CALIBRATING OPTIMAL TRANSPORTATION: A NEW PSEUDO-RIEMANNIAN GEOMETRY

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ABSTRACT. Given a transportation cost $c : M \times \bar{M} \to \mathbf{R}$, optimal maps minimize the total cost of moving masses from M to $\bar{M}$. We find a pseudo-metric and a calibration form on $M \times \bar{M}$ such that the graph of an optimal map is a calibrated maximal submanifold. We define the mass of space-like currents in spaces with indefinite metrics.

1. INTRODUCTION

The aim of this article is to adapt the notion of calibration (see [HL]) to a pseudo-Riemannian framework constructed to describe and explore the geometry of optimal transportation from a new perspective.

Given a smooth function $c : M \times \bar{M} \to \mathbf{R}$ (called the transportation cost), and probability densities ρ and $\bar{\rho}$ on two manifolds M and $\bar{M}$ (possibly with boundary), a natural variational problem is to find an optimal map $F : M \to \bar{M}$ that minimizes the total cost

$$\int_M c(x, F(x))\rho(x)dx \tag{1.1}$$

under the constraint that for every measurable function f on $\bar{M}$,

$$\int_M f(F(x))\rho(x)dx = \int_{\bar{M}} f(\bar{x})\bar{\rho}(\bar{x})d\bar{x}. \tag{1.2}$$

The last condition will be denoted by $F_{\#}\rho = \bar{\rho}$, in which case we say F *pushes forward* ρ to $\bar{\rho}$. This variational problem, called optimal transportation, dates back to Monge in 19th century and is currently undergoing a rapid and broad development, especially in relation to geometry; see Villani's recent book and the references it contains [V]. Our central result can be stated imprecisely as follows

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Theorem 1.1. *Given certain assumptions on the cost function and mass densities c, ρ and $\bar{\rho}$, there exist a pseudo-metric on $M \times \bar{M}$, depending on c, ρ and $\bar{\rho}$, such that the graph of the optimal map is a stable maximal surface.*

A new geometric aspect of optimal transportation was observed by the first two authors [KM]. Namely, the transportation cost c induces a certain pseudo-metric on $M \times \bar{M}$, (4.1), in such a way that positivity of its Riemannian curvature tensor on certain sections gives a necessary condition for the regularity of general optimal maps F. This result gives a geometric perspective on the fundamental regularity theory developed for optimal maps by Ma, Trudinger and Wang [MTW] [TW] and Loeper [L]. Moreover, the graph of the optimal map F is a Lagrangian submanifold with respect to the Kähler form of the pseudo-metric (see Section 4 or [KM] for more details).

On the other hand, the third author [W] showed that the graph of the gradient map of a convex function ϕ that solves the standard Monge-Ampère equation

$$\det D^2\phi = 1,$$

gives a calibrated submanifold in the product space $\mathbf{R}^n \times \mathbf{R}^n$ equipped with a pseudo-Euclidean metric.

In this paper, we provide a framework that combines and extends these results by providing another pseudo-metric defined in (4.5), which is conformally equivalent to the pseudo-metric in [KM] but depends on ρ and $\bar{\rho}$ in addition to c. With respect to this metric, the graphs of optimal maps are calibrated, thus give maximal submanifolds (see Theorem 4.1 and Corollary 4.2). This demonstrates how the functional extremality of optimal maps for (1.1)–(1.2) characterizes in a natural way the geometric extremality of their graphs.

To make Theorem 1.1 more precise, we adapt some standard assumptions on the cost and mass densities, in order to draw conclusions where the cost function is well behaved. Our assumptions imply reasonably nice behavior of the solutions, but certainly do not guarantee smoothness. Allowing for nonsmooth solutions, we will use the language of currents. The behavior of solutions to the problem of optimal transportation for general measures on general metric spaces can be quite wild. It is unclear how much of this theory can be adapted in the more general cases.

Our use of currents will require some apparently new definitions of mass for currents in a pseudo-Riemannian manifold, see definition 2.1. While the corresponding notions of mass from Euclidean geometric measure theory are expressed in terms of suprema, we will need to express mass in terms of infima, and will also need some sort of space orientation for these definitions to make sense. If these are reasonable definitions, they should recover the volume of a smooth space-like surface, which we verify by showing that any space-like plane is calibrated, see Proposition 2.1.

After laying down some pseudo-Riemannian geometric measure theory in section 2, we will define calibrations on spaces with pseudo-metrics. In the final section we show how the optimal transportation problem fits into this setting. The algebraic proof of Proposition 2.1 is in the appendix.

2. Spacelike Currents in an Indefinite (pseudo) Metric

We formulate some definitions which adapt the geometric measure theoretic notion of *mass* to oriented manifolds with indefinite metric. This notion will allow us to compare the mass of calibrated currents to homologous currents in Section 4. Let $(N^{n,m}, h, \tau)$ denote a smooth $(n+m)$-dimensional manifold, a metric with signature (n, m) and a space orientation n-form τ. The "unit sphere" of n-planes in an indefinite metric will have distinct connected components, since each such plane comes with two orientations distinguished by the sign of the space orientation form τ (see Proposition 5.1 in the appendix).

In the following, a *current* shall mean a de Rham n-current with compact support on N, that is, an element of the dual space $(\Omega^n N)^*$ of the space $(\Omega^n N)$ of smooth n-forms on N. Recall that a compact oriented n-submanifold defines a linear functional on the space of n-forms by integration, thus a nonsmooth submanifold with enough tangent spaces (i.e. with negligible singular set) is also a current, in an obvious way. Recall also that a plane P is simple if it is a single product of n vectors, i.e. $P = v_1 \wedge \ldots \wedge v_n$. If $h|_P > 0$ the plane is spacelike, in which case the n-dimensional h-volume of the parallelepiped formed by the generating vectors is denoted by

$$\|P\|_h := \sqrt{\det(h(v_i, v_j))_{1\leq i,j\leq n}}. \tag{2.1}$$

Finally, the support of current T is given by the following: Let U be the largest open set having the property that if φ is compactly supported inside U, then $T(\varphi) = 0$. The *support* of T, $\mathrm{supp}(T)$, is the complement of U.

Definition 2.1 (Mass of a current; comass of a form)**.** Define the set of simple τ*-oriented* space-like unit n-planes by

$$\mathcal{P}_x = \{P \in \Lambda^n T_x N |\ P \text{ simple}, \tau(P) > 0, h|_P > 0, \|P\|_h = 1\}.$$

Define the *oriented comass* of an n-form ψ_x at a point via

$$\|\psi_x\|_h^* = \inf_{P\in\mathcal{P}_x} \psi_x(P).$$

On any set $U \subset N$, define the *oriented comass* of a n-form ψ on U as

$$\|\psi\|_{(U,h)}^* = \inf_{x\in U} \|\psi_x\|_h^* .$$

Now define the *oriented mass* of a current $T \in (\Omega^n U)^*$ by

$$\|T\|_{(U,h)} = \inf_{\|\psi\|_{(U,h)}^* \geq 1} T(\psi).$$

Some remarks about the definitions: First, since the space $\mathcal{P}_x$ is noncompact, we observe that the values of all of the infima in Definition 2.1 may be $-\infty$. Time-like or negatively oriented planes are given infinite weight, thus any current with enough time-like or negatively oriented planes is given a mass of $-\infty$. Fortunately, this rules out certain pathologies that occur for pseudo-metrics (see examples in [W]), and recovers expected values of mass for smooth space-like sets.

For Riemannian (thus positive definite) metrics, a calibration is a closed p-form Ψ such that for all p-planes P, $\Psi_x(P) \leq \|P\|$ (see [HL]). A calibrated current S is one for which $S(\Psi) = \|S\|$. It follows from Stokes' theorem that smooth calibrated manifolds are minimal. In a pseudometric, a calibration needs to give an opposite inequality. To be precise, we define:

Definition 2.2 (Calibration for indefinite metric)**.** A calibration on $(N^{n,m}, h, \tau)$ is an n-form Ψ such that

$$d\Psi = 0$$

and

$$\Psi_x(P_x) \geq \|P_x\|_h$$

at each point x, for each τ-oriented space-like tangent plane P_x. Notice that this latter condition is equivalent to $\|\Psi\|_h^* \geq 1$.

The following proposition, whose proof is in the appendix, is used to verify that the volume of any smooth n-dimensional oriented space-like submanifold is realized as the mass of the corresponding current.

Proposition 2.1. *Let h be an signature (n, m) metric on $N^{n,m}$. Every spacelike n-plane is calibrated. Namely, for each $x \in N$, and for each space-like n-plane $P \in \mathcal{P}_x$, there exists an n-form $\psi \in \Omega_x^n N$ such that $\psi(Q) \geq \|Q\|_h$ for any $Q \in \mathcal{P}_x$ with equality when and only when $P = Q$.*

This proposition is used to show:

Proposition 2.2. *Let S be a smooth compact n-dimensional space-like submanifold. Let $\mathrm{vol}_h(S)$ denote the volume of S with respect to the pseudo-metric h. Then, $\mathrm{vol}_h(S) = \|S\|_h$.*

Proof. It is clear that the form Φ defined in Proposition 2.1 depends smoothly on the tangent space to S. Thus we can find a form $\psi \in \Omega^n S$ such that for each $x \in S$, $\psi_x(Q) \geq \|Q\|_h$ for all $Q \in \mathcal{P}_x$ with equality when and only when Q is the unit oriented tangent plane to S. Extend this smoothly to a form ψ_0 which has positive comass on a neighborhood V_0 of S. Dividing by the pointwise comass at each point where the comass is positive, we have that $\|\psi_0\|^*_{(V_0,h)} = 1$. Next, take a cover of N with open sets V_i, so that $V_i \cap S = \emptyset$, for $i \geq 1$, and so that on each V_i, there is a ψ_i with $\|\psi_i\|^*_{(V_i,h)} \geq 1$. Such V_i are easily locally available. Noting that the comass is superadditive, we may sum over a partition of unity and get a form, which we call ψ, with $\|\psi\|^*_{(N,h)} \geq 1$. Also, for any φ with $\|\varphi\|_h^* \geq 1$, $S(\varphi) \geq S(\psi)$, by the choice of ψ. It follows that

$$\inf_{\|\varphi\|_h^* \geq 1} S(\varphi) = \inf_{\|\varphi\|_h^* \geq 1} \int_S \varphi \geq \int_S dV = \mathrm{vol}_h(S).$$

while on the other hand,

$$S(\psi) = \int_S \psi = \mathrm{vol}_h(S).$$

This completes the proof. ∎

From the above proposition, it is legitimate to use the notion of mass of currents to make volume comparison between smooth spacelike submanifolds.

We are now ready to state the main result of this section. For the Riemannian analogue, see ([HL], Theorem 4.2, p. 59).

Corollary 2.3 (Maximality of calibrated currents)**.** *Let $T \subset N$ be an n-current, which satisfies $T(\Phi) = \|T\|_h$, for some calibration Φ. Then $\|T\|_h \geq \|S\|_h$ for any S homologous to T. In particular, if T is a smooth submanifold, then its mean curvature (with respect to h) vanishes.*

Proof. Using the fact that Φ is closed and S is homologous to T, $T(\Phi) = S(\Phi)$. But since Φ has comass 1, $S(\Phi) \geq \|S\|_h$ by the definition of mass. If the mean curvature does not vanish, then one could perform a compactly supported deformation preserving homology, smoothness and space-like qualities while increasing the volume infinitesimally, contradicting the maximality. □

3. Calibrations for Pseudo-Riemannian product spaces

In this section, we discuss calibrations for some pseudo-Riemannian spaces of product type. In particular, we define a calibration form Φ in Proposition 3.1 that is crucial in Section 4. Let $(M^n \times \bar{M}^n, h, \tau, \omega)$ be an oriented manifold with metric of the following block form

$$h(x, \bar{x}) = \frac{1}{2} \begin{pmatrix} 0 & A^T(x, \bar{x}) \\ A(x, \bar{x}) & 0 \end{pmatrix}, \tag{3.1}$$

for $(x, \bar{x}) \in M \times \bar{M}$, and a symplectic form

$$\omega(x, \bar{x}) = \frac{1}{2}\kappa(x, \bar{x}) \begin{pmatrix} 0 & A^T(x, \bar{x}) \\ -A(x, \bar{x}) & 0 \end{pmatrix}, \tag{3.2}$$

for some positive $\kappa(x, \bar{x})$, and a space orientation n-form τ that is locally expressed as

$$\tau = dx^1 \wedge \cdots \wedge dx^n,$$

with $\tau(x, \bar{x}) > 0$. Here, we use local coordinates $x^1, \cdots, x^n$, $\bar{x}^1, \cdots, \bar{x}^n$ for M, $\bar{M}$, respectively.

We will say a space-like plane P is τ-oriented if $\tau(P) > 0$. The following is the main result of this section.

Proposition 3.1. *Let M,$\bar{M}$ be equipped with volume forms $\rho = \rho(x)dx^1 \wedge \cdots \wedge dx^n$, $\bar{\rho} = \bar{\rho}(\bar{x})d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^n$, respectively. Suppose that $\sqrt{\det h(x, \bar{x})} = \rho(x)\bar{\rho}(\bar{x}) > 0$ along $M \times \bar{M}$. Define an n-form Φ in local expression as*

$$\Phi = \frac{1}{2}(\rho(x)dx^1 \wedge \cdots \wedge dx^n + \bar{\rho}(\bar{x})d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^n). \tag{3.3}$$

This form Φ is a calibration, that is, for any τ-oriented space-like n-plane P, we have

$$\|P\|_h \leq \Phi(P).$$

Further, equality holds if only if P is Lagrangian (i.e. $\omega\big|_P \equiv 0$) and

$$\rho(x)dx^1 \wedge \cdots \wedge dx^n|_P = \bar\rho(\bar x)d\bar x^1 \wedge \cdots \wedge d\bar x^n|_P.$$

Proof. It is clear that Φ is closed. Let $E_\alpha = (e_\alpha, \bar e_\alpha) \in T_xM \times T_{\bar x}\bar M$, $\alpha = 1, \cdots n$, be a basis for P. The metric on P with respect to E_α is then given by

$$h(E_\alpha, E_\beta) = \frac{1}{2}\left(\langle Ae_\alpha, \bar e_\beta\rangle + \langle A^T\bar e_\alpha, e_\beta\rangle\right)$$

where $\langle,\rangle$ is the standard Euclidean dot product. Now let $B_{\alpha\beta} = \langle Ae_\alpha, \bar e_\beta\rangle$, that is

$$B = \begin{pmatrix} \bar e_1 \\ \vdots \\ \bar e_n \end{pmatrix} A \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}$$

then it is easy to see that

$$\det B = \det A|e_1 \wedge \cdots \wedge e_n|\ |\bar e_1 \wedge \cdots \wedge \bar e_n|$$

where $|\cdot|$ is the norm defined on n-vectors with respect to the underlying Euclidean metric for our choice of local coordinates. Recall the fact (see [W], Lemma 3.1) that for each $n \times n$ matrix B with property $\langle Bv, v\rangle \geq 0$ for every v,

$$\det(\frac{1}{2}(B + B^t)) \leq \det B$$

and the equality holds if and only if $B = B^t$. From this, we get

$$\begin{aligned}
\|E_1 \wedge \cdots \wedge E_n\|_h &= \sqrt{\det(\frac{1}{2}(B + B^t))} \\
&\leq \sqrt{\det B} \\
&= \sqrt{\rho(x)\bar\rho(\bar x)|e_1 \wedge \cdots \wedge e_n|\ |\bar e_1 \wedge \cdots \wedge \bar e_n|} \\
&\leq \frac{1}{2}(\rho(x)|e_1 \wedge \cdots \wedge e_n| + \bar\rho(\bar x)|\bar e_1 \wedge \cdots \wedge \bar e_n|) \\
&= \Phi(E_1 \wedge \cdots \wedge E_n),
\end{aligned}$$

using

$$|\det A| = \sqrt{\det h(x, \bar x)} = \rho(x)\bar\rho(\bar x)$$

and the Cauchy-Schwarz inequality, respectively. It is clear that equality holds in the above chain of inequalities if and only if $B = B^t$, that is, P is Lagrangian, and $\rho(x)|e_1 \wedge \cdots \wedge e_n| = \bar\rho(\bar x)|\bar e_1 \wedge \cdots \wedge \bar e_n|$. This completes the proof. □

4. Application to Optimal Transportation

In this section, the results in the previous sections are combined and applied to optimal transportation. We define a pseudo-metric $h^{\rho,\bar\rho}$ in (4.5) and then show the graph of an optimal map is calibrated by the form Φ of (3.3).

For n-dimensional manifolds M and $\bar M$, let $c : N \subset M \times \bar M \to \mathbf{R}$ be a continuous cost function which is smooth almost everywhere, except on a set $\mathfrak{C} = M \times \bar M - N$ which we will call the "cut locus." (The reason for this terminology is clear if we use the distance squared function on a manifold as the cost function.) Let $D\bar{D}c$ be the $n \times n$ matrix given by

$$\left(D\bar{D}c\right)_{i\bar{j}}(x,\bar{x}) = \frac{\partial^2}{\partial x^i \partial \bar{x}^j} c(x,\bar{x}).$$

On N we will require that

$$\det\left(D\bar{D}c\right) \neq 0 \tag{A2}$$

and that the cost satisfies the *bi-twist* condition: For each x and $\bar{x}$ respectively,

$$\bar{x} \to Dc(x,\bar{x}) \tag{A1}$$

and

$$x \to \bar{D}c(x,\bar{x}) \tag{A$\bar{1}$}$$

are invertible, each with an inverse depending continuously on x and $\bar{x}$ respectively.

Now let h_c be the pseudo-metric on $N \subset M \times \bar{M}$ that is defined at each $T_{(x,\bar{x})}N = T_xM \times T_{\bar{x}}\bar{M}$ as the nondegenerate symmetric matrix

$$h_c = \begin{pmatrix} 0 & -D\bar{D}c \\ -\left(D\bar{D}c\right)^T & 0 \end{pmatrix} \tag{4.1}$$

(This metric was introduced by the first and second author in [KM].) Note that as a matrix, h_c has n positive and n negative eigenvalues. Let ρ, $\bar{\rho}$ be smooth (probability) volume densities on M, $\bar{M}$, respectively, which are locally expressed as $\rho = \rho(x)dx^1 \wedge \cdots \wedge dx^n$, $\bar{\rho} = \bar{\rho}(\bar{x})d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^n$ with $\rho(x), \bar{\rho}(\bar{x}) > 0$.

Given assumptions (A1) and (A2) together with some decay conditions on $\rho, \bar{\rho}$ if M, $\bar{M}$ are noncompact, we will always have a unique solution (see, for example, [V, Theorem 10.28]) to the optimal transport problem, namely, a map $F : M \to \bar{M}$ which satisfies $F_{\#}\rho = \bar{\rho}$ in the sense of measures.

The Kantorovich method for obtaining a solution to the optimal transportation is to maximize a functional, producing two potential functions $u \in C(M)$ and $v \in C(\bar{M})$, from which we may draw the optimal mapping $F_u : M \to \bar{M}$ and an inverse mapping $F_v : \bar{M} \to M$. Note that F_v solves the symmetric optimal transportation problem. Both are differentiable almost everywhere. Where differentiable, the potential u satisfies

$$Du(x) = -Dc(x, F_u(x))$$

and

$$D^2u(x) = -D^2c(x, F_u(x)) - D\bar{D}c(x, F_u(x))DF_u(x)$$

in particular

$$-D\bar{D}c(x, F_u(x))DF_u(x) = D^2u(x) + D^2c(x, F_u(x)) \tag{4.2}$$

the right-hand side of which is clearly symmetric. Further, it is known that

$$D^2u(x) \geq -D^2c(x, F_u(x)). \tag{4.3}$$

To satisfy $F_{\#}\rho = \bar{\rho}$, the map F_u must also satisfy, where differentiable

$$\bar{\rho}(F_u(x)) \det DF_u(x) = \rho(x). \tag{4.4}$$

Now consider the following symplectic form on $M \times \bar{M}$:

$$\omega_c = \begin{pmatrix} 0 & -D\bar{D}c \\ \left(D\bar{D}c\right)^T & 0 \end{pmatrix},$$

and the following conformal perturbation of h_c:

$$h^{\rho,\bar{\rho}} = \Big(\frac{\pi^*\rho \wedge \bar{\pi}^*\bar{\rho}}{d\,\mathrm{vol}_{h_c}}\Big)^{\frac{1}{n}} h_c. \tag{4.5}$$

First we show that the graph of F_u is Lagrangian, wherever it is differentiable, by the following. Pullback the form ω_c to M, and evaluate on any two tangent vectors

$$\begin{aligned}(Id \times F)^*\omega_c(\partial_i, \partial_j) &= \langle -D\bar{D}c\,(\partial_i F_u)\,, \partial_j\rangle + \langle \left(D\bar{D}c\right)^T \partial_i, (\partial_j F_u)\rangle \\ &= \langle -D\bar{D}c\,(\partial_i F_u)\,, \partial_j\rangle + \langle \partial_i, D\bar{D}c\,(\partial_j F_u)\rangle \\ &= \left(D\bar{D}cDF\right)_{ij} - \left(D\bar{D}cDF\right)_{ji}\end{aligned}$$

which vanishes by (4.2) .

Similarly, pulling back the metric h_c :

$$\begin{aligned}(Id \times F)^* h_c(\partial_i, \partial_j) &= \langle -D\bar{D}c\,(\partial_i F_u)\,, \partial_j\rangle + \langle \left(-D\bar{D}c\right)^T \partial_i, (\partial_j F_u)\rangle \\ &= -\left(D\bar{D}cDF\right)_{ij} - \left(D\bar{D}cDF\right)_{ji}\end{aligned}$$

which is nonnegative by (4.3). The measure preserving condition (4.4), along with (4.3) and (4.2) guarantee that this will be strictly positive wherever the map is differentiable. Noting that $(Id \times F)^*\tau$ is a positive multiple of the volume form on M, and that $h^{\rho,\bar{\rho}}$ satisfies in local coordinates $\det h^{\rho,\bar{\rho}} = \rho^2(x)\bar{\rho}^2(x)$, we are in position to apply Proposition 3.1 to an appropriate current.

Now we make sense of the graph of F as a current. To do this, recall that whenever a map f is differentiable, we may push forward a current, simply by evaluating the pullback of a given form. When our current is represented by a manifold with a tangent space, this is equivalent to evaluating the form on the oriented tangent space, pushed forward. In our case, we can define the current $T = (Id \times F)_\# M$ as

$$T(\psi) = \left[(Id \times F)_\# M\right](\psi) = \int_{M \setminus \mathrm{sing}(F)} (Id \times F)^*\psi$$

where $\mathrm{sing}(F)$ denotes the subset of M where F is not differentiable. Note that the tangent space of the graph of F is defined $\mathcal{H}^n$ almost everywhere on $\mathrm{supp}(T)$. To see this, let $\Sigma \subset \mathrm{supp}(T) \subset M \times \bar{M}$ be the set where the tangent space representing T does not exist. Now, $\rho dx(\pi_M(\Sigma)) = 0$, because F is differentiable a.e. The symmetric problem is also solvable, hence an inverse of F exists, and has the same properties (this is where we use (A$\bar{1}$)), therefore, $\bar{\rho}d\bar{x}(\,\pi_{\bar{M}}(\Sigma)) = 0$. From this, we conclude $\mathcal{H}^n(\Sigma) = 0$. Moreover, T is an n-dimensional integral rectifiable current, as follows from [V, Theorem 10.48].

We must make one final restriction: On portions of the current where we apply the calibration, the graph of the optimal map is assumed to not intersect the cut locus. Our arguments require that the current we are defining is compactly supported inside the region where the metric is smooth. If the metric is not smooth it is unlikely we will be able to recover the mass from our definitions, as n-forms may be forced to be badly behaved in order to maintain positive comass near the singularities. We note further that as calibration arguments require that we work with currents of compact support, for many situations the following result will not apply globally, (if say M is not compact) however it will apply locally. Note that given any optimal map $F : M \to \bar{M}$ the restriction of the optimal map to any subset is an optimal map onto its image. and satisfies the same relations (4.3) (4.2) and (4.4), hence locally is calibrated.

We are ready to state our main result.

Theorem 4.1. *Assume $c(x,\bar{x})$ satisfies (A1)(A$\bar{1}$)(A2) and $\rho,\bar{\rho}$, are smooth and positive. Let $M_0 \subset M$ be such that the integral current $T = (Id \times F)_{\#}M_0$ representing the graph of the optimal transport map F is compactly supported away from the cut locus of $c(x,\bar{x})$. Then with respect to the metric $h^{\rho,\bar{\rho}}$(given in 4.5), T is calibrated, and thus has the mass maximizing property $\|T\|_{h^{\rho,\bar{\rho}}} \geq \|S\|_{h^{\rho,\bar{\rho}}}$ for all compactly supported n-currents S which are homologous to T.*

Proof. Let $N = M \times \bar{M} - \mathfrak{C}$. Take Φ from Proposition 3.1. Wherever F is differentiable, we have $\bar{\rho}\det DF = \rho$. Thus on any tangent plane $P = (Id \times F)_{*}T_xM$ to the graph, $\rho(x)dx^1 \wedge \cdots \wedge dx^n|_P = \bar{\rho}(\bar{x})d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^n|_P$ which implies that $\Phi(\vec{T}_x) = \left\|\vec{T}_x\right\|_{h^{\rho,\bar{\rho}}}$, by Proposition 3.1. Now, since T is represented by such tangent spaces, we see that $T(\Phi) \leq T(\Psi)$ for any $\|\Psi\|^*_{h^{\rho,\bar{\rho}}} \geq 1$. It follows from the definition of mass that $T(\Phi) = \|T\|_{h^{\rho,\bar{\rho}}}$. Thus, T is calibrated and we may apply Corollary 2.3. □

The above theorem generalizes the result of Warren [W], which boils down to the case $c(x,\bar{x}) = -x\cdot\bar{x}$ and $\rho = 1_\Omega$, $\bar{\rho} = 1_{\bar{\Omega}}$ for bounded convex domains $\Omega, \bar{\Omega} \subset \mathbf{R}^n$.

Proposition 2.1 allows us to compare a smooth space-like submanifold to smooth variations of the submanifold, and we conclude:

Corollary 4.2. *At any point where the graph of the optimal transport map F is a C^1 submanifold in $(M \times \bar{M}, h^{(\rho,\bar{\rho})})$ the graph has zero mean curvature.*

Remark 1. If we evaluate the calibration Φ against the current $T = (Id \times F)_{\#}M$, by integrating on coordinates in M, we observe that

$$\|T\|_{h^{\rho,\bar{\rho}}} = T(\Phi) = \int_{M\setminus \mathrm{sing}(F)} \frac{1}{2}(\rho + \bar{\rho}\det DF) = \int_M \rho.$$

Thus when ρ is a probability density, the mass on the calibrated current always satisfies $\|T\|_{h^{\rho,\bar{\rho}}} = 1$.

Remark 2. We conclude with a remark on the Ma-Trudinger-Wang cost-curvature condition [MTW, A3 condition]. The first two authors [KM] expressed the MTW condition as a curvature condition on h_c, which may be described as follows: The weak (strong) MTW A3 condition holds if and only if, in each coordinate chart on M and $\bar{M}$, the Riemannian sectional curvature $R_{i\bar{j}\bar{j}i}$ corresponding to any vanishing component $(h_c)_{i\bar{j}} = 0$ of the metric tensor is nonnegative (positive).

With respect to the conformal metric $h^{\rho,\bar{\rho}}$, we have the Riemann curvature tensor

$$R^{\rho,\bar{\rho}}_{i\bar{j}\bar{j}i} = \Big(\frac{\pi^*\rho \wedge \bar{\pi}^*\bar{\rho}}{d\,\mathrm{vol}_{h_c}}\Big)^{\frac{1}{n}} (R_{i\bar{j}\bar{j}i} + \Lambda_{i\bar{j}}(h_c)_{\bar{j}i} - \Lambda_{ii}(h_c)_{\bar{j}\bar{j}} + \Lambda_{\bar{j}i}(h_c)_{i\bar{j}} - \Lambda_{\bar{j}\bar{j}}(h_c)_{ii})$$

for some Λ_{ij} involving derivatives of the conformal factor. If a metric component $(h_c)_{i\bar{j}} = 0$, we easily see the corresponding component of the Riemann tensor is given by

$$R^{\rho,\bar{\rho}}_{i\bar{j}\bar{j}i} = \Big(\frac{\pi^*\rho \wedge \bar{\pi}^*\bar{\rho}}{d\,\mathrm{vol}_{h_c}}\Big)^{\frac{1}{n}} R_{i\bar{j}\bar{j}i}.$$

Thus, the weak (strong) MTW A3 condition holds if and only if, in each coordinate chart on M and $\bar{M}$, whenever $h_{i\bar{j}}{}^{\rho,\bar{\rho}} = 0$, the sectional curvature $R^{\rho,\bar{\rho}}_{i\bar{j}\bar{j}i}$ is nonnegative (positive).

5. Appendix

Proposition 5.1. *Let h be an indefinite metric on $\mathbb{R}^{n+m}$ of the form $dx^2 - dy^2$. The "unit sphere" $\|P\|_h = 1$ of oriented space-like n-planes is disconnected.*

Proof. Let P be an n-plane given by $v_1 \wedge \cdots \wedge v_n$. The projection π of the independent set $\{v_i\}$ onto $\mathbb{R}^n$ gives an independent set, otherwise the plane would contain time-like vectors. Define $\tau(P) = \det(\pi(v_1 \wedge \cdots \wedge v_n))$. It is clear that the alternating form τ will be either positive or negative on any space-like plane. □

Proof of Proposition 2.1. Proposition 2.1 is a pointwise result, so we can assume that the metric is of a simple form

$$h = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}.$$

Given a spacelike n-plane we can always choose a basis $v_1 \ldots v_n$ so that these row vectors are of the form

$$\begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} = \begin{pmatrix} I & A^T \end{pmatrix}$$

where $A^T A < I$. Now define a calibration Φ by

$$\Phi\,(w_1 \wedge w_2 \wedge \cdots \wedge w_n) = \frac{1}{\sqrt{\det\,(I - A^T A)}} \det\left[\begin{pmatrix} I & A^T \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}\begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}\right].$$

On the original n-plane P, letting $w_1 = v_1$, etc, then

$$\begin{aligned}\Phi\,(v_1 \wedge v_2 \wedge \cdots \wedge v_n) &= \frac{1}{\sqrt{\det\,(I - A^T A)}} \det\left[\begin{pmatrix} I & A^T \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}\begin{pmatrix} I \\ A \end{pmatrix}\right] \\ &= \frac{1}{\sqrt{\det\,(I - A^T A)}} \det\big(I - A^T A\big) = \sqrt{\det\,(I - A^T A)} \\ &= \|(v_1 \wedge v_2 \wedge \cdots \wedge v_n)\|_h\,.\end{aligned}$$

Now for any space-like plane $w_1 \wedge w_2 \wedge \ldots \wedge w_n$ we can always rechoose the basis so that as column vectors they form a matrix

$$\left(\begin{array}{ccc} w_1 & \cdots & w_n \end{array} \right) = \left(\begin{array}{c} I \\ B \end{array} \right).$$

Here again, $B^T B < I$. Let

$$\Phi \left(w_1 \wedge w_2 \wedge \cdots \wedge w_n \right) = \frac{1}{\sqrt{\det \left(I - A^T A \right)}} \det \left(I - A^T B \right).$$

Now

$$\begin{aligned} \det \left(I - A^T B \right) &\geq \det \left(I - \frac{A^T B + B^T A}{2} \right) \\ &\geq \det \left(\frac{I - A^T A + I - B^T B}{2} \right) \\ &\geq \sqrt{\det \left(I - A^T A \right) \det \left(I - B^T B \right)} \end{aligned}$$

Here follows an explanation to this step. The first inequality is from the fact for any matrix Q with $\langle Qx, x \rangle \geq 0$, the symmetrized matrix will have smaller determinant (see [W]). For the nonnegativity of $I - A^T B$, notice that from Cauchy-Schwarz inequality

$$\langle A^T B x, x \rangle = \langle Bx, Ax \rangle \leq \frac{\langle Ax, Ax \rangle + \langle Bx, Bx \rangle}{2} = \langle \frac{A^T A + B^T B}{2} x, x \rangle.$$

The second inequality also follows from Cauchy-Schwarz. The final inequality is the Minkowski Determinant theorem for positive matrices (i.e. log concavity of the determinant). Combing the above, we see that

$$\Phi \left(w_1 \wedge w_2 \wedge \cdots \wedge w_n \right) \geq \sqrt{\det \left(I - B^T B \right)} = \left\| \left(w_1 \wedge w_2 \wedge \cdots \wedge w_n \right) \right\|_h.$$

with equality when and only when $B = A$. This completes the proof. □

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