

When do systematic gains uniquely determine the number of marriages between different types in the Choo-Siow matching model? Sufficient conditions for a unique equilibrium*

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Abstract

In a transferable utility context, Choo and Siow (2006) introduced a competitive model of the marriage market, and derived its equilibrium output, a marriage matching function. The marriage matching function defines the gains generated by a marriage between agents of prescribed types in terms of the observed frequency of such marriages within the population, relative to the number of unmarried individuals of the same types. Left open in their work is the question of whether, for a given population whose frequency of types is known, this gains data captures all of the statistical information used to define it. Equivalently, it is not known whether the Choo-Siow model of the marriage market admits a unique equilibrium. We resolve this question in the affirmative, assuming the norm of the gains matrix (viewed as an operator) to be less than two. The analytical difficulty of showing uniqueness of positive roots of polynomial systems has generated a growing literature that provides numerical techniques for tackling such problems. Our method adapts a strategy called the continuity method, more commonly used to solve elliptic partial differential equations, to the new setting of isolating positive roots of polynomial systems. Finally, the data estimated in [4] yields gains matrices with norms in the range to which our result applies.

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1 Introduction

In ‘Who Marries Whom and Why?’[4], Choo and Siow tackle the problem of identifying the relative systematic gains to individuals with different observable types who choose to marry each other, from statistical data concerning who marries whom in a given population. Choosing a transferable utility model, they propose a particular marriage matching function which defines the gains generated by a marriage between agents of prescribed types in terms of the observed frequency of such marriages within the population, relative to the number of unmarried individuals of the same types. The gains scheme defined by Choo and Siow is particularly convenient for comparing preferences and trends between population samples of different sizes, because — in addition to other virtues — it is scale-free. Until now, however, it has not been clear whether their gains scheme captures all of the statistical information used to define it, i.e. whether or not, it is in fact a recharacterization of this data. This question is of interest from an econometric and a demographic point of view; it will be called the *Choo-Siow Inverse Problem*. It is resolved affirmatively below, provided the operator norm of the gains matrix is strictly less than two.

The Choo-Siow inverse problem may also be interpreted in the context of a competitive model of the marriage market. Each type of individual has a utility for each type of potential spouse. In this framework, individuals seek to maximize utility by marrying a spouse that accords highest utility. This maximization is constrained by scarcity of all types of individuals, and by the preferences of potential spouses. Gains are an aggregation of the utility earned by the two spouses. The Choo-Siow inverse problem can be viewed as a competitive equilibrium problem. The marriage market comes into equilibrium when individuals make marital arrangements whose frequency generates the prescribed gains. In the theory of competitive equilibria, there are general theorems that guarantee existence of equilibria, but comparatively few results guaranteeing uniqueness of equilibrium; the theorem established below for the Choo-Siow model is one example of such a result.

1.1 Related literature, background, and motivation

A marriage matching function specifies the marital output in a society, taking as inputs population vectors, which are the distributions of different types of men and women in the population under study, and exogenous parameters. It is a production function of marital matches.

Given population data and exogenous parameters, a marriage matching function generates a marriage distribution which is a bivariate distribution of marriage matches by spousal type, and of unmarried individuals by type.

A thirty year old research agenda involves the search for non-parametric marriage matching functions which are econometrically identified and also allow for substitution effects as in Pollak [11], and Pollard [12]. Choo and Siow [4] proposed such a marriage matching function using a transferable utility model of the marriage market. This model has been used to study the effects of the

legalization of abortion on marital behavior in the United States (Choo-Siow [4]), the decomposition of marital behavior of famine born cohorts in China into quantity versus quality effects (Brandt, Siow and Vogel [2]), changes in marital matching in the United States in recent decades (Chiappori, Selanie and Weiss, [3]), and to test Becker's model of positive assortative matching (Siow [13]). Siow [15] surveys other applications.

Given any observed population vectors and marriage distribution, the parameters of the non-parametric Choo-Siow marriage matching function are point identified. What is unknown is whether any admissible set of parameters and population vectors generates a unique marriage distribution. This is the *Choo-Siow inverse problem*. The question of uniqueness is important for several reasons. First, if the *Choo-Siow inverse problem* has a unique solution, the estimated parameters are an alternative characterizations of the observed marriage distribution. The recharacterization is useful because the parameters of the Choo-Siow model have a behavioral interpretation.

Second, the Choo-Siow marriage matching function is the equilibrium outcome of a competitive market. There are few, if any, realistic environments with finitely many agent types and many commodities which are known to generate unique competitive equilibria. That a competitive marriage matching environment does so is a relevant contribution. Moreover in the Choo-Siow model, individuals are assumed to have McFadden [10] random utility functions over spousal choices. This logit class of discrete choice utility functions is the current standard in equilibrium discrete choice modeling as in Berry and Reiss [1]. So uniqueness of equilibrium in the Choo-Siow model has potentially wider applicability.

Finally, an affirmative solution to the *Choo-Siow inverse problem* means that the parameters, which are estimated from who marries whom, implicitly define a single valued marriage matching function. Researchers are often interested in predicting changes in the marriage distribution in response to changes in marriage market conditions. A single valued marriage matching function allows a unique prediction to be made.

1.2 The Choo-Siow marriage matching model

We begin by reviewing the Choo-Siow model. Consider an observation space that consists of M men and F women. The space can be partitioned into non-empty subsets according to the types of its constituents. There are I types of men, and J types of women. The number of men of type i is denoted m_i , and the number of women of type j is denoted f_j . The vector whose i^{th} component is m_i , and whose $(I + j)^{th}$ component is f_j , is denoted by ν . It has $(I + J)$ components and is called the *population vector*. In the observation space, some individuals are married, and others are single. Let μ_{ij} be the number of marriages of type i men to type j women. Let μ_{i0} be the number of unmarried men of type i and μ_{0j} be the number of unmarried women of type j . A specification of $\mu_{ij}, \mu_{i0}, \mu_{0j}$ for all i and j is called a *marital arrangement*. The following population constraints must be satisfied by all marital arrangements,

and are a consequence of the definitions:

$$\mu_{i0} + \sum_{j=1}^J \mu_{ij} = m_i, \quad (1)$$

$$\mu_{0j} + \sum_{i=1}^I \mu_{ij} = f_j. \quad (2)$$

The *relative systematic gains* (also called *payoff* or *welfare*) of a marriage of a type i man to a type j woman is denoted Π_{ij} , and is defined — apart from a logarithm irrelevant to the discussion below — by the equation

$$\Pi_{ij} = \frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}}. \quad (3)$$

Note that division by zero causes some values of Π_{ij} to be undefined, unless — as we henceforth assume — the sample set is large enough that there remain a positive number of unmarried men and unmarried women of each type. The $I \times J$ matrix of gains will be denoted by $\Pi = (\Pi_{ij})$. It is a function on the space of marital arrangements; a marital arrangement is said to *generate* gains Π .

We can now precisely formulate the Choo-Siow Inverse Problem:

Problem (Choo-Siow inverse problem) *Given a gains matrix $\Pi = (\Pi_{ij})$ and a population vector $\nu = (m, f)$, does there exist a unique marital arrangement generating Π ? In other words, assuming the entries Π_{ij} to be non-negative and m_i and f_j to be strictly positive, does exactly one matrix (μ_{ij}) with non-negative entries exist which satisfies (1)–(3)?*

One might also be interested in whether the entries μ_{ij} of the solution, when they exist, turn out to be integers. This question is not addressed here.

Remark 1 (Interpretation of the gains matrix) Choo and Siow show that equation (3) is the equilibrium outcome of a transferable utility model of a competitive marriage market. Choo and Siow call Π_{ij} , which are exogenous parameters, the systematic gains to an $\{i, j\}$ marriage relative to them not marrying. Π_{ij} has an intuitive interpretation. If it is larger, there will be more $\{i, j\}$ marriages and or less i type men or j type women remaining unmarried. Given an observed marriage distribution, Π_{ij} can be estimated. So a researcher can learn which marital matches generate larger systematic gains than others. Even if one does not fully subscribe to the interpretation of the Choo-Siow model, Π_{ij} is an alternative characterization of the marriage distribution if the *Choo-Siow inverse problem* has a unique solution.

Remark 2 (Special structure for ordered types) Many of the types commonly available to the researcher can be ordered. These include income, age, and years of education. In the ordinal case, the I types of men and J types of women can be considered as points on the real line, specified only up to order.

Given a type i we let $i + 1$ denote the next type. In the presence of ordering, a general identity for our problem,

$$\log \left(\frac{\mu(i', j')\mu(i, j)}{\mu(i', j)\mu(i, j')} \right) = \log \Pi(i', j') + \log \Pi(i, j) - \log \Pi(i, j') - \log \Pi(i', j) \quad (4)$$

as mentioned in [13], acquires special interpretation. Indeed, taking $i' = i + 1$ and $j' = j + 1$, the right hand side of equation (4) becomes a finite difference representation of the mixed partial derivative $\frac{\partial^2 \log \Pi}{\partial i \partial j}$. Galichon [5] realized that taking the continuum limit of the lattice of types on \mathbf{R}^2 , Π and μ may become smooth functions¹ satisfying the partial differential equation:

$$\frac{\partial^2 \log \Pi}{\partial i \partial j} - \frac{\partial^2 \log \mu}{\partial i \partial j} = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) (\log \Pi - \log \mu) = 0 \quad (5)$$

where $(t, x) = (i + j, i - j)/\sqrt{2}$. Equation (5) recasts the unique expression of marriage statistics (3) in terms of unmarried statistics as the unique solution of an inhomogeneous wave equation on the quadrant $|x| < t$. This reduction to boundary data is a continuum analog of Melino's observation (6). Determining the correct boundary data from Π and from ν is the question resolved below in the discrete setting.

The apparent simplification gained by working with ordinal or cardinal types comes at the cost of assuming more structure than is natural in certain settings. Types may be comprised of multiple parameters or include properties like religion and race have which no canonical ordering. In either case, there is no linear ordering of these types with respect to which we would expect an individual to have continuous marital preferences. This problem is eliminated if types are viewed abstractly.

1.3 Preliminaries

Our approach addresses types which are abstract and discrete. Let us begin with a reformulation of the problem; Siow [14] attributes this reformulation to Angelo Melino. Let $\alpha_{ij}^2 = \mu_{ij}$. In this new notation, the gains matrix and population constraints (1)–(3) take the form:

$$\alpha_{i0}\alpha_{0j}\Pi_{ij} = \alpha_{ij}^2 \quad (6)$$

$$\alpha_{i0}^2 + \sum_{j=1}^J \alpha_{ij}^2 = m_i \quad (7)$$

$$\alpha_{0j}^2 + \sum_{i=1}^I \alpha_{ij}^2 = f_j. \quad (8)$$

¹The discrete values of μ are determined from the data. Taking the continuum limit will involve smoothing the discrete data, or fitting it to a parameterized family of functions.

Borrowing terminology from quantum physics, we call any collection of α_{ij} 's which solve (6)–(8) *amplitudes* corresponding to Π . By substituting (6) into (7) and (8), we can eliminate all variables but those that correspond to unmarried men and women. This yields a system of $(I + J)$ quadratic polynomials in the $(I + J)$ variables $\{\alpha_{i0}\}_{i=1}^I, \{\alpha_{0j}\}_{j=1}^J$:

$$\begin{aligned}\alpha_{i0}^2 + \sum_{j=1}^J \alpha_{i0}\alpha_{0j}\Pi_{ij} &= m_i, & 1 \leq i \leq I \\ \alpha_{0j}^2 + \sum_{i=1}^I \alpha_{i0}\alpha_{0j}\Pi_{ij} &= f_j, & 1 \leq j \leq J.\end{aligned}\tag{9}$$

A solution to this system of equations is a vector of amplitudes α that has $(I + J)$ components. In principle its components can be real, complex, or both. The Choo-Siow Inverse Problem is equivalent to showing that the polynomial system (9) has a unique solution with real positive amplitudes for all gains matrices Π with non-negative entries and all population vectors $\nu = (m, f)$ with positive components. Our strategy is a proof by the continuity method, a technique used in the study of nonlinear elliptic partial differential equations [6]. We fix ν , and show that there is a unique non-negative solution when $\Pi = \mathbf{0}$. Then we show that uniqueness is preserved when Π is perturbed slightly. We iterate this process until we reach the desired Π or encounter an obstruction.

Take $\{\alpha_{i0}, \alpha_{0j}\}_{0 < i \leq I, 0 < j \leq J}$ to be coordinates in \mathbf{R}^{I+J} . The $I + J$ coordinate plans defined by setting each of these variables to zero bound 2^{I+J} open regions in \mathbf{R}^{I+J} . In analogy with the case $I = J = 1$, we call these regions *quadrants*.

Let Π^T denote the transpose of Π , and $\lambda_{\max}(\Pi^T\Pi)$ the largest eigenvalue of $\Pi^T\Pi$. Then $\|\Pi\|_{\text{op}} := |\lambda_{\max}(\Pi^T\Pi)|$ defines a norm, called the operator norm of the matrix Π . It is related the Euclidean (or Hilbert-Schmidt) norm $\|\Pi\| = \text{trace}(\Pi^T\Pi)$ of this $I \times J$ matrix by the well-known estimates $\frac{\|\Pi\|}{\sqrt{J}} \leq \|\Pi\|_{\text{op}} \leq \|\Pi\|$. The main result of this paper is the following theorem.

Theorem 1 (Sufficient conditions for unique equilibrium) *If the entries of $\Pi = (\Pi_{ij})$ are non-negative, and those of $(m_i)_{1 \leq i \leq I}$, $(f_j)_{1 \leq j \leq J}$ are positive, and $\|\Pi\|_{\text{op}} < 2$, then precisely one solution α of (9) lies in the positive quadrant of \mathbf{R}^{I+J} .*

Since each matrix (μ_{ij}) with non-negative entries solving (1)–(3) corresponds to a solution α of (9) having positive amplitudes $\alpha_{i0} = \sqrt{\mu_{i0}}$ and $\alpha_{0j} = \sqrt{\mu_{0j}}$, this theorem gives the sought characterization of (μ_{ij}) by Π — thus solving the Choo-Siow inverse problem provided $\|\Pi\|_{\text{op}} < 2$.

It turns out that when the operator norm of the gains matrix is bounded by 1, more can be said about the solutions to (9):

Theorem 2 (Sufficient conditions for separated real solutions) *If the entries of $\Pi = (\Pi_{ij})$ are non-negative, and those of $(m_i)_{1 \leq i \leq I}$, $(f_j)_{1 \leq j \leq J}$ are positive, and $\|\Pi\|_{\text{op}} < 1$, then exactly 2^{I+J} vectors $(\alpha_{10}, \dots, \alpha_{I0}, \alpha_{01}, \dots, \alpha_{0J})$ solve (9) in \mathbf{C}^{I+J} , their coefficients are real and non-vanishing, and precisely one solution lies in each quadrant of \mathbf{R}^{I+J} .*

The non-positive solutions to (9) are superfluous from the economic point of view. However, the technique used in its proof has potentially wider applicability to solving inverse problems of this type. The proof of both theorems can be viewed as an application of the continuity method more commonly used in the study of elliptic partial differential equations [6].

A second virtue of Theorem 2 is that the stated hypothesis $\|\Pi\|_{\text{op}} < 1$ is sharp for its conclusion:

Remark 3 (Sharpness of Theorem 2) *When there is a single type $I = 1 = J$ of each sex, the Choo-Siow inverse problem reduces to solving a system of two quadratic equations in two unknowns. In this case we see the spurious solutions with $\alpha_1\alpha_2 < 0$ diverge as $\Pi = \Pi_{11} \nearrow 1$. In this sense the hypothesis $\|\Pi\|_{\text{op}} < 1$ is sharp for the conclusions of Theorem 1 to be true. On the other hand, the relevant solutions $\alpha_1\alpha_2 > 0$ remain bounded as $\Pi_{11} \rightarrow 1$, and the Choo-Siow Inverse Problem turns out to have an affirmative answer for all $\Pi_{11} \geq 0$ in this case. Whether or not this affirmative answer extends to all componentwise non-negative Π for some $IJ \geq 2$ remains an interesting open question.*

We prove Theorem 1 and Theorem 2 in two separate sections below.

First, we develop a compact notation that makes the dependence of the polynomial equations (9) on various parameters more explicit. Recalling $\nu = (m, f)$, we see the quadratic system of equations (9) is equivalent to the following system:

$$\text{diag} \left[\begin{pmatrix} Id_I & s\Pi \\ s\Pi^T & Id_J \end{pmatrix} \alpha\alpha^T \right] = \nu \quad (10)$$

with parameter value $s = 1$. Here Id_J denotes the $J \times J$ identity matrix, and $\alpha\alpha^T$ denotes the rank-one matrix which gives a scaled projection of \mathbf{R}^{I+J} onto the vector of amplitudes α .

The strategy of our proof of both Theorem 1 and Theorem 2 is the following: when $s = 0$ the theorems are obviously true, since then $\alpha_{i0} = \pm\sqrt{m_i}$ and $\alpha_{0j} = \pm\sqrt{f_j}$ give the only solutions to (10). If we can show, for a fixed Π that satisfies the prescribed operator norm bound, that the set of $s \in [0, 1]$ for which the conclusions of each theorem hold true is both open and closed, then it must amount to the entire interval $[0, 1]$.

2 Proof of Theorem 1

A simple but crucial observation that facilitates the proof of Theorem 1 by continuity (or deformation) method is the following:

Lemma 3 (Separation lemma) *For every positive $\nu = (m, f)$ and $\Pi = (\Pi_{ij})$, no complex component of any amplitude α solving (9) vanishes.*

Proof. Suppose α is a vector of amplitudes, and $\alpha_{i0} = 0$. Since $m_i > 0$, the i^{th} equation in the polynomial system (9) is violated. Similarly, if $\alpha_{0j} = 0$, since $f_j > 0$, the $I + j^{\text{th}}$ equation in (9) is violated. ■

We shall use this lemma as follows: when $s = 0$ we have already remarked that the only positive solution is given by $\alpha_0 = (\sqrt{m_i}, \sqrt{f_j})$.

As $s \in [0, 1]$ evolves, solutions may continue to be real, or may become complex. By the Separation Lemma, solutions that remain real but non-positive can be bounded away from the positive quadrant. This line of reasoning will be repeatedly useful in what follows.

Proof of Theorem 1. Fix $\nu = (m, f)$ with positive components and $\Pi = (\Pi_{ij})$ with non-negative entries, having $\|\Pi\|_{\text{op}} < 2$. Let $S \subset [0, 1]$ denote the set of parameter values s for which there is a unique positive vector α of amplitudes solving (10). We have already remarked that S contains 0, hence is non-empty. If $S \subset [0, 1]$ can be shown to be both open and closed, it must contain $s = 1$.

To show S is open, we first need to verify the following claims, which will enable us to employ the continuity method.

Claim 4 (A priori bound on positive solutions) *Suppose α solves (10) for some $s \in [0, 1]$, and let $\alpha_{i0} > 0$ and $\alpha_{0j} > 0$ for all i , and for all j (which may henceforth be denoted by $\alpha > 0$). Then $\alpha_{i0} \leq \sqrt{m_i}$, and $\alpha_{0j} \leq \sqrt{f_j}$.*

Proof. When $\Pi = 0$, $\alpha_{i0} = \sqrt{m_i}$, and $\alpha_{0j} = \sqrt{f_j}$. When $\Pi > 0$, the square-free coefficients of s in the system (9), being non-negative and real, contribute non-negatively to the left hand side. Since the right-hand side is a fixed vector ν , the amplitudes must be no larger than they are when $\Pi = 0$. ■

We denote by Id_k the identity operator $id : \mathbf{R}^k \rightarrow \mathbf{R}^k$ in the standard basis of \mathbf{R}^k . Further, if $x \in \mathbf{R}^k$ has all of its components positive, we write $x > 0$.

Claim 5 (Implicit function hypothesis for real solutions) *Let*

$$F(\alpha, s) = \text{diag}((Id_{I+J} + sW_\infty)\alpha\alpha^T) - \nu$$

denote the difference between the two sides of (10), so that $F : (\alpha, s) \in \mathbf{R}^{I+J} \times [0, 1] \rightarrow \mathbf{R}^{I+J}$ and W_∞ is defined as in Lemma 9. Fix $s \in [0, 1]$ and recall the components of $\nu = (m, f)$ are all positive.

If α is any solution to $F(\alpha, s) = 0$, and if $\|\Pi\|_{\text{op}} \leq 1$, the matrix $D_\alpha F(\alpha, s) = (\partial F_k / \partial \alpha_\ell)_{1 \leq k, \ell \leq I+J}$ is invertible.

Alternately, if $F(\alpha, s) = 0$ for some $\alpha > 0$, then if $\|\Pi\|_{\text{op}} \leq 2$ the same conclusion holds: the matrix $D_\alpha F(\alpha, s) = (\partial F_k / \partial \alpha_\ell)_{1 \leq k, \ell \leq I+J}$ is invertible.

Proof. We first compute $D_\alpha F(\alpha, s)$, and then apply the a priori bound on positive solutions.

The map $D_\alpha F(\alpha, s) = \begin{pmatrix} \Delta_I & s\Pi_I \\ s\Pi_J^T & \Delta_J \end{pmatrix}$ is an $(I+J) \times (I+J)$ matrix that is conveniently partitioned into four submatrices: an $I \times I$ diagonal matrix Δ_I ; an $I \times J$ matrix Π_I ; a $J \times I$ matrix Π_J^T ; and a $J \times J$ diagonal matrix Δ_J . We shall verify the determinant of $D_\alpha F(\alpha, s)$ is non-vanishing.

The submatrices have the form:

$$\begin{aligned}
(\Delta_I)_{ii} &= 2\alpha_{i0} + \sum_{j=1}^J \Pi_{ij}\alpha_{0j} = \alpha_{i0} + m_i/\alpha_{i0}, \\
(\Delta_J)_{jj} &= 2\alpha_{0j} + \sum_{i=1}^I \Pi_{ij}\alpha_{i0} = \alpha_{0j} + f_j/\alpha_{0j}, \\
(\Pi_I)_{ij} &= \Pi_{ij}\alpha_{i0} = \alpha_I \bullet \Pi, \\
(\Pi_J^T)_{ij} &= \Pi_{ij}\alpha_{0j} = \alpha_J \bullet \Pi^T,
\end{aligned}$$

where we have used the equality $F(\alpha, s) = 0$ to simplify the diagonal terms. Here \bullet denotes row-wise scalar multiplication by the entries of α_I , given by $(\alpha_I)_i = (\alpha_{i0})$.

Because α_{0j} and α_{i0} are non-zero by hypothesis, we can divide each row of Π_J^T and Π_I by some α_{0j} and α_{i0} respectively, without changing the zeroes of the determinant. This process transforms $D_\alpha F$ into a matrix of the form $\begin{pmatrix} \widetilde{\Delta}_I & \Pi \\ \Pi^T & \widetilde{\Delta}_J \end{pmatrix}$, where $\widetilde{\Delta}_I$, and $\widetilde{\Delta}_J$ are diagonal and have entries all larger than one, namely $(\widetilde{\Delta}_I)_{ii} = 1 + m_i/\alpha_{i0}^2$, and $(\widetilde{\Delta}_J)_{jj} = 1 + f_j/\alpha_{0j}^2$.

There is a determinant formula for block matrices which asserts [7] that

$$\det \begin{pmatrix} \widetilde{\Delta}_I & \Pi \\ \Pi^T & \widetilde{\Delta}_J \end{pmatrix} = \det(\widetilde{\Delta}_I) \det(\widetilde{\Delta}_J) \det(\text{Id}_J - \widetilde{\Delta}_J^{-1} \Pi^T \widetilde{\Delta}_I^{-1} \Pi). \quad (11)$$

The preceding discussion shows $\widetilde{\Delta}_I^{-1}$ and $\widetilde{\Delta}_J^{-1}$ to be diagonal matrices whose entries are bounded between 0 and 1. It is immediate that if $\|\Pi\|_{\text{op}} \leq 1$, $D_\alpha F(\alpha, s)$ is invertible.

To verify the second part of the claim, we apply the a priori bound on positive solutions derived in Claim 4, and observe that the entries of these diagonal matrices $\widetilde{\Delta}_I^{-1}$ and $\widetilde{\Delta}_J^{-1}$ lie between 0 and $\frac{1}{2}$: since $\alpha_{i0} \leq \sqrt{m_i}$, and $\alpha_{0j} \leq \sqrt{f_j}$, it follows that $\frac{\alpha_{i0}^2}{\alpha_{i0}^2 + m_i} \leq \frac{1}{2}$, and $\frac{\alpha_{0j}^2}{\alpha_{0j}^2 + f_j} \leq \frac{1}{2}$. Thus the largest eigenvalue of $\widetilde{\Delta}_J^{-1}$ and of $\widetilde{\Delta}_I^{-1}$ is less than $\frac{1}{2}$.

Therefore, if operator norm of Π^T and of Π are both less than 2, the argument of the Jacobian matrix as computed above is positive definite, and so the final determinant in (11) is positive. Thus the product of the three determinants is non-vanishing as desired. However $\|\Pi\|_{\text{op}} = \|\Pi^T\|_{\text{op}}$, so we merely require $\|\Pi\|_{\text{op}} < 2$. ■

We are now equipped to show that the set S defined above is both open and closed.

Claim 6 (Openness) *The set $S \subset [0, 1]$ of parameter values s for which the conclusions of Theorem 1 hold true is open.*

Proof. If $s_0 \in S$, let $\alpha = \alpha_0$ be the unique real solution with positive components to the equation $F(\alpha, s_0) = 0$. By hypothesis $\|\Pi\|_{\text{op}} < 2$, we can apply the implicit function theorem at the point (α_0, s_0) . This follows from Claim 5. The theorem provides an open interval $U \subset \mathbf{R}$ centered at s_0 , and an open neighbourhood $V \subset \mathbf{R}^{\mathbf{I}+\mathbf{J}}$ centered at α_0 , such that on $V \times U$, all zeroes of F lie on a smooth curve $(\alpha(s), s)$ (with domain U), such that $\alpha(s_0) = \alpha_0$.

Consider the set $B = \{s \in U \mid \exists \alpha'_s \neq \alpha(s) \text{ such that } F(\alpha'_s, s) = 0\}$. Our goal is to show that B is empty. Suppose otherwise. Define $D(s) = |s - s_0|$. Let \tilde{s} be such that $D(\tilde{s}) = \inf_{s \in B} \{D(s)\}$.

This choice of $\tilde{s} \in U$ guarantees some sequence $s_i \rightarrow \tilde{s}$ with $s_i \in B$. To each s_i correspond the distinct amplitudes $\alpha(s_i)$, and α'_{s_i} . We consider separately the sequences $\alpha(s_i)$, and α'_{s_i} .

By Claim 4, which provides an a priori bound on positive solutions, the sequence α'_{s_i} has a convergent subsequence, whose limit we denote by α'_∞ . Smoothness of $\alpha(s)$ implies the convergence of $\lim_{i \rightarrow \infty} \alpha(s_i)$ to $\alpha_\infty := \alpha(\tilde{s})$ also. These limits both lie in the positive real quadrant, for by Lemma 3 no solution may lie on its boundary. It follows from continuity of F that $F(s_0, \alpha'_\infty) = 0 = F(s_0, \alpha_\infty)$.

There are now two cases to consider. First, suppose $\alpha'_\infty = \alpha_\infty$. Then arbitrarily close to $(\alpha_\infty, \tilde{s})$ are solutions $(\alpha'_i, s_i) \in V \times U$ which lie outside the curve $s \in U \mapsto (\alpha(s), s)$, contradicting the discussion above.

Thus it must be the case that $\alpha'_\infty \neq \alpha(\tilde{s})$, and therefore $\tilde{s} \in B$. Then we may apply the implicit function theorem to $(\tilde{s}, \alpha'_\infty)$. In doing so, we obtain a smooth curve defined in a neighbourhood of $\tilde{s} \in U$ the images of which is separated by a positive distance from the curve $\alpha(s)$ at the point \tilde{s} . By continuity of these curves, there is some small ball about \tilde{s} on which the image of the curves are separated by a positive distance. This violates the minimality of \tilde{s} . Hence S is open. ■

Claim 7 (Closedness) S is closed.

Proof. Let s_i be a sequence of points in S such that $s_i \rightarrow s_\infty$. We show that $s_\infty \in S$.

Given the sequence (s_i, α_i) we may apply the a priori bound established in Claim 4 to the positive solutions α_i , and extract a convergent subsequence $(s_{i(k)}, \alpha_{i(k)}) \rightarrow (s_\infty, \alpha_\infty)$. By continuity, $F(s_\infty, \alpha_\infty) = 0$. By Claim 3, the vector α_∞ lies in the positive real quadrant, for it may not lie on the boundary. We apply the implicit function theorem to $(s_\infty, \alpha_\infty)$, and obtain a smooth, unique local solution to $F(\alpha, s) = \mathbf{0}$, which we denote $(\alpha(s), s)$.

Arbitrarily close to s_∞ are members of S . Within some neighborhood of s_∞ these points, and the amplitudes that correspond to them, must lie on $(\alpha(s), s)$. Suppose there were an additional solution, $(\tilde{\alpha}, s_\infty)$, with $\tilde{\alpha} \neq \alpha(s_\infty)$. Then we apply the implicit function theorem at this point, to get a contradiction to the fact that $s_{i(k)} \in S$ for k large has a unique solution $\alpha(s_{i(k)})$ far from $\tilde{\alpha}$. Hence S is closed. ■

3 Proof of Theorem 2

In this section we use the continuity method to provide a proof of Theorem 2. As is always the case, the method relies on the implicit function theorem to prove the set of parameters $s \in S$ for which the hypothesis of Theorem 2 are true form an open subset of $[0, 1]$, and a priori bounds on solutions to prove that the same set is closed.

Like in the proof of the previous theorem, Lemma 3 is crucial. We shall use it as follows: when $s = 0$ we have already remarked that the complete list of solutions is given by $\alpha_0 = (\pm\sqrt{m_i}, \pm\sqrt{f_j})$. There are 2^{I+J} complex solutions, each consists of real amplitudes, and each resides in a distinct quadrant of \mathbf{R}^{I+J} . If each solution α_s remains real as $s \in [0, 1]$ evolves, by the Separation Lemma, these solutions remain in distinct quadrants. As long as no new solutions emerge from complex space, it is guaranteed that there is a unique solution with real amplitudes in each quadrant. As it turns out, we need not worry about additional complex solutions: they are all accounted for by Bezout's Theorem [16]:

Theorem 8 (Bezout's theorem) *Given n polynomials in n variables, with degrees d_1, \dots, d_n , the maximum number of isolated complex solutions is $\prod_{i=1}^n d_i$.*

Since (10) consists of $(I+J)$ equations in $(I+J)$ variables, and each polynomial is a quadratic, there can be no more than 2^{I+J} complex solutions. Thus as long as we can extend the 2^{I+J} real solutions corresponding from $s = 0$ through $s > 0$ to $s = 1$, uniqueness of equilibrium is guaranteed.

To understand whether this extension can be accomplished, it is useful to know whether the matrix

$$W = \begin{pmatrix} Id_I & \Pi \\ \Pi^T & Id_J \end{pmatrix} \quad (12)$$

governing the solutions to (10) is positive-definite. The following lemma and its corollary answer this question. In it, Π^* denotes the adjoint — the transpose of the complex conjugate of Π . The same lemma has applications in statistical physics and quantum chemistry; there Π_{ij} governs the rate at which a quantum particle in state i makes a transition into state j . It has been exploited in those contexts in [9] and the references there.

Lemma 9 (Diagonalization of the welfare matrix) *Given an $I \times J$ matrix Π with complex coefficients, let $W_\infty = \begin{pmatrix} 0 & \Pi \\ \Pi^* & 0 \end{pmatrix}$. If (e, f) is an eigenvector of W_∞ with eigenvalue $\lambda \neq 0$, then $(e, -f)$ is an eigenvector of W_∞ with eigenvalue $-\lambda$, e is an eigenvector of $\Pi\Pi^*$ with eigenvalue λ^2 , and f is an eigenvector of $\Pi^*\Pi$ with eigenvalue λ^2 . Conversely, if f is an eigenvector of $\Pi^*\Pi$ with eigenvalue $\lambda^2 > 0$, then $(\Pi f/\lambda, f)$ is an eigenvector of W_∞ with eigenvalue λ . As a consequence, W_∞ has at least $|J - I|$ zero eigenvalues.*

Proof. Since W_∞ is self-adjoint, it has real eigenvalues and a complete set of orthogonal eigenvectors. Moreover, the special structure of W_∞ implies that for

(e, f) to be an eigenvector with eigenvalue λ means $\Pi f = \lambda e$ and $\Pi^* e = \lambda f$. If $\lambda \neq 0$, this forces both $f \neq 0$ and $e \neq 0$. Apart from its final assertion, the lemma follows immediately. Since every pair of eigenvalues $\pm \lambda \neq 0$ of the matrix W_∞ leads to one of the I eigenvalues of the matrix $\Pi\Pi^*$, and one of the J eigenvalues of the matrix $\Pi^*\Pi$, it follows that W_∞ has at most $\min\{I, J\}$ non-zero eigenvalue pairs (counted always with multiplicity). On the other hand W_∞ has $I + J$ eigenvalues total, so at least $|J - I|$ of them must vanish. ■

Corollary 10 (Positive-definiteness of the welfare matrix) *The matrix W defined by (12) is positive definite if and only if $\|\Pi\|_{\text{op}} < 1$.*

Proof. Since the coefficients of Π are real, the preceding lemma shows the eigenvalues of $W = W_\infty + Id_{I+J} = \begin{pmatrix} Id_I & \Pi \\ \Pi^T & Id_J \end{pmatrix}$ take the form $\lambda = 1$ (with multiplicity greater than $|J - I|$), and $\lambda_i = 1 \pm \sqrt{\delta_i}$, where $\delta_i \geq 0$ are the eigenvalues of the non-negative definite symmetric $J \times J$ matrix $\Pi^T\Pi$. Hence W has all positive eigenvalues if and only if $\delta_i < 1$ for all i , if and only if $\|\Pi\|_{\text{op}} < 1$. ■

Proof of Theorem 2. Fix $\nu = (m, f)$ with positive components and $\Pi = (\Pi_{ij})$ with non-negative entries, having $\|\Pi\|_{\text{op}} < 1$. Let $S \subset [0, 1]$ denote the set of parameter values s for which there are (at least) 2^{I+J} solutions α to (10), one in each quadrant of \mathbf{R}^{I+J} . We have already remarked that S contains 0, hence is non-empty. If $S \subset [0, 1]$ can be shown to be both open and closed, it must contain $s = 1$. Since there are at most 2^{I+J} solutions by Bezout's theorem, all solutions will have been accounted for and the theorem established. To show S is open, we will invoke the first part of Claim 5, which will enable us to apply the implicit function theorem. We also require an a priori bound on positive solutions when $\|\Pi\|_{\text{op}} < 1$.

Claim 11 (A priori bound on solutions) *Let $W - \lambda Id_{I+J}$ be non-negative definite, where $W = \begin{pmatrix} Id_I & \Pi \\ \Pi^T & Id_J \end{pmatrix}$ and $\lambda > 0$. If $s \in [0, 1]$ and the components of $\nu = (m, f)$ are non-negative, then all solutions to $F(\alpha, s) = 0$ satisfy $|\alpha|^2 \leq (I + J)^{1/2} |\nu| / \lambda$.*

Proof of claim. Given $\nu \in \mathbf{R}^{I+J}$, set $|\nu|_p := (\sum_{k=1}^{I+J} |\nu_k|^p)^{1/p}$ and recall the elementary inequalities $(I + J)^{-1/2} |\nu|_1 \leq |\nu|_2 \leq |\nu|_1$. Since any solution to (9) satisfies $\nu = \text{diag}(W\alpha\alpha^T)$, the matrix $W\alpha\alpha^T$ has non-negative entries on its diagonal. Thus $|\nu|_1 = \text{trace}(W\alpha\alpha^T) \geq \lambda |\alpha|_2^2$, which concludes the lemma in case $s = 1$.

In case $s \in [0, 1]$, Lemma 9 implies that the eigenvalues of $W(s) := \begin{pmatrix} Id_I & s\Pi \\ s\Pi^T & Id_J \end{pmatrix}$ range from $1 - s\sqrt{\delta_1}$ to $1 + s\sqrt{\delta_1}$, where $\delta_1 \geq \delta_2 \geq \dots \geq \delta_J$ are the eigenvalues of $\Pi^T\Pi$. Non-negative definiteness of $W - \lambda Id_{I+J}$ therefore implies the same for $W(s) - \lambda Id_{I+J}$, and the claim follows by applying the preceding paragraph to $W(s)$. ■

We can now show that the set S is open and closed, completing the proof of Theorem 2 by the continuity method.

Claim 12 (Openness) $S \subset [0, 1]$ is open.

Proof. Suppose $s_0 \in S$. That means there are 2^{I+J} vectors α_0 satisfying $F(\alpha_0, s_0) = 0$; one in each of the open quadrants of \mathbf{R}^{I+J} . For any one of these solutions α_0 , the implicit function theorem provides a small neighbourhood around s_0 within which $F(\alpha, s) = 0$ admits a solution α in the same quadrant as α_0 , provided the matrix $D_\alpha F(s_0, \alpha)$ of partial derivatives of F with respect to α is invertible. This invertibility was verified in Claim 5. The intersection of these 2^{I+J} neighborhoods yields an open interval in S containing s_0 . ■

Claim 13 (Closedness) S is closed.

Proof. We must show that if $s_i \rightarrow s_\infty$, and each $s_i \in S$, then $s_\infty \in S$. For each s_i there are 2^{I+J} vectors α_i^k ($1 \leq k \leq 2^{I+J}$) such that $F(\alpha_i^k, s_i)$ vanishes — one in each quadrant of \mathbf{R}^{I+J} . Since Corollary 10 implies the matrix W is positive-definite, Claim 11 asserts all the sequences $\{\alpha_i^k\}$ are contained in a sufficiently large closed ball. Hence for each k some subsequence $\{\alpha_{i(j)}^k\}$ converges to α_∞^k . Continuity of F implies $F(\alpha_\infty^k, s_\infty) = 0$. Moreover α_∞^k must lie in the k -th quadrant, and not on its boundary, according to Lemma 3. This shows the 2^{I+J} solutions α_∞^k are distinct, hence $s_\infty \in S$ as desired. ■

4 Discussion

The gains matrix Π has no clear economic interpretation as a linear transformation. It was introduced as a convenient way of arranging the various gains functions Π_{ij} . In our method of proof it takes on a geometric significance. We showed that the Choo-Siow Inverse problem is solved in the affirmative when $\|\Pi\|_{\text{op}} < 2$. Moreover, when Π is a strict contraction we are able to account for the signs of all 2^{I+J} complex solutions. It would be interesting to find an interpretation of Π as a linear operator — perhaps it governs some related dynamics — which leads to a deeper understanding of these results.

Finally our result has application to the existing empirical literature. In [4], Choo and Siow estimate Π , dividing the male and female population into seven age bins in the period 1971/1972. Hence, Π is a square matrix of dimension 7. It satisfies the bound $\|\Pi\|_{\text{op}} < 0.25$, and so falls well within the range of applicability of our theorems.

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