THE PRINCIPAL-AGENT MODEL
AND INCENTIVE COMPATIBILITY IN
MICROECONOMIC THEORY

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Rochet & Choné Econometrica 1998
G. Carlier PhD (Dauphine) 2000
Carlier & Lachand-Robert CPAM 2001
**Principal's Problem:**

Maximize profits, by selective pricing of product lines designed to differentiate customers (i.e., "agents") based on e.g., preferences,
- location
- wealth
- willingness to pay...
- access to substitute products

E.g., - Airline tickets prices
- product differentiation (calculators)
- contracts
- taxation
Product (e.g. car) types: \( y = (y_1, \ldots, y_n) \in \mathbb{M}^n \subset \mathbb{R}^n \)

- \( y_1 = \text{fuel efficiency} \)
- \( y_2 = \text{safety} \)
- \( y_3 = \text{appearance} \)
- \( y_4 = \text{comfort} \)

Agent (i.e. buyer) types: \( x = (x_1, \ldots, x_m) \in \mathbb{M}^m \subset \mathbb{R}^m \)

- \( x_1 = \text{environmental consciousness} \)
- \( x_2 = \text{size of family} \)
- \( x_3 = \text{profession / desire to impress} \)
- \( x_4 = \text{pursimanioussness} \)
- \( x_5 = \text{income} \)

\[ 
\]
EXOGENOUS DATA

\[ w(x,y) = \text{utility (i.e. value in$)} \] of
\[ \text{car } y \in M^- \text{ to agent } x \in M^+ \]
\[ = -c(x,y) \]

\[ \mu^+ : \text{distribution of agent types in population} \]
\[ = \text{a Borel probability measure on } M^+ \]

\[ k(y) : \text{principal's cost to manufacture car } y \]
**Principal (Monopolist, Manufacturer)**

Chooses a price menu \( V: M^- \to \mathbb{R} U \{0\} \) designed to maximize profits = revenue - costs.

\( V(y) \) = price at which she wants to sell car \( y \in M^- \)

**Agent \( x \in M^+ \)**: Decides whether to buy a car, and if so which one, by calculating the "\( u \)-convex" function (or indirect utility)

\[
U(x) = V^u(x) = \sup_{y \in M^-} u(x, y) - V(y)
\]

If \( U(x) > 0 \), \( x \) buys car \( y_0(x) \) attaining sup

If \( U(x) \leq 0 \), \( x \) does not buy any car.

"Participation constraint"
SET-UP

utility $u : M^+ \times M^- \rightarrow \mathbb{R}^{\text{max}} \rightarrow \mathbb{R} \cup \{\text{inf}\}$ of $y$ to $x$

measure $\mu^+ \geq 0$ on $M^+$

cost $k : M^- \rightarrow \mathbb{R} \cup \{\text{inf}\}$ to manufacture

PRINCIPAL: chooses $V : M^- \rightarrow \mathbb{R} \cup \{\text{inf}\}$ to maximize

$$\pi[V] = \text{revenues} - \text{costs}$$

AGENT $x$: buys a car $y_0(x)$ which achieves maximal

$$U(x) = V^u(x) = \sup_{y \in M^-} u(x,y) - V(y) \quad \text{iff} \quad U_b(x) > 0$$

PROFIT:

$$\pi[V] := \int_{\{x \in M^+ \mid V^u(x) > 0\}} \left[ V(y_0(x)) - k(y_0(x)) \right] \, d\mu^+ (x)$$
SOME HISTORY

SPENCE '74 (signal) \( n = m = 1 \)

MIRLEES '76 (Hex)

\[(X) \quad \frac{\partial y}{\partial x} > 0 \Rightarrow y_0(x) \text{ non-decreasing on } x \in M^+ \cap \mathbb{R} \]

\( \Rightarrow \) Compactness, existence, characterization, example...

"Assortative matching": higher quality agents choose better products (and pay more)

LORENTZ '55 (Rearrangement inequalities)

\[ F_+^*(x) = \int_{-\infty}^{x} dp^*(s) = \int_{-\infty}^{x} f^*(s) \, ds \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(x,y) f^*(x) f^*(y) \, dx \, dy \leq \int_{0}^{2} c(F^+_*(w), F^-_* (w)) \, dw \]

\( \forall \) p.d.f.'s \( f^* \iff (X) \)
**Theorem (Rockafellar & Monnéd):** \( M^g \subseteq \mathbb{R}^n \)

\[ u(x,y) = \langle x, y \rangle \quad \text{and} \quad k(\vec{0}) = 0 \]

\[ \pi[V] = \int_{\{ x \in \mathbb{R}^n : u(x) > 0 \}} \left[ \langle x, u(x) \rangle - u(x) - k(u(x)) \right] \, dp^+ \]

where \( u(x) = V^*(x) = \sup_{y \in M^-} \langle x, y \rangle - V(y) \)

**Enormously simplifies variational problem:**

\[ \max \int_{\mathbb{R}^n} \left[ \langle x, u(x) \rangle - u - k(u(x)) \right] \, dp^+ \]

for \( u: \mathbb{R}^n \rightarrow [0, \infty] \) convex \( u(\mathbb{R}^n) \notin M^- \cup \{ \vec{0} \} \)

\[ \text{i.e., } \partial u(\mathbb{R}^n) \cap \partial M^- = \partial (M^g \cup \{ \vec{0} \}) \]
Proof: \( U(x) = V^*(x) = \sup_{y \in \mathbb{N}} \langle x, y \rangle - V(y) \)

attained at \( y_0(x) = \alpha U(x) \) (or \( y_0 \in \alpha U(x) \)),

where \( U(x) + V(\alpha U(x)) = \langle x, \alpha U(x) \rangle \). Profit

\[
\pi [V] = \int \left[ V(y_0) - k(y_0) \right] \, dp^+(x) \\
\{ V^+ > 0 \}
\]

\[
= \int \left[ \langle x, \alpha u \rangle - u - k(\alpha u) \right] \, dp^+(x) \\
\{ u > 0 \}
\]

\[
= \int_{\mathbb{R}^2} \left[ \langle x, vU_+ \rangle - U_+ - k(\alpha U_+) \right] \, dp^+(x) \\
\]

where \( U_+(x) = \max \{ U(x), 0 \} \)
Numerical Example

Example: ROCHET-CHONE '98

\( n = 2 \)

\[ d \rho^+(x) = \chi_{[0,1]^2}(x) \, d\mathcal{H}^2(x) \]

\( u(x,y) = \langle x, y \rangle \)

\( k(y) = \frac{1}{2} y^2 \)

\( M^+ = (\mathbb{R}_+)^2 \) = positive quadrant

\[ \min \int \int \left[ \frac{1}{2} |\nabla u|^2 + u - \langle x, du \rangle \right] \, dx \, dx \]

\( u \geq 0 \) convex

\( \nabla \mathcal{H}(\mathbb{R}^2) \in M^- \)

Dirichlet energy
CALCULUS OF VARIATIONS
UNDER CONVEXITY CONSTRAINTS

- numerical challenges (see Magy '01)
- theoretical: polar dual to cone of convex functions

**THM:** (Carlier & Lachand-Robert '01)

\[ M^+ \subset \mathbb{R}^n \] bounded convex domain
\[ dp^+(x) = f^+(x) \chi_{M^+}(x) \, dx \quad \| \log f^+ \|_{\infty} < \infty \]
\[ M^- = (\mathbb{R}_+)^n = \text{positive constant} \]

\[ \min_{U \geq 0 \text{ convex}} \int_{M^+} \left[ \frac{1}{2} |u|^2 + u - \langle x, du \rangle \right] f^+(dx) \, d\nu \]
\[ u_{xy} \geq 0 \]
\[ u_{x \geq 0} \]

unique minimizer \( U_0 \in C^1(M^+) \)

OPEN: What about \( k(y) = \frac{1}{p} |y|^p \)?
ROBUSTNESS: OTHER COSTS?

Assume (as usual) \( u : M^+ \times M^- \to \mathbb{R} \) satisfies:

\[ \forall x \in M^+, \ p \in \mathbb{R}^m \ \exists \ \text{at most one} \ y = Y(x, p) \in M^- . \]

\[ \delta_{1u}(x, y) = p \]

+ suitable smoothness

Agent \( x \in M^+ \) will choose \( y^*_x (x) = Y(x, u(x)) \)

where

\[ u(x) = V^u(x) = \sup_{y \in M^-} u(x, y) - V(y) \]
\[ \sup_{V: M^+ \to \mathbb{R}} \int_{\{x \in M^+ : V^I \leq 0\}} \left[ V(Y(x, \varpi U(x))) - k(Y(x, \varpi U(x))) \right] d\mu^I(x) \]

\[ \max_{U: M^+ \to \mathbb{R}} \int_{\{x \in M^+ : U(x) > 0\}} \left[ U(x, Y(x, \varpi U(x))) - U(x) - k(Y(x, \varpi U(x))) \right] d\mu^I(x) \]

\(Y(x, \varpi U(x)) \ni M^-\)

and is attained (because of the compactifying property of \(V \to U = V^u \) supremal convolutions)

Characterization? Examples?
3D Rotating Stratified Fluid (Incompressible)

Velocity

Geopotential

Potential temperature

\[
(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} + \nu \nabla^2 \mathbf{v} + \mathbf{J} \mathbf{v} + \theta \mathbf{g} = 0
\]

\[
(\partial_t + \mathbf{v} \cdot \nabla) \theta = 0
\]

\[
\mathbf{v} \cdot \nabla = 0
\]

\[
\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}
\]

\[
\mathbf{V} + \mathbf{Y} \in \mathbb{R}^3
\]

\[
\mathbf{V} = \mathbf{V}_H + \mathbf{V}_T = -\mathbf{J}^2 \mathbf{V} + \mathbf{V}_T \mathbf{g}
\]
3D Incompressible Semi-geostrophic Theory

Geostrophy: \( \mathbf{v}_n = \mathbf{J} \nabla \mathbf{p} \quad \Theta = \frac{\partial \mathbf{p}}{\partial y_2} \)

Semi-geostrophy: \[ (\mathbf{\alpha} + \mathbf{\nabla} \cdot \mathbf{v}) \cdot \mathbf{v} = \mathbf{J} \partial_t \mathbf{p} + \mathbf{J} \mathbf{v} \]

\( \mathbf{\nabla} \cdot \mathbf{v} = 0 \)

Conserved Quantities

Energy: \( E = \frac{d}{dt} \int \frac{1}{2} \partial \mathbf{v}^2 \, d^3y \)

Potential

Vertical

\( \Theta = \left( \frac{2}{5t} + \mathbf{\nabla} \cdot \mathbf{v} \right) \det (\partial \mathbf{p} + J) \)

\( \Rightarrow \) convexity of \( P(t, \mathbf{y}) \) preserved

(Allen i.e., Proser "stability")
Dual Formulation in Geostrophic Variables.

\[ V(t, \hat{y}) = P(t, \hat{y}) + \frac{1}{2} \hat{y}^2 \]

\[ x = X(t, \hat{y}) = \phi V(t, \hat{y}) \]

\[ \psi(t, \hat{z}) = \psi \hat{z} \cdot \hat{y} - V(t, \hat{y}) = \psi \hat{z} \]

\[ = U(t, \hat{z}) \]

\[ = \psi (t, \hat{z}) + \frac{1}{2} \hat{z}^2 \]

\[ x = \partial \psi (t, \hat{z}) \hat{y} \]

\[ p(\hat{z}, \hat{x}) = \det [I + D^2 \psi (t, \hat{z})] \]

\[ = \text{Monge-Ampère elliptic} \]

\[ \left( \frac{\partial}{\partial t} - \text{J}_\psi \cdot \hat{v} \right) p(\hat{z}, \hat{x}) = 0 \]

Active scalar advected in layers by a divergence free vector field.

\[ \text{d}_2(x, y) p(t, z) = \text{constant energy} \]