assertion 2) is shown in Section 4. Assertion 3) and 4) are easy facts which are explained in Section 3 in detail.  $\square$

**Remark 1.3.** Following our announcement of Theorem 1.2 [KM], Figalli and Rifford [FR] gave an alternate proof of result 1) in a different form, slightly stronger than the present statement but complementary to the *almost positivity* shown in Definition 4.6 and Theorem 6.2

As a byproduct of our method, we obtain an O'Neill type inequality for cross-curvature in Riemannian submersions (see Theorem 4.5). This verifies that Riemannian submersion quotients of the round sphere all satisfy **A3s**; they have strict convex domains for their exponential maps by Theorem 4.9. As a result, optimal maps are continuous between positively bounded densities, and higher regularity follows on notable examples such as on complex projective spaces  $\mathbf{CP}^n$  with the Fubini-Study metric: see Section 5. These are new results which not covered by other discrete quotient cases of Delanoë and Ge [DG] or Figalli and Rifford [FR].

As another important consequence, the Riemannian product of round spheres and Euclidean space  $\mathbf{S}^{n_1} \times \cdots \times \mathbf{S}^{n_k} \times \mathbf{R}^l$  and its *Riemannian submersion quotients*, all satisfy cross-curvature non-negativity and thus **A3w**. Since this rules out the known counterexamples to regularity [L1], optimism combines with a lack of imagination to lead to the conjecture that regularity of optimal mappings also holds in such settings. Together with the perturbations [DG][LV][FR] of the round sphere and its discrete quotients discussed by Delanoë and Ge, Loeper and Villani, and Figalli and Rifford, for which the continuity [FR] or regularity [LV][DG] of optimal maps is already shown, these presently form the only examples of non-flat Riemannian manifolds which are known to be **A3w**. (In flat manifold case, **A3w**

is trivial, and in fact the cross-curvature vanishes everywhere. Regularity of optimal maps in this flat case is known by Cordero-Erausquin [Co] applying Caffarelli's result [Ca1][Ca2].)

As far as we know, it remains an open challenge to show regularity of optimal maps on the new tensor product type examples  $\mathbf{S}^{n_1} \times \cdots \times \mathbf{S}^{n_k} \times \mathbf{R}^l$  (when the supports of the source and target measure are the whole domain). However, Loeper's maximum principle (and a stronger convexity statement) is easily verified on these examples using the results and methods of [KM]: see Corollary 2.12 and Remark 3.4. We hope these key ingredients will make it possible to address the regularity issue in a subsequent work.

This paper is organized as follows: Section 2 is devoted to preliminary notions and facts. Some important geometric implications of cross-curvature non-negativity (Theorem 2.11 and Corollary 2.12) are shown also. In Section 3, the tensor product construction of costs is explained. Section 4 discusses the relation between Riemannian submersion and cross-curvature. Notably, we derive an O'Neill type inequality for cross-curvature (see Theorem 4.5) and show **A3w** implies the heredity of domain convexity for the exponential map. Section 5 shows continuity of optimal maps on all Riemannian submersions of the round sphere, and higher regularity in particular examples such as the complex projective spaces  $\mathbf{CP}^n$ . Section 6 establishes the cross-curvature non-negativity of the standard round sphere.

**Acknowledgements.** We thank Philippe Delanoë, Yuxin Ge, Alessio Figalli, Grégoire Loeper, Neil Trudinger, Cedric Villani and Xu-Jia Wang for useful discussions and timely exchanges of preprints. We are also grateful to the participants of Fields Analysis Working Group Seminar 2006–2008 for the stimulating research environment they helped to create.

## 2. PRELIMINARIES

In this section, we recall basic terminology and facts from [KM] (also [MTW] [TW1] [L1]). Theorem 2.11 and Corollary 2.12 are not stated there, but are easy consequences of the method in [KM], to which we also refer the reader for proofs of the other results summarized below. For improvements to these results, see also [V2]. Let  $N \subset M \times \bar{M}$  be an open set where the cost function  $c \in C^4(N)$ .

**Definition 2.1 (visible sets, twist condition and non-degeneracy).** We define the visible sets:

$$\begin{aligned} N(\bar{x}) &= \{x \in M \mid (x, \bar{x}) \in N\}, \\ \bar{N}(x) &= \{\bar{x} \in \bar{M} \mid (x, \bar{x}) \in N\}. \end{aligned}$$

Throughout this section it is assumed as in [MTW] that for all  $(x, \bar{x}) \in N$ ,

- A1**  $Dc(x, \cdot) : \bar{N}(x) \rightarrow T_x^*M$ ,  $\bar{D}c(\cdot, \bar{x}) : N(\bar{x}) \rightarrow T_{\bar{x}}^*\bar{M}$  are injective;
- A2**  $D\bar{D}c$  is non-degenerate.

We also call **A1** the *(bi-)twist condition* and **A2** *non-degeneracy*.

We recall an important map of Ma, Trudinger & Wang [MTW], called the *cost-exponential* by Loeper [L1], which coincides with the Riemannian exponential map for  $c = \frac{1}{2}\text{dist}^2$ .

**Definition 2.2. (cost exponential)** If  $c \in C^2(N)$  is twisted **(A1)**, we define the *c-exponential* on

$$(2.1) \quad \begin{aligned} \text{Dom}(c\text{-Exp}_x) &:= -Dc(x, \bar{N}(x)) \\ &= \{p^* \in T_x^*M \mid p^* = -Dc(x, \bar{x}) \text{ for some } \bar{x} \in \bar{N}(x)\} \end{aligned}$$

by  $c\text{-Exp}_x p^* = \bar{x}$  if  $p^* = -Dc(x, \bar{x})$ . Non-degeneracy **(A2)** then implies the *c-exponential* is a diffeomorphism from  $\text{Dom}(c\text{-Exp}_x) \subset T_x^*M$  onto  $\bar{N}(x) \subset \bar{M}$ .

**Remark 2.3 (differential of c-Exp).** Linearizing the cost exponential  $\bar{x} = c\text{-Exp}_x p^*$  around  $p^* \in T_x^*M$  we obtain a map  $c\text{-Exp}_{(x, \bar{x})^*} : T_x^*M \rightarrow T_{\bar{x}}\bar{M}$  given explicitly by

$$(2.2) \quad c\text{-Exp}_{(x, \bar{x}(t))^*}(\dot{p}^*(t)) = \dot{\bar{x}}(t) \text{ for } p^*(t) = -Dc(x, \bar{x}(t)).$$

Equivalently  $c\text{-Exp}_{(x, \bar{x})^*} = -\bar{D}Dc(x, \bar{x})^{-1}$ , meaning the inverse tensor  $\frac{1}{2}h^{-1}$  to the metric (1.1) — which gives the pseudo-Riemannian correspondence between the tangent and cotangent spaces to  $N \subset M \times \bar{M}$  — also carries covectors forward through the cost-exponential.

The following lemma characterizes Ma, Trudinger & Wang's *c-segments* as geodesics of  $h$ .

**Lemma 2.4. (the c-segments of [MTW] are geodesics)** *Use a twisted (A1) and non-degenerate (A2) cost  $c \in C^4(N)$  to define a pseudo-metric (1.1) on the domain  $N \subset M \times \bar{M}$ . Fix  $x \in M$ . For each line segment  $(1-s)p^* + sq^* \in \text{Dom}(c\text{-Exp}_x)$ ,  $s \in [0, 1]$ , the curve*

$$s \in [0, 1] \longrightarrow \sigma(s) := (x, c\text{-Exp}_x((1-s)p^* + sq^*))$$

*is an affinely parameterized null geodesic in  $(N, h)$ . Conversely, every geodesic segment in the totally geodesic submanifold  $\{x\} \times \bar{N}(x)$  can be parameterized locally in this way.*

To see some relevant geodesic equations in local coordinates, given  $s_0 \in [0, 1]$ , introduce coordinates on  $M$  and  $\bar{M}$  around  $\sigma(s_0)$  so that nearby, the curve  $\sigma(s)$  can be represented in the form  $(x^1, \dots, x^n, x^{\bar{1}}(s), \dots, x^{\bar{n}}(s))$ . Differentiating the definition of the cost exponential

$$(2.3) \quad 0 = (1-s)p_i^* + sq_i^* + c_i(\sigma(s))$$

twice with respect to  $s$  yields

$$(2.4) \quad 0 = c_{i\bar{j}}\ddot{x}^{\bar{j}} + c_{i\bar{j}\bar{k}}\dot{x}^{\bar{j}}\dot{x}^{\bar{k}}$$

for each  $i = 1, \dots, n$ . This equation will be used later.

Regarding curvature of the metric  $h$ , the following fact is fundamental.

**Lemma 2.5. (Non-tensorial expression for curvature)** *Use a non-degenerate cost  $c \in C^4(N)$  to define a pseudo-metric (1.1) on the domain  $N \subset M \times \bar{M}$ . Let  $(s, t) \in [-1, 1]^2 \rightarrow (x(s), \bar{x}(t)) \in N$  be a surface containing two curves  $\sigma(s) = (x(s), \bar{x}(0))$  and  $\tau(t) = (x(0), \bar{x}(t))$  through  $(x(0), \bar{x}(0))$ . Then  $0 \oplus \dot{\bar{x}}(0)$  defines a parallel vector-field along  $\sigma(s)$ . Moreover, if  $s \in [-1, 1] \rightarrow \sigma(s) \in N$  is a geodesic in  $(N, h)$  then*

$$(2.5) \quad -2 \frac{\partial^4}{\partial s^2 \partial t^2} \Big|_{s=0=t} c(x(s), \bar{x}(t)) = \text{cross}_{(x(0), \bar{x}(0))}(\dot{x}(0), \dot{\bar{x}}(0)).$$

Note in this lemma that only one curve  $\sigma(s)$  needs to be geodesic in  $(N, h)$ . As a consequence of this result, the conditions **A3w/s** can alternately be characterized by the concavity/strong concavity for each  $x \in M$  and  $q_0^* \in \text{Dom}(c\text{-Exp}_x)$  of the function

$$(2.6) \quad q^* \in T_x^*M \rightarrow p^i p^j c_{ij}(x, c\text{-Exp}_x(q_0^* + q^*))$$

restricted to  $q^*$  in the nullspace of  $p \in T_x M$ . Here *strong* concavity refers to negative-definiteness of the Hessian of this function, also called 2-uniform concavity. Cross-curvature nonnegativity asserts this concavity extends to all  $q^*$  (not necessarily in the nullspace of  $p$ ) such that  $q_0^* + q^* \in \text{Dom}(c\text{-Exp}_x)$ .

Before recalling the important geometric implications of the curvature properties of  $h$ , let us define:

**Definition 2.6. (Illuminated set)** Given  $(x, \bar{x}) \in N$ , let  $V(x, \bar{x}) \subset M$  denote those points  $y \in N(\bar{x})$  for which there exists a geodesic curve from  $(x, \bar{x})$  to  $(y, \bar{x})$  in  $N(\bar{x}) \times \{\bar{x}\}$ .

We now state a version of Loeper's maximum principle. Although the statements below are from [KM], where a direct elementary proof is given using pseudo-Riemannian geometry, the fundamental form of the equivalence (Theorem 2.7) below was deduced by Loeper [L1] in the simpler setting  $N = M \times \bar{M} \subset \mathbf{R}^n \times \mathbf{R}^n$  from results of Trudinger and Wang [TW1][TW2]. We visualize his maximum principle as asserting that the *double-mountain*  $\max[f_0, f_1]$  stays *above* the *sliding mountain*  $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$ , hence refer to it by the acronym DASM.

**Theorem 2.7 (A3w  $\Leftrightarrow$  local DASM).** *Let  $h$  be the pseudo-Riemannian metric on  $N \subset M \times \bar{M}$  induced from the non-degenerate cost  $c \in C^4(N)$  as in (1.1). The following are equivalent.*

1.  $(N, h)$  satisfies **A3w**.
2. **(local DASM)** *For any  $h$ -geodesic  $\sigma : t \in [0, 1] \rightarrow (x, \bar{x}(t)) \in N$  and sufficiently small neighbourhood  $U \subset V_\sigma$  of  $x$ , where  $V_\sigma := \cap_{0 \leq t \leq 1} V(x, \bar{x}(t))$  is from Definition 2.6, the sliding mountain  $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$  satisfies the maximum principle*

$$f_t(y) \leq \max[f_0, f_1](y) \quad \text{for } 0 \leq t \leq 1.$$

Then one can prove as in [KM] the following.

**Theorem 2.8 (A3w +  $c$ -convexity of domains  $\Rightarrow$  DASM).** *Let  $h$  be the pseudo-Riemannian metric on  $N \subset M \times \bar{M}$  induced by the non-degenerate cost  $c \in C^4(N) \cap C(M \times \bar{M})$  as in (1.1). Suppose  $(N, h)$  is **A3w**, and the set  $V_\sigma := \bigcap_{0 \leq t \leq 1} V(x, \bar{x}(t))$  from Definition 2.6 is dense in  $M$  for some  $h$ -geodesic  $\sigma : t \in [0, 1] \rightarrow (x, \bar{x}(t)) \in N$ . For any  $y$  in  $M$ , the sliding mountain  $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$  satisfies the maximum principle*

$$(2.7) \quad \text{DASM:} \quad f_t(y) \leq \max[f_0, f_1](y) \quad \text{for } 0 \leq t \leq 1.$$

**Remark 2.9.** In Theorem 2.8, if we further assume that  $(N, h)$  is **A3s**, then the inequality (2.7) becomes strict whenever  $y \neq x$  and  $t \in ]0, 1[$ .

**Remark 2.10 (Relating density of the illuminated set to  $c$ -convexity of the domain).** In case  $N = M \times \bar{M}$ , the density hypothesis of the preceding theorem holds whenever  $\bar{D}c(M, \bar{x}) \subset T_{\bar{x}}^* \bar{M}$  is convex for each  $\bar{x} \in \bar{M}$ , since then  $V_\sigma = M$ . This is the case considered initially by Trudinger and Wang [TW1] and Loeper [L1]. For our argument, it is enough that  $\bar{D}c(M, \bar{x}(t))$  be star-shaped around  $\bar{D}c(x, \bar{x}(t))$  for each  $t \in [0, 1]$ . Conversely,  $V(x, \bar{x}) = M$  for each  $x \in M$  implies convexity of  $\bar{D}c(M, \bar{x})$ .

Our pseudo-Riemannian method [KM] makes it equally possible to deduce further geometric implications of the cross-curvature condition, as the next theorem and corollary show.

**Theorem 2.11 (nonnegative cross-curvature  $\Leftrightarrow$  local time-convex sliding mountain).** *Let  $h$  be the pseudo-Riemannian metric on  $N \subset M \times \bar{M}$  induced from the non-degenerate cost  $c \in C^4(N)$  as in (1.1). The following are equivalent.*

1.  $(N, h)$  is non-negatively cross-curved.
2. For each  $h$ -geodesic  $\sigma : t \in [0, 1] \rightarrow (x, \bar{x}(t)) \in N$  and sufficiently small neighborhood  $U \subset V_\sigma$  of  $x \in U$ , where  $V_\sigma := \bigcap_{0 \leq t \leq 1} V(x, \bar{x}(t))$  is from Definition 2.6, the sliding mountain  $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$  is a convex function of  $t \in [0, 1]$  for each  $y \in U$ , i.e.,  $\frac{\partial^2}{\partial t^2} f_t(y) \geq 0$  for  $0 \leq t \leq 1$ .

*Proof: (1 implies 2).* Fix an arbitrary  $h$ -geodesic  $t : [0, 1] \rightarrow (x, \bar{x}(t)) \in N$ . Let  $U$  be chosen open so that  $x \in U \subset V_\sigma$ . The existence of such a  $U$  is elementary, tedious, and independent of hypothesis 1: it requires checking — at least for  $\delta$  sufficiently small depending on  $T \in [0, 1]$  — that  $\bigcap_{T-\delta \leq t \leq T+\delta} V(x, \bar{x}(t))$  contains a neighbourhood of  $x$ , as we now do. Choose coordinates on  $M$  near  $x$ . Since the geodesic  $\sigma$  is compact in the open set  $N$ , some coordinate ball satisfies  $B_r(x) \times \{\bar{x}(t)\} \subset N$  for all  $t \in [0, 1]$ . Since the coordinate charts  $x \in M \rightarrow \bar{D}c(x, \bar{x}(t)) \in T_{\bar{x}(t)}^* \bar{M}$  are  $C^2$  smooth functions of  $(x, t) \in M \times [0, 1]$ , taking  $r = r(T)$  and  $\delta(T) > 0$  sufficiently small ensure  $\bar{D}c(B_r(x), \bar{x}(t))$  is convex for each  $t$  within  $\delta(T)$  of  $T$ . This convexity implies  $B_r(x) \subset V(x, \bar{x}(t))$ . Extracting a finite subcover  $\bigcup_{i=1}^N (T_i - \delta(T_i), T_i + \delta(T_i))$  of  $[0, 1]$  and taking  $U = \bigcap_{i \leq N} B_{r(T_i)}(x)$  will suffice.

Now define  $f_t(\cdot) := -c(\cdot, \bar{x}(t)) + c(x, \bar{x}(t))$ . Fix arbitrary  $t_0 \in [0, 1]$  and  $y \in U$ . The fact  $U \subset V_\sigma$  guarantees an  $h$ -geodesic  $s : [0, 1] \rightarrow (x(s), \bar{x}(t_0)) \in N$  with  $x(0) = x$  and  $x(1) = y$ . Define an auxiliary function  $g(s) := \frac{\partial^2}{\partial t^2} f_t(x(s))|_{t=t_0}$ ,

which shall be shown to be non-negative for  $s \in [0, 1]$ . By Lemma 2.5 and the cross-curvature non-negativity,

$$(2.8) \quad \frac{d^2 g}{ds^2} \geq 0, \text{ for } s \in [0, 1].$$

In particular,  $g(s)$  is convex. It is clear that  $g(0) = \frac{\partial^2}{\partial t^2} \Big|_{t=t_0} f_t(x) = 0$ . We also claim  $g'(0) = 0$ : introducing coordinates  $x^1, \dots, x^n$  around  $x = x(0)$  on  $M$  and  $\bar{x}^1, \dots, \bar{x}^n$  around  $\bar{x}(t_0)$  on  $\bar{M}$ , we compute

$$(2.9) \quad \begin{aligned} \frac{dg}{ds} \Big|_{s=0} &= -(c_{\bar{i}k}(x(0), \bar{x}(t_0)) \ddot{x}^{\bar{i}} + c_{\bar{i}\bar{j}k}(x(0), \bar{x}(t_0)) \dot{x}^{\bar{i}} \dot{x}^{\bar{j}}) \dot{x}^k \\ &= 0 \end{aligned}$$

by the  $h$ -geodesic equation (2.4) for  $t \in [0, 1] \rightarrow (x, \bar{x}(t)) \in N$ . From (2.8), this shows  $g(s) \geq 0$ , for  $0 \leq s \leq 1$ , in particular at  $s = 1$ ,

$$g(1) = \frac{\partial^2}{\partial t^2} f_t(y) \Big|_{t=t_0} \geq 0.$$

Since we have used  $U \subset V_\sigma$  but not the openness or smallness of  $U$ , we have actually deduced convexity of  $t \in [0, 1] \rightarrow f_t(y)$  for all  $y \in V_\sigma$ . A fortiori,  $1 \implies 2$ .  $\triangle$

*Proof: (2 implies 1).* Fix  $(x, \bar{x}) \in N$ ,  $p \oplus \bar{p} \in T_{(x, \bar{x})}N$ . It shall be shown that  $\text{sec}_{(x, \bar{x})}((p \oplus 0) \wedge (0 \oplus \bar{p})) \geq 0$ . Choose an  $h$ -geodesic  $t \in [-1, 1] \rightarrow (x, \bar{x}(t)) \in N$ , with  $\bar{x}(0) = \bar{x}$ ,  $\dot{\bar{x}}(0) = \bar{p}$ , and a curve  $s \in [-1, 1] \rightarrow (y(s), \bar{x}) \in N$ , with  $y(0) = x$ ,  $\dot{y}(0) = p$ . Let  $f_t(\cdot) := -c(\cdot, \bar{x}(t)) + c(x, \bar{x}(t))$ . Suppose

$$(2.10) \quad g(s) := \frac{\partial^2}{\partial t^2} \Big|_{t=0} f_t(y(s)) \geq 0,$$

for  $s \in [-1, 1]$ : this holds if the property 2 is assumed and the curve  $y(s)$  is chosen inside the neighborhood  $U$  of  $x$  constructed at the outset. Note  $g(0) = \frac{\partial^2}{\partial t^2} \Big|_{t=0} f_t(x) = 0$ . Thus, from (2.10),  $g'(0) = 0$  and

$$\begin{aligned} 0 &\leq \frac{d^2}{ds^2} \Big|_{s=0} g(s) \\ &= \frac{1}{2} \text{cross}_{(x, \bar{x})}(\dot{y}(0), \dot{\bar{x}}(0)). \end{aligned}$$

The last equality comes from Lemma 2.5. This completes the proof  $2 \implies 1$ .  $\square$

**Corollary 2.12 (non-negative cross-curvature +  $c$ -convexity of domains  $\implies$  time-convex sliding mountain).** *Suppose the pseudo-Riemannian metric  $h$  induced by the non-degenerate cost  $c \in C^4(N) \cap C(M \times \bar{M})$  on  $N \subset M \times \bar{M}$  in (1.1) is non-negatively cross-curved. If for some  $h$ -geodesic  $\sigma : t \in [0, 1] \rightarrow (x, \bar{x}(t)) \in N$  the set  $V_\sigma := \cap_{0 \leq t \leq 1} V(x, \bar{x}(t))$  from Definition 2.6 is dense in  $M$ , then for each  $y \in M$ , the sliding mountain  $f_t(y) := -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$  satisfies*

$$(2.11) \quad f_t(y) \leq (1-t)f_0(y) + tf_1(y) \quad \text{for } 0 \leq t \leq 1.$$

*Proof.* For  $y \in V_\sigma$ , inequality (2.11) was established while deriving implication  $1 \implies 2$  of Theorem 2.11. The inequality extends to  $y \in M$  by the density of  $V_\sigma$  and the continuity of  $c$  on  $M \times \bar{M}$ .  $\square$

**Remark 2.13.** The density condition of Theorem 2.8 and Corollary 2.12 can be further weakened by the works of Figalli, Loeper and Villani [FV][LV] (see [V2, Theorem 12.36]).

### 3. TENSOR PRODUCTS OF PSEUDO-RIEMANNIAN METRICS

This section contains the proofs of assertions 3) and 4) in Theorem 1.2. Assertion 3) is actually a corollary of a more general theorem, which does not require the transportation cost  $c$  or manifold  $M$  to be Riemannian:

**Theorem 3.1. (Products preserve non-negative cross-curvature)** *Let  $c_{\pm} \in C^4(N_{\pm})$  be non-degenerate non-negatively cross-curved costs on two manifolds  $N_{\pm} \subset M_{\pm} \times \bar{M}_{\pm}$ . Then  $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$  is non-degenerate and non-negatively cross-curved on  $(x_+, \bar{x}_+, x_-, \bar{x}_-) \in N_+ \times N_-$ , but never **A3s**.*

*Proof.* If the cost functions  $c_{\pm} : N_{\pm} \subset M_{\pm} \times \bar{M}_{\pm} \rightarrow \mathbf{R}$  define non-degenerate metrics  $h_{\pm}$ , then so does the cost  $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$  on  $N = N_+ \times N_-$ , since the corresponding metric  $h$  separates into block anti-diagonal components  $h_{\pm}$  with non-vanishing determinants. In this sense the geometry  $(N, h)$  is the pseudo-Riemannian tensor product of the geometries  $(N_{\pm}, h_{\pm})$ . It follows that the product  $(\gamma_+(s), \gamma_-(s))$  of geodesics  $s \in [-1, 1] \rightarrow \gamma_{\pm}(s)$  in  $N_{\pm}$  is a geodesic in  $N = N_+ \times N_-$ . Lemma 2.5 then implies

$$(3.1) \quad \text{cross}(p_+ \oplus p_-, \bar{p}_+ \oplus \bar{p}_-) = \text{cross}_+(p_+, \bar{p}_+) + \text{cross}_-(p_-, \bar{p}_-)$$

for all  $(x_+, \bar{x}_+, x_-, \bar{x}_-) \in N$  and  $(p_+ \oplus p_-) \oplus (\bar{p}_+ \oplus \bar{p}_-) \in T_{(x_+, \bar{x}_+, x_-, \bar{x}_-)}N$ . The proof is completed by observing that non-negativity of both summands guarantees the same for their sum. Choosing  $p_- = 0$  and  $\bar{p}_+ = 0$  in identity (3.1) demonstrates that the tensor product cost  $c$  cannot satisfy **A3s**.  $\square$

**Proof of Theorem 1.2(3).** Given two Riemannian manifolds  $M_{\pm} = \bar{M}_{\pm}$ , denote their geodesic distances squared by  $c_{\pm}(x_{\pm}, \bar{x}_{\pm}) = \frac{1}{2} \text{dist}_{\pm}^2(x_{\pm}, \bar{x}_{\pm})$  respectively, and let  $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = \frac{1}{2} \text{dist}^2((x_+, x_-), (\bar{x}_+, \bar{x}_-))$  denote the geodesic distance squared on  $M_+ \times M_-$  equipped with the Riemannian product metric. Then  $c(x_+, x_-, \bar{x}_+, \bar{x}_-) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$ , and the non-negative cross-curvature of  $c$  on  $N = N_+ \times N_-$  follows from Theorem 3.1, taking  $N_{\pm}$  to be the domains where  $c_{\pm}$  are smooth. This concludes the proof of Theorem 1.2(3).  $\square$

The assertion 4 in Theorem 1.2 follows easily from the next lemma.

**Lemma 3.2 (Certain cross-curvatures vanish in any Riemannian setting).** *Define the pseudo-metric (1.1) using the cost  $c = d^2/2$  on a Riemannian manifold  $(M, d)$ . Each point  $(x, \bar{x}) \in N = M \times M \setminus \text{cut-locus}$  with  $x \neq \bar{x}$  admits a 2-plane whose cross-curvature vanishes. For example, letting  $t \in [0, 1] \rightarrow \gamma(t) \in M$  be a geodesic from  $x = \gamma(0)$  to  $\bar{x} = \gamma(1)$  yields  $\text{cross}(\dot{\gamma}(0), \dot{\gamma}(1)) = 0$  but*

$$h(\dot{\gamma}(0) \oplus \dot{\gamma}(1), \dot{\gamma}(0) \oplus \dot{\gamma}(1)) = d(x, \bar{x})^2.$$

*Proof.* Any (affinely parameterized) Riemannian geodesic satisfies  $d(\gamma(s), \gamma(t)) = |s - t|d(\gamma(0), \gamma(1))$ . Defining  $f(s, t) = c(\gamma(s), \gamma(t)) = |s - t|^2 d(x, \bar{x})^2 / 2$  allows us to compute

$$(3.2) \quad h(\dot{\gamma}(0) \oplus \dot{\gamma}(1), \dot{\gamma}(0) \oplus \dot{\gamma}(1)) = - \frac{\partial^2 f}{\partial s \partial t} \Big|_{(s,t)=(0,1)}$$

immediately. Moreover, the fact that  $t \in [0, 1] \longrightarrow (x, \gamma(t))$  is an  $h$ -geodesic yields the vanishing of  $\text{cross}(\dot{\gamma}(0), \dot{\gamma}(1)) = -2\partial^4 f / \partial s^2 \partial t^2 \Big|_{(s,t)=(0,1)}$  via Lemma 2.5.  $\square$

**Proof of Theorem 1.2(4).** Without loss of generality, assume  $M_+$  fails to be non-negatively cross-curved. Then  $\text{cross}_+(p_+, \bar{p}_+) < 0$  for some  $(p_+, \bar{p}_+) \in T_{(x_+, \bar{x}_+)} N_+$ . Noting

$$h_+(p_+ \oplus \pm \bar{p}_+, p_+ \oplus \pm \bar{p}_+) = \pm h_+(p_+ \oplus \bar{p}_+, p_+ \oplus \bar{p}_+)$$

but  $\text{cross}_+(p_+, \pm \bar{p}_+) = \text{cross}_+(p_+, \bar{p}_+)$ , we may assume that  $h_+(p_+ \oplus \bar{p}_+, p_+ \oplus \bar{p}_+) < 0$ . Pick any nontrivial geodesic  $\gamma_-$  in  $M_-$ . Noting  $h_-(\dot{\gamma}_-(0) \oplus \dot{\gamma}_-(1), \dot{\gamma}_-(0) \oplus \dot{\gamma}_-(1)) = d^2(\gamma(0), \gamma(1)) > 0$  from (3.2), one can choose  $\lambda \in \mathbf{R}$  so that

$$\begin{aligned} & h((p_+ \oplus \lambda \dot{\gamma}_-(0)) \oplus (\bar{p}_+ \oplus \lambda \dot{\gamma}_-(1)), (p_+ \oplus \lambda \dot{\gamma}_-(0)) \oplus (\bar{p}_+ \oplus \lambda \dot{\gamma}_-(1))) \\ &= h_+(p_+ \oplus \bar{p}_+, p_+ \oplus \bar{p}_+) + \lambda^2 h_-(\dot{\gamma}_-(0) \oplus \dot{\gamma}_-(1), \dot{\gamma}_-(0) \oplus \dot{\gamma}_-(1)) \\ &= 0. \end{aligned}$$

However, from (3.1) and Lemma 3.2,

$$\begin{aligned} & \text{cross}(p_+ \oplus \lambda \dot{\gamma}_-(0), \bar{p}_+ \oplus \lambda \dot{\gamma}_-(1)) \\ &= \text{cross}_+(p_+, \bar{p}_+) + \lambda^4 \text{cross}_-(\dot{\gamma}_-(0), \dot{\gamma}_-(1)) \\ &= \text{cross}_+(p_+, \bar{p}_+) < 0. \end{aligned}$$

This completes the proof that  $(N_+ \times N_-, h_+ \oplus h_-)$  fails to be weakly regular **A3w** unless both  $(N_\pm, h_\pm)$  are non-negatively cross-curved.  $\square$

At present, the authors do not know any Riemannian manifold which is **A3w** yet fails to be non-negatively cross-curved. On the other hand, the proof of Theorem 1.2 (4) adapts to non-Riemannian cost functions as in the next example, where it shows that the tensor product of two (or more) costs on manifolds which are not all non-negatively cross-curved is surely not **A3w**.

**Example 3.3 (A3s  $\times$  A3s  $\not\Rightarrow$  A3w).** Let  $c_\pm(x, \bar{x}) = -\log|x - \bar{x}|$  on  $M_\pm \times M_\pm \setminus \Delta$ ,  $M_\pm = \mathbf{R}^n$ ,  $\Delta := \{(x, x) \mid (x, x) \in M_\pm \times M_\pm\}$ . This logarithmic cost function is known to be **A3s** [MTW] [TW1] but it does not induce non-negatively cross-curved pseudo-metric as indicated, e.g. in [KM, Example 3.5]. To see this fact one uses (2.6), whose righthand-side coincides for this logarithmic cost with

$$p^i p^j f_{ij} \Big|_{(Df)^{-1}(-q_0^* - q^*)} = 2((q_0 + q)_i^* p^i)^2 - |p|^2 |(q_0 + q)^*|^2$$

where  $f(x - \bar{x}) := -\log|x - \bar{x}|$ . Here the righthand-side is strictly convex with respect to  $q^*$  along the Euclidean line parallel to  $p$ , but strictly concave along the

nullspace of  $p$ . As a result  $\text{cross}_\pm(p_\pm, \lambda \bar{p}_\pm) < 0$  for  $p_\pm$  parallel to  $\bar{D}Dc(x, \bar{x})\bar{p}_\pm$  as Euclidean vectors and  $\lambda \neq 0$ . On the other hand,

$$h_+(p_+ \oplus \bar{p}_+, p_+ \oplus \bar{p}_+) + h_-(p_- \oplus \lambda \bar{p}_-, p_- \oplus \lambda \bar{p}_-)$$

vanishes for some non-zero  $\lambda \in \mathbf{R}$ . From (3.1), the pseudo-Riemannian metric  $h$  induced by the cost  $c((x, \bar{x}), (y, \bar{y})) = c_+(x_+, \bar{x}_+) + c_-(x_-, \bar{x}_-)$  on  $(M_+ \times M_-) \times (M_+ \times M_-)$  then fails to be **A3w**.

**Remark 3.4 (tensor product examples, Loeper’s maximum principle, and time-convexity of the sliding mountain).** As mentioned in the introduction, Loeper’s maximum principle (**DASM**) is a key property for the regularity of optimal maps. The conclusions of Theorem 2.8 and Corollary 2.12 (hence **DASM**) hold for the distance squared cost on the Riemannian product of round spheres  $M = \bar{M} = \mathbf{S}^{n_1} \times \cdots \times \mathbf{S}^{n_k} \times \mathbf{R}^l$  — thus also on its Riemannian submersions (see Theorem 4.8). To see this, first note that by the result of present section and Section 6,  $M$  satisfies non-negative cross-curvature on  $N = M \times \bar{M} \setminus \text{cut-locus}$ . The density condition of  $\cap_{0 \leq t \leq 1} V(x, \bar{x}(t))$  is easily checked since the cut locus of one point in this example is a smooth submanifold of codimension greater than or equal to 2. This new global result illustrates an advantage of our method over other approaches [TW1] [TW2] [L1], where one would require a regularity result for optimal maps (or some a priori estimates) to obtain the conclusion of Theorem 2.8. For example, to implement these other approaches for the manifolds of this tensor product example, one would need to establish that an optimal map remains *uniformly away from the cut locus*, as is currently known only for a single sphere  $M = \bar{M} = \mathbf{S}^n$  from work of Delanoë & Loeper [DL] (alternately [L1] or Appendix of [KM]), for the case of perturbations [DG][LV][V3], and for the Riemannian submersion quotients of  $\mathbf{S}^n$  which have no purely focal points (see the discussion in Section 5), including the complex projective spaces  $\mathbf{CP}^n$  with Fubini-Study metric. To the best of our knowledge, no one has yet succeeded in establishing regularity results for this tensor product example, though as mentioned above we expect to resolve this in a subsequent work [FKM2].

#### 4. RIEMANNIAN SUBMERSIONS AND CROSS-CURVATURE

In this section we prove the assertion 2) in Theorem 1.2. The key result is an O’Neill type inequality contained in Theorem 4.5 which compares cross-curvatures in Riemannian submersion. Theorem 4.8 deals with the survival of global properties such as Loeper’s maximum principle (**DASM**) and time-convexity of the sliding mountain, under Riemannian submersion. These results are applied to show the regularity of optimal maps on the Riemannian submersion quotients of the round sphere, for example, the complex projective space  $\mathbf{CP}^n$  with Fubini-Study metric (see Section 5). From now on we focus on the case of manifolds with costs given by strictly convex increasing functions of Riemannian distance (such as  $c = \frac{1}{2} \text{dist}^2$ ). In this case the  $c$ -exponential map coincides with the Riemannian exponential map:  $c\text{-Exp} = \text{exp}$ .

Recall the definition and basic facts of Riemannian submersion.

**Definition 4.1 (Riemannian submersion).** (See [CE].) A surjective differentiable map  $\pi : M \rightarrow B$  from a Riemannian manifold  $M$  onto a Riemannian manifold  $B$  is said to be a *Riemannian submersion* if the following hold:

- $\pi$  is a submersion, i.e.  $d\pi : T_x M \rightarrow T_{\pi(x)} B$  is surjective for each  $x \in M$ ;
- for the orthogonal decomposition  $T_x M = \ker d\pi \oplus (\ker d\pi)^\perp$  for each  $x \in M$ ,  $d\pi|_{(\ker d\pi)^\perp}$  is an isometry.

The subspaces  $\mathcal{V} := \ker d\pi$ ,  $\mathcal{H} := (\ker d\pi)^\perp$  are called *vertical* and *horizontal* subspaces, respectively. For each  $v \in T_b B$ ,  $b \in B$ , there exists its unique *horizontal lift*  $\tilde{v} \in T_x M \cap \mathcal{H}$  for each  $x \in \pi^{-1}(b)$  such that  $d\pi(\tilde{v}) = v$ . We use the metric identifications  $T_b^* B = T_b B$  and  $T_x^* M = T_x M$  to extend the definition of a horizontal lift to cotangent vectors. For each piecewise smooth curve  $\gamma : [0, 1] \rightarrow B$  and  $x \in \pi^{-1}(\gamma(0))$ , there exists its *horizontal lift*  $\tilde{\gamma} : [0, 1] \rightarrow M$  such that  $\gamma(0) = x$ ,  $\pi(\tilde{\gamma}) = \gamma$ ,  $\dot{\tilde{\gamma}}(t) \in \mathcal{H}$ . Moreover,  $\gamma$  is a geodesic if and only if its horizontal lift  $\tilde{\gamma}$  is. If  $\gamma$  is minimal, then so is  $\tilde{\gamma}$ . This property yields, for the horizontal lifts  $\tilde{v} \in T_x M \cap \mathcal{H}$  and  $x \in \pi^{-1}(b)$  of  $v \in T_b B$  and  $b \in B$ ,

$$(4.1) \quad \pi(\exp_x \tilde{v}) = \exp_b v.$$

Regarding Riemannian distance,

$$(4.2) \quad \text{dist}_M(x, y) \geq \text{dist}_B(\pi(x), \pi(y)).$$

We call  $M$  the *total space* of the Riemannian submersion and  $B$  the *base* of the Riemannian submersion or the *Riemannian submersion quotient*.

Many examples of Riemannian submersions may be found in Cheeger & Ebin [CE], Bess [Be] or more recent book by Falcitelli, Ianus and Pastore [FIP]. Every Riemannian covering projection is obviously a Riemannian submersion. Other important examples are Hopf fibrations such as complex and quaternionic projective spaces  $\mathbf{CP}^m$  and  $\mathbf{HP}^m$ , where the standard round sphere  $S^n$  (sectional curvature  $\equiv 1$ ) is the total space.

**Example 4.2 (Hopf fibrations).** (See [Be] pages 257–258.)  $\pi : S^{2m+1} \rightarrow \mathbf{CP}^m$ ,  $\pi : S^{4m+3} \rightarrow \mathbf{HP}^m$ . The base spaces  $\mathbf{CP}^m$  and  $\mathbf{HP}^m$  have real dimensions  $2m$ ,  $4m$ , respectively, and have non-isotropic sectional curvatures  $K$ ,  $1 \leq K \leq 4$ .

Our goal in this section is to compare the cross-curvature of the base space with that of the total space of Riemannian submersion. For this purpose, we use the definition above to assign to each pair of points in the base space a corresponding pair of points horizontally lifted to the total space.

**Definition 4.3 (horizontal lift of a pair of points).** Let  $\pi : M \rightarrow B$  be a Riemannian submersion. For each pair of points  $(x, \bar{x}) \in N_B = B \times B \setminus \text{cut-locus}$ , the unique minimal geodesic  $\gamma : [0, 1] \rightarrow B$  with  $\gamma(0) = x$ ,  $\gamma(1) = \bar{x}$  has its horizontal lift  $\tilde{\gamma}$  with  $\tilde{\gamma}(0) = \tilde{x}$ ,  $\tilde{\gamma}(1) = \tilde{\bar{x}}$ . Then the pair  $(\tilde{x}, \tilde{\bar{x}}) \in N_M = M \times M \setminus \text{cut-locus}$  is said to be a *horizontal lift of*  $(x, \bar{x})$ .

To have a bit more general conclusion in the next theorem, we observe, as in [M]:

**Lemma 4.4 (radial cost-exponential).** *[c.f. [M, Theorem 13]] Let  $f \in C^4(\mathbf{R})$  be a strictly convex smooth function, so that the derivative  $f'$  has an inverse function. Further assume that  $f''(0) > 0$  and  $f'(0) = f'''(0) = 0$ , so that  $f$  is strictly increasing on  $\mathbf{R}_+$ . Let  $M = \bar{M}$  be a Riemannian manifold, and  $c(x, \bar{x}) = f(\text{dist}(x, \bar{x}))$  a cost function defined on  $M \times \bar{M}$ , where  $\text{dist}$  denotes the Riemannian distance. By the properties of  $f$  this cost  $c$  is smooth on  $N_M := M \times M \setminus \text{cut-locus}$  and is **A2** non-degenerate and **A1** bi-twisted. Thus, it induces a pseudo-Riemannian metric  $h_M$  on  $N_M$  as in (1.1). The  $c$ -Exp and  $\text{exp}$  have the relation*

$$(4.3) \quad c\text{-Exp}_x(p^*) = \begin{cases} \exp_x\left(\frac{(f')^{-1}(|p|)}{|p|}p\right) & \text{for } 0 \neq p^* \in \text{Dom } c\text{-Exp}_x, \\ x & \text{for } p^* = 0 \end{cases}$$

where the vector  $p$  and the co-vector  $p^*$  are identified by the Riemannian metric.

*Proof.* For  $f(\rho) = \rho^2$ , this fact is well-known. For general  $f$ , it follows from

$$-Dc(x, \bar{x}) = -\frac{f'(\text{dist}(x, \bar{x}))}{\text{dist}(x, \bar{x})}D\left(\frac{\text{dist}^2}{2}\right)(x, \bar{x}),$$

where the inverse  $(f')^{-1}$  exists for strictly convex function  $f$ . Observe that the conditions **A1** and **A2** together are equivalent to  $c\text{-Exp}_x$  being a diffeomorphism on  $\text{Dom } c\text{-Exp}_x$  for each  $x \in M$ . From (4.3) and Definition 2.1, the properties **A1** **A2** follow. This completes the proof.  $\square$

Now we are ready to show our main theorem in this section.

**Theorem 4.5 (cross-curvature and Riemannian submersion).** *Let  $\pi : M \rightarrow B$  be a Riemannian submersion from  $M$  to  $B$ . Let  $f \in C^4(\mathbf{R})$  be a strictly convex function satisfying  $f''(0) > 0$  and  $f'(0) = f'''(0) = 0$ . Let  $c_M = f \circ \text{dist}_M$  and  $c_B = f \circ \text{dist}_B$  be cost functions defined on  $M \times M$  and  $B \times B$ , where  $\text{dist}$  denotes the Riemannian distance of the corresponding manifold;  $c_M$  and  $c_B$  induce the pseudo-Riemannian metrics  $h_M$  and  $h_B$  on  $N_M := M \times M \setminus \text{cut-locus}$  and  $N_B := B \times B \setminus \text{cut-locus}$  respectively, as in (1.1). Fix  $(x, \bar{x}) \in N_B$  and let  $(\tilde{x}, \tilde{\bar{x}}) \in N_M$  be a horizontal lift of  $(x, \bar{x})$ . Given  $v \oplus \bar{v} \in T_{(x, \bar{x})}N_B$  there exists  $\tilde{w} \oplus \tilde{\bar{w}} \in T_{(\tilde{x}, \tilde{\bar{x}})}N_M$  with  $d\pi_{\tilde{x}}(\tilde{w}) = v$ ,  $d\pi_{\tilde{\bar{x}}}(\tilde{\bar{w}}) = \bar{v}$ , such that*

$$(4.4) \quad h_B(v \oplus \bar{v}, v \oplus \bar{v}) = h_M(\tilde{w} \oplus \tilde{\bar{w}}, \tilde{w} \oplus \tilde{\bar{w}})$$

and

$$(4.5) \quad \text{cross}_{(x, \bar{x})}^{(N_B, h_B)}(v, \bar{v}) \geq \text{cross}_{(\tilde{x}, \tilde{\bar{x}})}^{(N_M, h_M)}(\tilde{w}, \tilde{\bar{w}}).$$

For example, it suffices to take  $\tilde{w}^* = -\bar{D}Dc_M(\tilde{x}, \tilde{\bar{x}})\tilde{\bar{w}} \in T_{\tilde{x}}^*M$  to be the horizontal lift of  $v^* = -\bar{D}Dc_B(x, \bar{x})\bar{v} \in T_x^*B$  and  $\tilde{\bar{w}}^* = -D\bar{D}c_M(\tilde{x}, \tilde{\bar{x}})\tilde{w} \in T_{\tilde{\bar{x}}}^*M$  to be the horizontal lift of  $\bar{v}^* = -D\bar{D}c_B(x, \bar{x})v \in T_{\bar{x}}^*B$ .

*Proof of (4.5).* Let  $(\tilde{x}, \tilde{\bar{x}})$  be a horizontal lift of  $(x, \bar{x}) \in N_B$ , and define  $q^*$  and  $\bar{q}^*$  by  $c_B\text{-Exp}_x q^* = \bar{x}$  and  $c_B\text{-Exp}_{\bar{x}} \bar{q}^* = x$ . Then  $\text{dist}_B(x, \bar{x}) = \text{dist}_M(\tilde{x}, \tilde{\bar{x}})$  and it follows from (4.3) that the horizontal lifts  $\tilde{q}^*$  of  $q^*$  and  $\tilde{\bar{q}}^*$  of  $\bar{q}^*$  satisfy  $c_M\text{-Exp}_{\tilde{x}} \tilde{q}^* = \tilde{\bar{x}}$  and  $c_M\text{-Exp}_{\tilde{\bar{x}}} \tilde{\bar{q}}^* = \tilde{x}$ . To fixed  $v \in T_x B$  and  $\bar{v} \in T_{\bar{x}} B$  correspond  $\bar{v}^* \in T_{\bar{x}}^* B$  and

$v^* \in T_x^*B$  such that  $\bar{v}^* = -D\bar{D}c_B(x, \bar{x})v$  and  $v^* = -\bar{D}Dc_B(x, \bar{x})\bar{v}$ . Equivalently,  $v^* \oplus \bar{v}^* = 2h_B(v \oplus \bar{v}, \cdot)$ .

Now let  $\Sigma : (s, t) \in [-1, 1]^2 \longrightarrow (x(s), \bar{x}(t)) \in N_B$  be the surface given by  $x(s) = c_B\text{-Exp}_{\bar{x}}(\bar{q}^* + s\bar{v}^*)$  and  $\bar{x}(t) = c_B\text{-Exp}_x(q^* + tv^*)$ . By Lemma 2.4, the curves  $\sigma(s) = (x(s), \bar{x})$  and  $\tau(t) = (x, \bar{x}(t))$  through  $(x(0), \bar{x}(0)) = (x, \bar{x})$  are  $h_B$ -geodesics. Lift  $\Sigma$  to  $\tilde{\Sigma} : (s, t) \in [-1, 1]^2 \longrightarrow (\tilde{x}(s), \tilde{\bar{x}}(t)) \in N_M$  by setting  $\tilde{x}(s) = c_M\text{-Exp}_{\tilde{\bar{x}}}(\tilde{q}^* + s\tilde{w}^*)$ , and  $\tilde{\bar{x}}(t) = c_M\text{-Exp}_{\tilde{x}}(\tilde{q}^* + t\tilde{w}^*)$ , where  $\tilde{w}^*$  and  $\tilde{w}^*$  are the horizontal lifts of  $\bar{v}^*$  and  $v^*$  respectively. Thus,  $\tilde{\sigma}(s) = (\tilde{x}(s), \tilde{\bar{x}})$  and  $\tilde{\tau}(t) = (\tilde{x}, \tilde{\bar{x}}(t))$  are  $h_M$ -geodesics, with  $\tilde{\sigma}(0) = (\tilde{x}, \tilde{\bar{x}}) = \tilde{\tau}(0)$ . Moreover,  $\pi(\tilde{x}(s)) = x(s)$  and  $\pi(\tilde{\bar{x}}(t)) = \bar{x}(t)$  from (4.3) and (4.1), so taking  $\tilde{w} := \dot{\tilde{x}}(0)$  and  $\tilde{\tilde{w}} := \dot{\tilde{\bar{x}}}(0)$  yields  $d\pi_{\tilde{x}}(\tilde{w}) = \dot{x}(0) = v$  and  $d\pi_{\tilde{\bar{x}}}(\tilde{\tilde{w}}) = \dot{\bar{x}}(0) = \bar{v}$ . Notice that

$$\begin{aligned} -D\bar{D}c_B(x(0), \bar{x})\dot{x}(0) &= \bar{v}^*, & -\bar{D}Dc_B(x, \bar{x}(0))\dot{\bar{x}}(0) &= v^* ; \\ -D\bar{D}c_M(\tilde{x}(0), \tilde{\bar{x}})\dot{\tilde{x}}(0) &= \tilde{w}^*, & -\bar{D}Dc_M(\tilde{x}, \tilde{\bar{x}}(0))\dot{\tilde{\bar{x}}}(0) &= \tilde{w}^*. \end{aligned}$$

Define an auxiliary function

$$F(s, t) := c_M(\tilde{x}(s), \tilde{\bar{x}}(t)) - c_B(x(s), \bar{x}(t)).$$

From Lemma 2.5, the inequality  $\frac{\partial^4}{\partial s^2 \partial t^2} F(s, t)|_{(0,0)} \geq 0$  will imply (4.5). First, observe from (4.2) and the monotonicity of  $f$  that  $c_B(x(s), \bar{x}(t)) \leq c_M(\tilde{x}(s), \tilde{\bar{x}}(t))$ , thus,

$$(4.6) \quad F(s, t) \geq 0 \quad \text{for } (s, t) \in [-1, 1]^2.$$

We shall verify the desired inequalities by computing the Taylor expansion of  $F$  at  $(0, 0)$  to fourth order. First observe that  $F(s, 0) = 0 = F(0, t)$  for all  $|s|, |t| \leq 1$ , so

$$F(s, t) = f_{11}st + f_{21}s^2t + f_{12}st^2 + f_{31}s^3t + f_{22}s^2t^2 + f_{13}st^3 + O(|s| + |t|)^5$$

as  $|s| + |t| \rightarrow 0$ . Since  $F(s, \pm s) \geq 0$  for  $|s| \leq 1$ , we deduce the vanishing of  $f_{11}, f_{12}$  and  $f_{21}$  in turn, and the inequalities  $f_{31} + f_{22} + f_{13} \geq 0$  and  $f_{22} - f_{31} - f_{13} \geq 0$ . This implies  $f_{22} \geq 0$  as desired. (Although not needed here,  $f_{31} = 0$  follows from  $F(s, s^2) \geq 0$ , and  $f_{13}$  vanishes similarly.) Noting

$$\begin{aligned} f_{11} &= \frac{\partial^2 F}{\partial s \partial t}(0, 0) \\ &= \dot{\tilde{x}}(0)\bar{D}Dc_M(\tilde{x}, \tilde{\bar{x}})\dot{\tilde{\bar{x}}}(0) - \dot{x}(0)\bar{D}Dc_B(x, \bar{x})\dot{\bar{x}}(0) \\ &= -h_M(\dot{\tilde{x}}(0) \oplus \dot{\tilde{\bar{x}}}(0), \dot{\tilde{x}}(0) \oplus \dot{\tilde{\bar{x}}}(0)) + h_B(\dot{x}(0) \oplus \dot{\bar{x}}(0), \dot{x}(0) \oplus \dot{\bar{x}}(0)) \end{aligned}$$

we have established (4.4) en passant to complete the proof.  $\square$

Before stating a corollary of this theorem, let us make a provisional definition which can serve as a strict cross-curvature condition for a Riemannian manifold. Notice from Lemma 3.2 that for each pair of points in a Riemannian manifold there are tangent vectors with zero cross-curvature.

**Definition 4.6 (almost positive cross-curvature).** A Riemannian manifold  $M$  with positive sectional curvature is said to be *almost positively cross-curved* if for

each  $(x, \bar{x}) \in N = M \times M \setminus \text{cut-locus}$  such that  $x \neq \bar{x}$  and  $p \oplus \bar{p} \in T_{(x, \bar{x})}N$ ,

$$(4.7) \quad \text{cross}(p, \bar{p}) \geq 0$$

and the equality holds if and only if  $p$  and  $\bar{p}$  are parallel to the velocity vectors  $\dot{\gamma}(0)$  and  $\dot{\gamma}(1)$ , respectively, for the unique geodesic  $t \in [0, 1] \rightarrow \gamma(t) \in M$  from  $x$  to  $\bar{x}$ .

For example, the standard round sphere is almost positively cross-curved as shown in Section 6.

**Corollary 4.7 (A3w/A3s, non-negative/almost positive cross-curvature survive Riemannian submersion).** *Let  $\pi : M \rightarrow B$  be a Riemannian submersion. If the distance squared cost on  $M$  satisfies **A3w**, **A3s**, non-negative cross-curvature, or almost positive cross-curvature condition, then  $B$  satisfies the same condition, respectively.*

*Proof.* The relevant inequalities for the cross-curvature follow directly from (4.5). Let us consider especially the equality case of almost positive cross-curvature. Assume  $M$  is almost positively cross-curved. Suppose

$$\text{cross}_{(x, \bar{x})}^{(N_B, h_B)}(p, \bar{p}) = 0$$

for  $x \neq \bar{x}$ ,  $(x, \bar{x}) \in N_B$ ,  $p \oplus \bar{p} \in T_{(x, \bar{x})}N_B$ . Lift the unique minimizing geodesic  $\gamma$  linking  $\gamma(0) = x$  to  $\gamma(1) = \bar{x}$  to a horizontal geodesic  $\tilde{\gamma}$  joining  $\tilde{x} = \tilde{\gamma}(0)$  to  $\tilde{\bar{x}} = \tilde{\gamma}(1)$ . There is a unique choice  $\tilde{p} \oplus \tilde{\bar{p}} \in T_{(\tilde{x}, \tilde{\bar{x}})}N_M$  such that each component  $\tilde{p}^*$  and  $\tilde{\bar{p}}^*$  of  $\tilde{p}^* \oplus \tilde{\bar{p}}^* = h_M(\tilde{p} \oplus \tilde{\bar{p}}, \cdot)$  is the horizontal lift of the corresponding component of  $p^* \oplus \bar{p}^* = h_B(p \oplus \bar{p}, \cdot)$ . The preceding theorem asserts  $d\pi_{\tilde{x}}(\tilde{p}) = p$  and  $d\pi_{\tilde{\bar{x}}}(\tilde{\bar{p}}) = \bar{p}$ . Apply (4.5) to get  $\text{cross}_{(\tilde{x}, \tilde{\bar{x}})}^{(N_M, h_M)}(\tilde{p}, \tilde{\bar{p}}) = 0$ . Then by almost positive cross-curvedness of  $M$ , the vectors  $\tilde{p}$  and  $\tilde{\bar{p}}$  are parallel to the velocities  $\dot{\tilde{\gamma}}(0)$  and  $\dot{\tilde{\gamma}}(1)$ , respectively. This implies that the vector projections  $p$  and  $\bar{p}$  are parallel to  $\dot{\gamma}(0) = d\pi_{\tilde{x}}(\dot{\tilde{\gamma}}(0))$  and  $\dot{\gamma}(1) = d\pi_{\tilde{\bar{x}}}(\dot{\tilde{\gamma}}(1))$ , respectively, to complete the proof.  $\square$

Let us now turn to more global aspects of the distance squared cost function under Riemannian submersion. Though **local DASM/local time-convex sliding mountain** are equivalent to **A3w**/nonnegative cross-curvature, respectively (see Theorem 2.7 and Theorem 2.11), their global counterparts **DASM/time-convex sliding mountain** require additional conditions on the geometry of the domain (see Theorem 2.8 and Corollary 2.12). The following theorem, uses a simple comparison of distance to give a direct proof that both Loeper's maximum principle and time-convexity of the sliding mountain survive Riemannian submersion even in the absence of restrictions on domain geometry.

**Theorem 4.8 (Loeper's maximum principle and time-convexity of the sliding mountain survive Riemannian submersion).** *Let  $\pi : M \rightarrow B$  be a Riemannian submersion. Let  $f \in C^4(\mathbf{R})$  be a strictly convex function satisfying  $f''(0) > 0$  and  $f'(0) = f'''(0) = 0$ . Compose it with the Riemannian distance on  $M$  to define a cost function  $c_M = f \circ \text{dist}_M$ . It induces a pseudo-Riemannian metric  $h_M$  on  $N_M := M \times M \setminus \text{cut-locus}$  as in (1.1) which is **A2** non-degenerate*

and **A1** bi-twisted. Similarly  $c_B = f \circ \text{dist}_B$  defines a non-degenerate and bi-twisted pseudo-metric  $h_B$  on  $N_B := B \times B \setminus \text{cut-locus}$ . Suppose that for each  $h_M$ -geodesic of the form  $t \in [0, 1] \rightarrow (\tilde{x}, \tilde{x}(t))$  in  $N_M$ ,  $\tilde{f}_t(\tilde{y}) = -c_M(\tilde{y}, \tilde{x}(t)) + c(x, \tilde{x}(t))$  satisfies (2.7) (or (2.11) respectively) for each  $\tilde{y} \in M$ . Then for each  $h_B$ -geodesic  $t \in [0, 1] \rightarrow (x, \bar{x}(t)) \in N_B$ ,  $f_t(y) = -c_B(y, \bar{x}(t)) + c_B(x, \bar{x}(t))$  satisfies the same inequality for each  $y \in B$ .

*Proof.* Let  $t \in [0, 1] \rightarrow \sigma(t) = (x, \bar{x}(t)) \in N_B$  be an  $h_B$ -geodesic and set  $\bar{x} := \bar{x}(0)$ . Define the sliding mountain  $f_t(\cdot) := -c_B(\cdot, \bar{x}(t)) + c_B(x, \bar{x}(t))$  on  $B$ . Identify tangent vectors with co-tangent vectors by the Riemannian metric. By Lemma 2.4 there exist  $p, q \in T_x B$  such that  $\bar{x}(t) = c\text{-Exp}_x(p + tq)$ . Lift  $p, q$  to horizontal vectors  $\tilde{p}, \tilde{q}$  at  $\tilde{x} \in \pi^{-1}(x)$ . Let  $\tilde{x}(t) = c\text{-Exp}_{\tilde{x}}(\tilde{p} + t\tilde{q})$ . From the Riemannian submersion property and Lemma 4.4,

$$c_B(x, \bar{x}(t)) = f \circ (f')^{-1}(|p + tq|_B) = c_M(\tilde{x}, \tilde{x}(t)) \text{ for all } t \in [0, 1];$$

and  $(\tilde{x}, \tilde{x}(t)) \in N_M$  for  $t \in [0, 1]$ . This last property comes from the fact that Riemannian submersions lift the minimal geodesic from  $x$  to  $\bar{x}(t)$  to the minimal geodesic from  $\tilde{x}$  to  $\tilde{x}(t)$ . Thus,  $t \in [0, 1] \rightarrow (\tilde{x}, \tilde{x}(t)) \in N_M$  gives an  $h_M$ -geodesic. Define  $\tilde{f}_t(\cdot) = -\tilde{c}(\cdot, \tilde{x}(t)) + c(\tilde{x}, \tilde{x}(t))$ . Fix  $t \in [0, 1]$ ,  $y \in B$ . Let  $\gamma : [0, 1] \rightarrow B$  be a geodesic from  $\gamma(0) = \bar{x}(t)$  to  $\gamma(1) = y$ . Let  $\tilde{\gamma}$  be the horizontal lift of  $\gamma$  such that  $\tilde{\gamma}(0) = \tilde{x}(t)$ . Let  $\tilde{y} := \tilde{\gamma}(1) \in \pi^{-1}(y)$ . Notice that

$$c_B(y, \bar{x}(t)) = c_M(\tilde{y}, \tilde{x}(t)); \quad c_B(y, \bar{x}(s)) \leq c_M(\tilde{y}, \tilde{x}(s)) \text{ for all } s \in [0, 1].$$

The last inequality is from (4.2) and the monotonicity of  $f$ . Therefore,

$$(4.8) \quad f_t(y) = \tilde{f}_t(\tilde{y}); \quad \tilde{f}_s(\tilde{y}) \leq \tilde{f}_s(y) \text{ for all } s \in [0, 1].$$

Now, assume  $s \in [0, 1] \rightarrow \tilde{f}_s(\tilde{y})$  is convex. Choosing  $s = 0, 1$ , from (4.8),

$$\begin{aligned} f_t(y) = \tilde{f}_t(\tilde{y}) &\leq (1-t)\tilde{f}_0(\tilde{y}) + t\tilde{f}_1(\tilde{y}) \\ &\leq (1-t)f_0(y) + tf_1(y). \end{aligned}$$

Since  $t \in [0, 1]$  was arbitrary, the same convexity holds for  $t \in [0, 1] \rightarrow f_t(y)$ . The survival of Loeper's maximum principle (**DASM**) follows by a similar argument.  $\square$

For the regularity of optimal maps, the so-called  $c$ -convexity of domains is crucial. In the Riemannian setting this condition corresponds to convexity of domain of the exponential maps. The following theorem addresses the heredity of this condition under Riemannian submersion.

**Theorem 4.9 ( $c$ -convexity survives Riemannian submersion).** *Let  $\pi, M, B, N_M, N_B, f, c_M, c_B$  be given as in Theorem 4.8. Suppose  $\text{Dom}(c\text{-Exp}_{\tilde{x}})$  is convex for each  $\tilde{x} \in M$ . Then,  $\text{Dom}(c\text{-Exp}_x)$  is convex for each  $x \in B$ . If we further assume that  $\text{Dom}(c\text{-Exp}_{\tilde{x}})$  is strictly convex and the cost  $c_M$  satisfies **A3s**, then  $\text{Dom}(c\text{-Exp}_x)$  is strictly convex.*

*Proof.* We first show that convexity is inherited by the submersion. Fix a point  $x \in B$  and two distinct vectors  $p_0, p_1 \in T_x B$  in the boundary  $\partial \text{Dom}(c\text{-Exp}_x)$ . Let  $t \in ]0, 1[$  and  $p_t = (1-t)p_0 + tp_1$ . It is enough to show

$$c_B(x, c\text{-Exp}_x p_t) = f(|p_t|).$$

Suppose to the contrary  $c_B(x, c\text{-Exp}_x p_t) < f(|p_t|)$ . Then, there exists  $p' \neq p_t \in T_x B$  such that  $c\text{-Exp}_x p' = c\text{-Exp}_x p_t$  and  $f(|p'|) = c_B(x, c\text{-Exp}_x p_t)$ . Choose a point  $\tilde{x} \in \pi^{-1}(x)$ . Let  $\tilde{p}_0, \tilde{p}_1$  and  $\tilde{p}_t$  be the horizontal liftings to  $M$  at  $\tilde{x}$  of  $p_0, p_1$  and  $p_t$  respectively. The convexity of  $\text{Dom}(c\text{-Exp}_{\tilde{x}})$  in  $T_{\tilde{x}} M$  implies  $c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}} \tilde{p}_t) = f(|p_t|)$ . Consider the curve  $s \in [0, 1] \mapsto c\text{-Exp}_{\tilde{x}} s p'$ . Lift it horizontally to  $M$  at  $c\text{-Exp}_{\tilde{x}} \tilde{p}_t$ . Let  $\tilde{x}'$  be the other end of this lifted curve so that  $\tilde{x}' \in \pi^{-1}(x)$ . Therefore,

$$c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}} \tilde{p}_i) \geq c_B(x, c\text{-Exp}_x p_i), \quad i = 0, 1.$$

Moreover, the choice of  $p'$  implies  $c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}} \tilde{p}_t) = f(|p'|) = c_B(x, c\text{-Exp}_x p_t)$ . Now, use Loeper's maximum principle **DASM** to see

(4.9)

$$\begin{aligned} -c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}} \tilde{p}_t) + c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}} \tilde{p}_t) &\leq \max_{i=0,1} [-c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}} \tilde{p}_i) + c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}} \tilde{p}_i)] \\ &\leq \max_{i=0,1} [-c_B(x, c\text{-Exp}_x p_i) + c_B(x, c\text{-Exp}_x p_i)] \\ &= 0. \end{aligned}$$

Thus, from the left-hand side of the inequality we see

$$(4.10) \quad f(|p_t|) \leq c_B(x, c\text{-Exp}_x p_t),$$

a contradiction. This finishes the proof for the survival of convexity under Riemannian submersion.

Let us show that strict convexity also survives Riemannian submersion when  $c_M$  is **A3s**. Use the same setting and notation (for  $p_i, \tilde{p}_i, i = 0, 1$  and  $p_t, \tilde{p}_t$ , etc) as before and suppose  $p_t \notin \text{Dom}(c\text{-Exp}_x)$ . As before, there exists  $p' \neq p_t$  for which the same arguments and constructions show (4.10). By Remark 2.9 and hypothesis **A3s**, inequalities (4.9) and (4.10) become strict unless  $\tilde{x}' = \tilde{x}$ . But, by construction of  $\tilde{x}'$ ,  $c_M(\tilde{x}', c\text{-Exp}_{\tilde{x}} \tilde{p}_t) \leq c_M(\tilde{x}, c\text{-Exp}_{\tilde{x}} \tilde{p}_t)$ , which forces  $\tilde{x}' = \tilde{x}$ . On the other hand, the strict convexity of  $\text{Dom}(c\text{-Exp}_{\tilde{x}}) \subset T_{\tilde{x}} M$  implies there is only one minimizing geodesic linking  $\tilde{x}$  to  $c\text{-Exp}_{\tilde{x}} p_t$ , from which we conclude  $\tilde{p}' = \tilde{p}_t$ . Back in the quotient  $B$ , this contradicts to the assumption  $p' \neq p_t$ . This shows  $p_t$  lies in the open set  $\text{Dom}(c\text{-Exp}_x)$ , establishing strict convexity of the latter to complete the theorem.  $\square$

## 5. REGULARITY OF OPTIMAL MAPS ON THE RIEMANNIAN SUBMERSION QUOTIENTS OF THE ROUND SPHERE

In this section, we show continuity of optimal maps (between positively bounded densities (5.1)) on any Riemannian submersion quotient of the round sphere. The smoothness of optimal maps also follows if we further assume nowhere differentiability of the distance function along the cut-locus; this condition is equivalent to asserting that the manifold is not purely focal, which means each pair of points

in the cut locus are connected by two or more distinct length minimizing geodesics. The Riemannian manifolds in Example 4.2 all satisfy this condition, implying smoothness of optimal maps between smooth positive source and target densities, in particular on  $\mathbf{CP}^n$ . In these examples, the domain of the exponential map is the ball of radius  $\frac{\pi}{2}$  in the tangent space, and the conjugate locus coincides with the cut locus. Thus, these manifolds are focal (but, not purely-focal), in contrast to the nonfocal manifolds analyzed by Loeper and Villani in the preprint version of [LV].<sup>1</sup> In the case of  $\mathbf{CP}^n$ , the exponential map gives a submersion of the sphere of radius  $\pi/2$  in  $T_x\mathbf{CP}^n$  onto the cutlocus  $\mathbf{CP}^{n-1}$  of  $x$ , in which the fibre over each point in the base is a topological circle.

From Corollary 4.7 and Theorem 4.8, any Riemannian submersion quotient  $B$  of the round sphere satisfies, in particular, **A3s** and Loeper's maximum principle (**DASM**). By Theorem 4.9, the domain of exponential map  $\text{Dom}(\text{Exp}_x)$  is strictly convex for each  $x \in B$ . Here the domain of exponential map coincides with the special case of (2.1) with  $c = \frac{1}{2}\text{dist}^2$ . We also identify vectors and co-vectors using the Riemannian metric. Exactly the same method as in Appendix E of the arXiv version of [KM], which uses Loeper's argument [L1], shows the continuity of optimal maps for the source and target measures  $\rho, \bar{\rho}$  on  $B$ , assuming there exists  $\lambda > 0$  such that

$$(5.1) \quad \lambda \leq \frac{d\bar{\rho}}{d\text{vol}} \quad \text{and} \quad \frac{d\rho}{d\text{vol}} \leq 1/\lambda,$$

or any of the weaker hypotheses proposed by Loeper [L1]. Let us further assume that the distance function is nowhere differentiable along the cut-locus (which holds on  $\mathbf{CP}^n$  in particular, as explained above). Then the continuity result we obtained also shows that optimal maps stay uniformly away from the cut-locus, enabling one to conclude Hölder continuity as in Appendix E of the arXiv version of [KM] or [L1]; for smoother source and target measures higher regularity may then be obtained using the method of Ma, Trudinger and Wang [MTW] as Loeper did for the sphere [L2].

For completeness we mention that for the case of a covering map (Riemannian submersion with discrete fibers) of the round sphere, it is known that lifting of the measures on  $B$  to the total space  $M$  can be applied to show the regularity of optimal maps using established regularity results [L2] on the round sphere. This was discovered independently by Delanoë and Ge in Appendix C of [DG]. An alternative approach (in the same spirit to our discussion above) to this covering case has also been given by Figalli and Rifford [FR].

## 6. SPHERE IS ALMOST POSITIVELY CROSS-CURVED

In this section we show our final result, namely that the standard round sphere is almost positively cross-curved. This represents a significant advance over Loeper's

---

<sup>1</sup>Remark added in revision: Shortly after we communicated the present manuscript to Loeper and Villani, we learned they had revised [LV] to address **A3s** manifolds which are not purely-focal. Our results of the preceding section establish  $\mathbf{CP}^n$  and  $\mathbf{HP}^n$  to be **A3s**, hence examples of such manifolds.

discovery that the round sphere satisfies **A3s**. Its proof will require the following elementary lemma.

**Lemma 6.1** (Calculus fact). *For  $0 \leq \rho \leq \pi$ , the function*

$$a(\rho) := \sin^2 \rho + \rho \sin \rho - \rho^2(1 + \cos \rho)$$

*satisfies  $a(\rho) \geq 0$ . Moreover  $a(\rho) = 0$  if and only if  $\rho = 0, \pi$ .*

*Proof.* Reparameterize  $\rho := \pi/2 + \arcsin(\lambda)$  by  $|\lambda| < 1$ . Then

$$\begin{aligned} & a(\pi/2 + \arcsin(\lambda)) \\ &= 1 - \lambda^2 + (\pi/2 + \arcsin(\lambda))\sqrt{1 - \lambda^2} - (\pi/2 + \arcsin(\lambda))^2(1 - \lambda). \end{aligned}$$

Define

$$b(\lambda) := \frac{a(\pi/2 + \arcsin(\lambda))}{1 - \lambda}.$$

The assertion holds if  $b(\lambda) > 0$ , for  $|\lambda| < 1$ . From

$$(1 - \lambda)b'(\lambda) = 2 - \lambda + \left(\frac{\pi}{2} + \arcsin \lambda\right)(2\lambda - 1)/(1 - \lambda^2)^{1/2}$$

one can check  $b(-1) = 0$ ,  $b'(1) = 0$ , and  $b'(\lambda) \geq 1$  if  $\frac{1}{2} \leq \lambda \leq 1$ . Moreover,

$$\sqrt{1 - \lambda^2}(1 - 2\lambda)^2 \frac{d}{d\lambda} \left( \frac{(1 - \lambda)\sqrt{1 - \lambda^2}}{1 - 2\lambda} \frac{db}{d\lambda} \right) = 2(1 - \lambda)(1 + \lambda)^2 > 0$$

for  $-1 < \lambda < \frac{1}{2}$ , which shows  $b'(\lambda)$  increases monotonically in this range. Thus  $b'(\lambda)$  and  $b(\lambda)$  both remain positive throughout  $|\lambda| < 1$ , completing the proof.  $\square$

**Theorem 6.2 (Sphere is almost positively cross-curved.).** *The  $n$ -dimensional sphere  $M = S^n$  with the standard round metric (i.e., sectional curvature  $K \equiv 1$ ) is almost positively cross-curved (4.7), a fortiori non-negatively cross-curved.*

*Proof.* This theorem follows from the following nontrivial (and tedious) calculations. Let us first set up the geometric configuration we are going to analyze. Let  $x$  be a point in the round sphere  $S^n$  of diameter  $\pi$ . Fix two unit tangent vectors  $q, w \in T_x S^n$ ,  $|q|, |w| > 0$ . For  $t \in \mathbf{R}$ , with  $|t|$  sufficiently small, let  $r(t)$  be a line in  $T_x S^n$  with  $\dot{r}(t) = q$ ,  $\ddot{r}(t) = 0$ ,  $|r(t)| < \pi$ , where  $\dot{f}, \ddot{f}$  denote the time derivatives  $\frac{d}{dt}f(t)$ ,  $\frac{d^2}{dt^2}f(t)$  of a function  $f$ . Let  $\bar{x}(t)$  be the  $c$ -segment  $\bar{x}(t) := \exp_x r(t)$ . Denote  $\bar{x} = \bar{x}(0)$ ,  $\rho = |r(t)|$  (thus  $0 \leq \rho < \pi$ ), and  $\hat{r} = \frac{r(t)}{|r(t)|}$ . Let  $\langle \cdot, \cdot \rangle$  denote the Riemannian inner product.

To apply Lemma 2.5, define

$$(6.1) \quad H := \text{Hess} \left( \frac{\text{dist}(\cdot, \bar{x}(t))^2}{2} \right) \Big|_x (w, w).$$

To prove (4.7) we will first show  $-\ddot{H} = -\frac{d^2}{dt^2}H \geq 0$ , and then the equality case shall be determined. By continuity we may assume  $0 < \rho < \pi$  without loss of generality. From a standard Riemannian geometry calculation (for example see [DL][L2]), one can show that

$$(6.2) \quad H = |w|^2 - IG,$$

where

$$I := |w|^2 - \langle \hat{r}, w \rangle^2, \quad G := 1 - \frac{\rho \cos \rho}{\sin \rho}.$$

**Step 1: reduction to 2-dimensional case.** One of the key points of the proof is to rearrange the expression of  $-\ddot{H}$  in a clever way to enable further analysis. Before differentiating  $H$ , let us list some preliminary computations in the order of complexity. Define a function

$$g(u) = -\frac{u \cos u}{\sin u}, \quad u \in ]0, \pi[.$$

Then,

$$g'(u)\Big|_{u=\rho} = \frac{1}{\sin \rho} B, \quad g''(u)\Big|_{u=\rho} = \frac{\rho}{\sin^3 \rho} A.$$

where

$$A := \frac{2(\sin \rho - \rho \cos \rho)}{\rho}, \quad B := \frac{\rho - \cos \rho \sin \rho}{\sin \rho}.$$

Here one can check that  $A, B > 0$  for  $0 < \rho < \pi$  and this will be important later. We use these and the identities

$$\dot{\rho} = \langle \hat{r}, q \rangle, \quad \ddot{\rho} = \frac{1}{\rho} (|q|^2 - \langle \hat{r}, q \rangle^2)$$

to do the following differentiations and rearrangements:

$$(6.3) \quad \begin{aligned} G &= \frac{\rho}{2 \sin \rho} A, \\ \dot{G} &= \frac{1}{\sin \rho} B \langle \hat{r}, q \rangle, \\ \ddot{G} &= \frac{\rho}{\sin^3 \rho} A \langle \hat{r}, q \rangle^2 + \frac{1}{\rho \sin \rho} B (|q|^2 - \langle \hat{r}, q \rangle^2), \\ \dot{I} &= \frac{2}{\rho} (-\langle \hat{r}, w \rangle \langle q, w \rangle + \langle \hat{r}, w \rangle^2 \langle \hat{r}, q \rangle), \\ \ddot{I} &= \frac{2}{\rho^2} (4 \langle \hat{r}, q \rangle \langle \hat{r}, w \rangle \langle q, w \rangle - 4 \langle \hat{r}, w \rangle^2 \langle \hat{r}, q \rangle^2 - \langle q, w \rangle^2 + \langle \hat{r}, w \rangle^2 |q|^2). \end{aligned}$$

Key observations here are first,

$$\ddot{G} > 0$$

and second, the quantities  $\dot{I}, \ddot{I}$  are independent of the normal component  $w^\perp$  of  $w$  to the plane  $\Sigma \subset T_x S^n$  generated by  $\hat{r}$  and  $q$ . Let  $w_1 = w - w^\perp$  be the projection of  $w$  to  $\Sigma$ . By separating  $I = |w^\perp|^2 + |w_1|^2 - \langle \hat{r}, w \rangle^2$ , one sees

$$(6.4) \quad \begin{aligned} -\ddot{H} &= \ddot{G}I + 2\dot{G}\dot{I} + G\ddot{I} \\ &= \ddot{G}|w^\perp|^2 - \ddot{H}_1 \\ &\geq -\ddot{H}_1 \end{aligned}$$

where  $H_1$  is the quantity defined by replacing  $w$  in (6.1) with  $w_1$ , thus independent of  $w^\perp$ . Notice that the quantity  $-\ddot{H}_1$  becomes identical to  $-\ddot{H}$  of  $r, q, w_1$  viewed

as tangent vectors of the 2-dimensional round sphere that is the exponential image of  $\Sigma$  in the original sphere. This reduces the consideration to two dimensions.

We will need the following key expression obtained by (6.3) and rearrangement:

$$(6.5) \quad -\ddot{H}_1 = \frac{1}{\rho \sin \rho} \left\{ \left[ A \frac{\rho^2}{\sin^2 \rho} \langle \hat{r}, q \rangle^2 + B(|q|^2 - \langle \hat{r}, q \rangle^2) \right] (|w_1|^2 - \langle \hat{r}, w_1 \rangle^2) \right. \\ \left. + 4(B - A)(\langle \hat{r}, q \rangle^2 \langle \hat{r}, w_1 \rangle^2 - \langle \hat{r}, q \rangle \langle \hat{r}, w_1 \rangle \langle q, w_1 \rangle) \right. \\ \left. + A(\langle \hat{r}, w_1 \rangle^2 |q|^2 - \langle q, w_1 \rangle^2) \right\}.$$

Note that

$$(6.6) \quad B - A = \frac{\rho^2 + \rho \sin \rho \cos \rho - 2 \sin^2 \rho}{\rho \sin \rho} > 0 \quad \text{for } 0 < \rho < \pi,$$

as can be checked by taking the fourth-order derivative of the numerator. At this point, Loeper's result in [L2] that  $S^n$  is **A3s** can be obtained by substituting  $\langle q, w \rangle = \langle q, w_1 \rangle = 0$  into the expression (6.5) and using the second line of (6.4).

**Step 2: two dimensional case.** From (6.4) it suffices to show  $-\ddot{H}_1 \geq 0$ . From now on, we assume without loss of generality the dimension is two, and let  $\hat{r} = (0, 1)$ ,  $q = (\cos \theta, \sin \theta)$ ,  $w_1 = (\cos \psi, \sin \psi)$  in  $\mathbf{R}^2 \cong T_x S^2$ , with  $0 \leq \theta, \psi \leq 2\pi$ . Let  $T := \tan \theta$ ,  $S := \tan \psi$ ,  $-\infty \leq T, S \leq +\infty$ . One checks from (6.5),

$$(6.7) \quad -\ddot{H}_1 = \frac{\cos^2 \theta \cos^2 \psi}{\rho \sin \rho} P,$$

where

$$P = A S^2 - 2(2B - A) T S + A \frac{\rho^2}{\sin^2 \rho} T^2 + B - A.$$

Thus it suffices to show  $P > 0$ .  $P$  is a convex (since  $A > 0$ ) quadratic polynomial in  $S$  with discriminant

$$D := 4(2B - A)^2 T^2 - 4A \left( A \frac{\rho^2}{\sin^2 \rho} T^2 + B - A \right) \\ = 4 \left\{ ((2B - A)^2 - A^2 \frac{\rho^2}{\sin^2 \rho}) T^2 - A(B - A) \right\}.$$

We show  $D < 0$  (regardless of  $T$ ), which implies  $P > 0$ . Since  $A(B - A) > 0$ ,  $D$  is always negative if

$$0 \geq (2B - A)^2 - A^2 \frac{\rho^2}{\sin^2 \rho} \\ = \left( 2B - A + A \frac{\rho}{\sin \rho} \right) \left( 2B - A - A \frac{\rho}{\sin \rho} \right).$$

The first factor is positive, and the second factor is negative since

$$2B - A - A \frac{\rho}{\sin \rho} = -\frac{2}{\rho \sin \rho} a(\rho),$$

where

$$a(\rho) := \sin^2 \rho + \rho \sin \rho - \rho^2(1 + \cos \rho)$$

which is positive from Lemma 6.1 (since  $0 < \rho < \pi$ ). This establishes the desired inequality (4.7).

**Step 4: equality case.** Let us analyze the cases of equality, to conclude the almost positive cross-curvature (4.7) of  $S^n$ . We only need to show for  $0 < \rho < \pi$  that  $-\ddot{H} = 0$  holds if and only if the three vectors  $q$ ,  $w$ ,  $\hat{r}$  at  $T_x S^n$  are all parallel. The necessity is easy to verify. For sufficiency, suppose  $-\ddot{H} = 0$ . From (6.4),  $w^\perp = 0$ . Thus it reduces to two dimensional case as in Step 2. Now, from (6.7) and  $P > 0$ ,  $\cos \theta \cos \psi = 0$ . Thus either  $q$  or  $w$  is parallel to  $\hat{r}$ . In either case examining with (6.5) shows the other vector is also parallel to  $\hat{r}$ . This establishes almost positivity of the cross-curvature of  $S^n$ .  $\square$

## REFERENCES

- [Be] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
- [Br] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44** (1991) 375–417.
- [Ca1] L. A. Caffarelli, The regularity of mapping with a convex potential. *J. Amer. Math. Soc.* **5** (1992) 99–104.
- [Ca2] L. A. Caffarelli, Boundary regularity of maps with convex potentials II, *Ann. Math.* **144** (1996) 453–496.
- [CE] J. Cheeger and D. G. Ebin, Comparison theorems in Riemannian geometry, North-Holland Mathematical Library, vol. 9, North-Holland, Amsterdam; American Elsevier, New York, 1975.
- [CNM] P.-A. Chiappori, R.J. McCann, and L. Nesheim, Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness, to appear in *Econom. Theory*.
- [Co] D. Cordero-Erausquin, Sur le transport de mesures périodiques, *C.R. Acad. Sci. Paris Sr. I Math.* **329** (1999) 199–202.
- [D1] P. Delanoë, Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator, *Ann. Inst. Henri Poincaré-Anal. Non Lin.* **8** (1991) 443–457.
- [D2] P. Delanoë, Gradient rearrangement for diffeomorphisms of a compact manifold *Diff. Geom. Appl.* **20** (2004) 145–165.
- [DG] P. Delanoë and Y. Ge, Regularity of optimal transportation maps on compact, locally nearly spherical, manifolds, to appear in *J. Reine Angew. Math.*
- [DL] P. Delanoë and G. Loeper, Gradient estimates for potentials of invertible gradient mappings on the sphere, *Calc. Var. Partial Differential Equations* **26** (3) (2006) 297–311.
- [F] A. Figalli, Existence, uniqueness, and regularity of optimal transport maps, *SIAM J. Math. Anal.* **39** (1) (2007) 126–137.
- [FIP] M. Falcitelli, S. Ianus and A.M. Pastore, *Riemannian Submersions and Related Topics*, World. Scientific, Singapore, 2004.
- [FKM1] A. Figalli, R. J. McCann and Y.-H. Kim, Continuity and injectivity of optimal maps for non-negatively cross-curved costs, preprint at [www.math.toronto.edu/mccann](http://www.math.toronto.edu/mccann)
- [FKM2] A. Figalli, R. J. McCann and Y.-H. Kim, Regularity of optimal transport maps: multiple products of spheres, in preparation.
- [FL] A. Figalli and G. Loeper,  $C^1$ -regularity of solutions of the Monge-Ampère equation for optimal transport in dimension two, *Calc. Var. Partial Differential Equations* **35** (4) (2009) 537–550.
- [FR] A. Figalli and L. Rifford, Continuity of optimal transport maps on small deformations of  $S^2$ , to appear in *Comm. Pure Appl. Math.*

- [FV] A. Figalli and C. Villani, An approximation lemma about the cut locus, with applications in optimal transport theory. *Methods Appl. Anal.* **15** (2) (2008) 149–154.
- [K] Y.-H. Kim, Counterexamples to continuity of optimal transportation on positively curved Riemannian manifolds, *Int. Math. Res. Not. IMRN* Art. ID **rnn120** (2008) 15 pp.
- [KM] Y.-H. Kim and R. J. McCann, Continuity, curvature, and the general covariance of optimal transportation, to appear *J. Eur. Math. Soc. (JEMS)*. (An older version at arXiv:0712.3077).
- [L] J. Liu, Hölder regularity of optimal mappings in optimal transportation, *Calc Var. Partial Differential Equations* **34** (2009) 435–451.
- [LTW] J. Liu, N.S. Trudinger and X.-J. Wang, Interior  $C^{2,\alpha}$  regularity for potential functions in optimal transportation, To appear in *Comm. Partial Differential Equations*.
- [L1] G. Loeper, On the regularity of solutions of optimal transportation problems, *Acta Math.* **202** (2) (2009) 241–283.
- [L2] G. Loeper, On the regularity of solutions of optimal transportation problems II: the sphere case and the reflector antenna, to appear in *Arch. Ration. Mech. Anal.*
- [LV] G. Loeper and C. Villani, Regularity of optimal transport in curved geometry: the nonfocal case, to appear in *Duke Math. J.*
- [M] R. J. McCann, Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* **11** (2001) 589–608.
- [MTW] X.-N. Ma, N. Trudinger, and X.-J. Wang, Regularity of potential functions of the optimal transport problem, *Arch. Ration. Mech. Anal.* **177** (2) (2005) 151–183.
- [S] K. Th. Sturm, *Convex functionals of probability measures and nonlinear diffusions on manifolds*, *J. Math. Pures Appl.* **84** (9) (2005) 149–168.
- [TW1] N. Trudinger and X.-J. Wang, On the second boundary value problem for Monge-Ampère type equations and optimal transportation, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8** (1) (2009) 143–174.
- [TW2] N. Trudinger and X.-J. Wang, On strict convexity and continuous differentiability of potential functions in optimal transportation, *Arch. Ration. Mech. Anal.* **192** (3) (2009) 403–418.
- [U] J. Urbas, On the second boundary value problem for equations of Monge-Ampère type, *J. Reine Angew. Math.* **487** (1997) 115–124.
- [V1] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics, vol. 58. American Mathematical Society, Providence, RI, 2003.
- [V2] C. Villani, *Optimal Transport, Old and New*, Grundlehren der Mathematischen Wissenschaften 338. Springer-Verlag, Berlin, 2009.
- [V3] C. Villani, Stability of a 4th-order curvature condition arising in optimal transport theory, *J. Funct. Anal.* **255** (9) (2008) 2683–2708.

YHK AND RJM: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO CANADA M5S 2E4

YHK: *current address* DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER BC CANADA V6T 1Z2

*E-mail address:* YHK: yhkim@math.ubc.ca, RJM: mccann@math.toronto.edu