COMMENT ON "IRONING, SWEEPING AND MULTIDIMENSIONAL SCREENING"

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ABSTRACT. In their study of price discrimination for a monopolist selling heterogeneous products to consumers having private information about their own multidimensional types, Rochet and Choné (1998) discovered a new form of screening in which consumers with intermediate types are bunched together into isochoice groups of various dimensions incentivized to purchase the same product. They analyzed a particular example involving customer types distributed uniformly over the unit square. For this example, we prove that their proposed solution is not selfconsistent, and we indicate how consistency can be restored.

Keywords. *Principal-Agent problem, Rochet-Choné, asymmetric information, adverse selection, monopolist nonlinear pricing, multidimensional screening, bilevel optimization, free boundary, bunching*

1. INTRODUCTION

Let potential consumers be parameterized by types $x \in \mathbf{R}^n$ and products by types $y \in [0, \infty)^n$, with $y = (0, \ldots, 0)$ representing the null product or outside option. Taking $b(x, y) = x \cdot y$ to be the direct utility of product y to agent x, Rochet and Choné (1998) study the price menu $v(y)$ a monopolist will select to maximize her profits when the price $v(0, \ldots, 0) = 0$ of the outside option is constrained, assuming the distribution $d\mu(x)$ of agents and the monopolist's cost $c(y)$ to produce each product y are both known. They show v can be taken to be the convex dual function of the consumers' indirect utility $u(x)$, which in turn maximizes the profit functional

$$\Phi[u] := \int_{\mathbf{R}^n} [x \cdot Du(x) - u(x) - c(Du(x))] \, d\mu(x) \tag{1.1}$$

among non-negative convex functions $u : \mathbf{R}^n \longrightarrow [0, \infty]$ which are coordinatewise nondecreasing. The product $y = Du(x)$ selected by consumer type x coincides with the gradient of u. They give an abstract characterization of the maximizing u, and explore its implications for the particular example in which $c(y) = |y|^2/2$, $n = 2$ and μ is distributed uniformly over the square $\Omega := [a, a+1]^2$ for fixed $a > 0$. Although their abstract characterization is no doubt correct, we demonstrate an inconsistency in their subsequent analysis of the square example; we show it is possible to restore consistency by modifying their solution to accommodate an overlooked market segment. In this example, Rochet and Choné assert the unique maximizer $u \in C^1(\Omega)$ and divides Ω into three regions

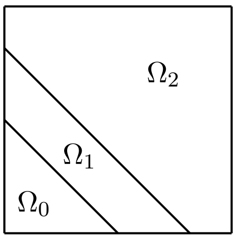

FIGURE 1. Partition of Ω (according to the rank of D^2u) given in Rochet and Choné (1998).

$$\Omega_0 = \{(x_1, x_2) \in \Omega : x_1 + x_2 \leq t_{0.5}\} \tag{1.2}$$

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$$\Omega_1 = \{(x_1, x_2) \in \Omega : t_{0.5} < x_1 + x_2 \le t_{1.5}\} \tag{1.3}$$

$$\Omega_2 = \{(x_1, x_2) \in \Omega : t_{1.5} < x_1 + x_2\} \tag{1.4}$$

of qualitatively different behaviour: a triangle Ω_0 of excluded customers on which $u(x) = 0$; a strip Ω_1 foliated by lines $x_1 + x_2 = t$ of customers each of whom chooses a product $y = (U'(t), U'(t))$ from the diagonal, where $u(x_1, x_2) = u_1(x_1, x_2) = U(x_1 + x_2)$ satisfies

$$U(t) = \frac{3}{8}t^2 - \frac{1}{2}at - \frac{1}{2}\log|t - 2a| + C_0, \tag{1.5}$$

with matching conditions $U(t_{0.5}) = 0 = U'(t_{0.5})$ selecting the constants C_0 and $t_{0.5}$; and a third region Ω_2 on which $u = u_2$ is strictly convex (so that each agent gets a customized product) and satisfies the mixed Dirichlet / Neumann problem for the Poisson equation

$$\begin{cases} \Delta u_2 := (\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}) = 3, & \text{on } \mathrm{Int}(\Omega_2), \\ (Du_2(x) - x) \cdot \hat{n}(x) = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ u_2 - u_1 = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega_2; \end{cases} \tag{1.6}$$

here $\hat{n}(x)$ denotes the outer unit normal to the interior $\mathrm{Int}(\Omega_2)$ of the domain at $x \in \partial\Omega_2$, and the additional boundary condition

$$D(u_2 - u_1) \cdot \hat{n}(x) = 0, \text{ on } \partial\Omega_1 \cap \partial\Omega_2 \tag{1.7}$$

is supposed to select and be satisfied by some constant $t_{1.5} \in \mathbf{R}$.

Subsequent numerics by Ekeland and Moreno-Bromberg (2010) and Mirebeau (2016) suggest this description is mostly but not entirely correct: in Figure 2, the region Ω_1 appears not to be a strip, but to have a more complicated upper boundary, parameterized by a nonsmooth curve $t_{1.5}(\,\cdot\,)$ over the anti-diagonal:

$$\Omega_1 = \{(x_1, x_2) \in \Omega : t_{0.5} < x_1 + x_2 \le t_{1.5}(x_1 - x_2)\}. \tag{1.8}$$

Below, we prove rigorously that Rochet and Choné (1998)'s ansatz $t_{1.5} \equiv const$ cannot be correct. Before doing so, we explain how to correct it and make it consistent with the theoretical and numerical evidence: assuming temporarily that we know Ω_1 (and hence Ω_2), we first augment Rochet and Choné's description of $u = u_1$ in Ω_1; we claim that (Ω_1, u_2) solves the boundary value problem (1.6)–(1.8). In McCann and Zhang (2023+), we give a nonrigorous justification of this claim, along with the rigorous proof that only one such pair (Ω_1, u_2) solving (1.6)–(1.8) can yield u convex throughout Ω. Setting aside the degree of rigor of the justification, this gives a unique characterization of the solution. Since finding the edge $\partial\Omega_1 \cap \partial\Omega_2$ of the unknown domain is half of the challenge, this is called a free boundary problem in the mathematical literature.

We begin with the ansatz that $\Omega_1 = \Omega_1^0 \cup \Omega_1^+ \cap \Omega_1^-$ splits into three regions: a strip

$$\Omega_1^0 := \{(x_1, x_2) \in \Omega_1 : x_1 + x_2 \in (t_{0.5}, t_{1.0}]\}, \tag{1.9}$$

plus two regions

$$\Omega_1^{\pm} := \{(x_1, x_2) \in \Omega_1 \setminus \Omega_1^0 : \pm(x_1 - x_2) \ge 0\}, \tag{1.10}$$

below and above the diagonal. The region Ω_1^0 is foliated by anti-diagonal isochoice sets, and the solution there $u(x_1, x_2) = U(x_1 + x_2)$ is exactly as Rochet and Choné describe (1.5). However, the region Ω_1^- and its reflection Ω_1^+ below the diagonal are foliated by isochoice segments making continuously varying angles θ with the horizontal.

We describe the solution $u = u_1^-$ in this region using an Euler-Lagrange equation derived in McCann and Zhang (2023+). Index each isochoice segment in Ω_1^- by its angle $\theta \in (-\frac{\pi}{4}, \frac{\pi}{2}]$. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to this boundary point $(a, h(\theta))$. Along the hypothesized length $R(\theta)$ of this segment assume u increases linearly with slope $m(\theta)$ and offset $b(\theta)$:

$$u_1^-\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta). \tag{1.11}$$

Given a constant $t_{1.0} \in [2a, 2a+1]$ and $R : \left[-\frac{\pi}{4}, \frac{\pi}{2}\right] \to \left[0, \sqrt{2}\right)$ locally Lipschitz on its interval of positivity, with $R\left(-\frac{\pi}{4}\right) = (t_{1.0} - 2a)/\sqrt{2}$, solve

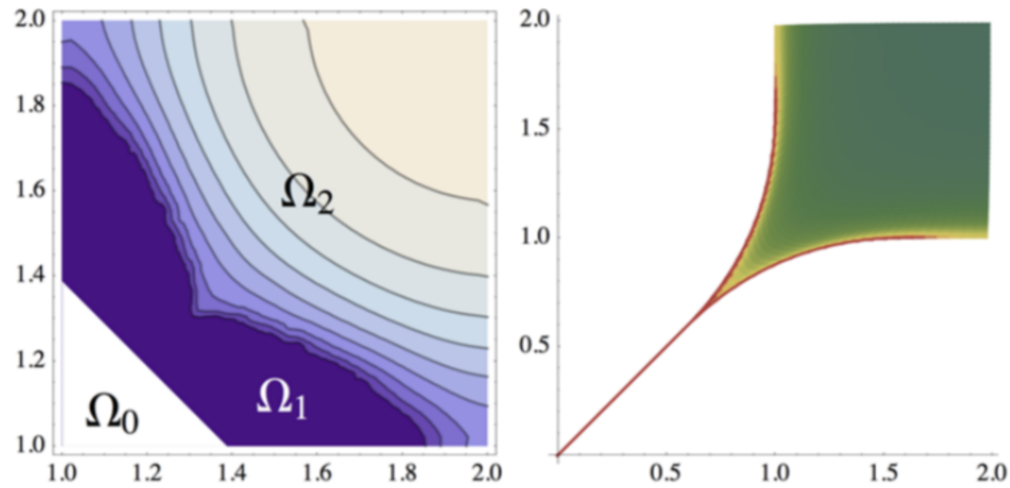

FIGURE 2. Numerics from Mirebeau (2016). Left: level sets of $\det D^2u$ with $u = 0$ on Ω_0 and $\det D^2u = 0$ on $\Omega_0 \cup \Omega_1$; Right: intensity of products sold by the monopolist.

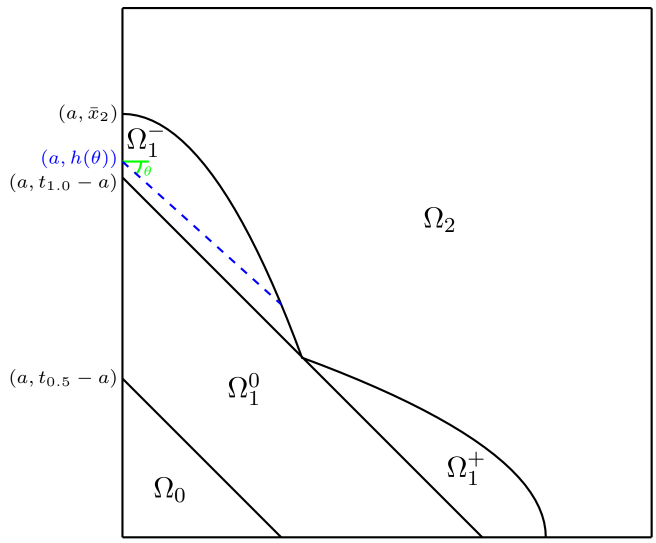

FIGURE 3. Revised partition of the space Ω of agent types, and coordinates in Ω_1^-.

$$m(-\tfrac{\pi}{4}) = 0, \qquad m'(-\tfrac{\pi}{4}) = \sqrt{2}U'(t_{1.0}) \qquad \text{such that} \tag{1.12}$$

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta. \tag{1.13}$$

Then set

$$h(\theta) = (t_{1.0} - a) + \frac{1}{3}\int_{-\pi/4}^{\theta} (m''(\vartheta) + m(\vartheta) - 2R(\vartheta))\frac{d\vartheta}{\cos\vartheta}, \tag{1.14}$$

$$b(\theta) = U(t_{1.0}) + \int_{-\pi/4}^{\theta} (m'(\vartheta)\cos\vartheta + m(\vartheta)\sin\vartheta)h'(\vartheta)d\vartheta. \tag{1.15}$$

Given $t_{1.0}$ and $R(\cdot)$ as above, the triple (m, b, h) satisfying (1.13)–(1.15) exists and is unique on the interval where $R(\cdot) > 0$. Thus the shape of Ω_1^- — or equivalently $t_{1.5}(\cdot)$ from (1.8) — and the value of u_1^- on it will be uniquely determined by $t_{1.0} \in [2a, 2a+1]$ and $R : \left(-\frac{\pi}{4}, \frac{\pi}{2}\right] \to \left[0, \sqrt{2}\right)$. We henceforth restrict our attention to choices of $t_{1.0}$ and $R(\cdot)$ for which the resulting set Ω_1^- lies above the diagonal and in the square Ω. In this case Ω_1^+ and the value of $u = u_1^+$ on Ω_1^+ are determined by reflection symmetry $x_1 \leftrightarrow x_2$ across the diagonal. Together, $u_1^{\pm}$ and (1.5) define $u = u_1$ on Ω_1 and provide the boundary data on $\partial\Omega_1 \cap \partial\Omega_2$ needed for the boundary value problem (1.6) which determines u_2. Finally, as claimed above, for only one choice of $t_{1.0}$ and $R(\cdot)$ can u (pieced together from u_0, u_1 and u_2) be convex and satisfy the extra boundary condition (1.7); when it exists (and we argue conditionally in McCann and Zhang (2023+) that it does) this choice uniquely solves Rochet and Choné (1998)'s square model.

The solution we propose also appears consistent with phenomena observed numerically and discussed in an investment-to-match taxation model proposed by Boerma, Tsyvinski, and Zimin (2022+) simultaneously and independently of the present work. In their terminology Ω_1 decomposes into a blunt bunching region Ω_1^0 in which the optimal product selected does not differentiate between buyers according to the sign $x_1 - x_2$ distinguishing their dominant trait, as opposed to the targeted bunching regions $\Omega_1^{\pm}$ in which the product selected sorts along the dimension of their dominant trait and bunches in the other dimension. In our case, the two regions can also be distinguished by the fact that the indirect utility $u(x)$ is constant on each bunch in the blunt bunching region Ω_1^0, whereas it varies along generic bunches in the targeted bunching regions $\Omega_1^{\pm}$.

As shown in Figure 1, Rochet and Choné (1998) hypothesized that the regions (1.2)–(1.4) are separated by two segments parallel to the anti-diagonal, so $\Omega_1 = \{(x_1, x_2) \in \Omega : t_{0.5} < x_1 + x_2 \le t_{1.5}\}$ with $t_{0.5} = \frac{4a+\sqrt{4a^2+6}}{3}$ and $t_{1.5} = 2a + \frac{\sqrt{6}}{3}$. Thus, they do not consider the possibility of a non-empty subset $\Omega_1^{\pm} \subset \Omega_1$ where $u(x)$ does not just depend on $x_1 + x_2$ (nor do they consider any system of equations comparable to (1.11)–(1.15)). Apart from that, their proposed solution is identical to ours, except that they fail to take into account that enforcing both the Dirichlet and Neumann conditions (1.6)–(1.7) on a line separating Ω_1 from Ω_2 overdetermines the problem and prevents the free interface from being a line segment. As a result, we now show their proposed solution to be inconsistent with the continuous differentiability $u \in C^1(\Omega)$ up to the boundary claimed by Rochet and Choné (1998), and also by Carlier and Lachand-Robert (2001).

Lemma 1.1. *If $u : \Omega \longrightarrow [0, \infty)$ convex nondecreasing satisfies (1.2) –(1.7) (so that $\Omega_1^{\pm}$ are empty), then $u \notin C^1(\Omega)$ hence cannot maximize (1.1) for $c(y) = |y|^2/2$ and $d\mu(x) = 1_\Omega(x)dx$.*

Proof. Rochet and Choné (1998) showed that if u convex and (coordinatewise) nondecreasing satisfies (1.2) – (1.7) (so $\Omega_1^{\pm}$ are empty), then

$$\Omega_1 = \{(x_1, x_2) \in \Omega \mid t_{0.5} \le x_1 + x_2 \le t_{1.5}\}$$

is bounded by $t_{0.5} = \frac{4a+\sqrt{4a^2+6}}{3}$ and $t_{1.5} = 2a + \frac{\sqrt{6}}{3} = t_{1.0}$.

Differentiating (1.5) at $x_1 + x_2 = t_{1.5}$ implies their solution to (1.6) also satisfies

$$Du(x) = (a, a) \text{ on } \partial\Omega_1 \cap \partial\Omega_2. \tag{1.16}$$

Assume that Rochet and Choné's solution $u \in C^1(\Omega)$ exists and is convex. This convexity implies $u_{x_1x_1} \ge 0$ on the interior $\mathrm{Int}(\Omega_2)$ of Ω_2, hence the Poisson equation implies $u_{x_2x_2} \le 3$ there.

Set $x' = (a, a + \frac{\sqrt{6}}{3}) \subset \partial\Omega \cap \partial\Omega_1 \cap \partial\Omega_2$. From (1.16), $u_{x_2}(x') = a$. Since $u \in C^1(\Omega)$, there exists a point $x'' \in \mathrm{Int}(\Omega_2)$ with the same x_2 coordinate as x' such that $u_{x_2}(x'') \le a + \frac{1}{10}$.

Denote by $x''' = (x_1'', a+1) \in \partial\Omega$ the point on the top edge of the square having the same x_1 coordinate as x''. Then the Neumann condition (1.6) implies $u_{x_2}(x''') = a + 1$.

But

$$\begin{aligned} u_{x_2}(x''') - u_{x_2}(x'') &= \int_{a+\sqrt{2/3}}^{a+1} u_{x_2x_2}(x_1'', x_2)dx_2 \\ &\le 3[1 - \sqrt{2/3})] < \frac{3}{5}, \end{aligned}$$

contradicting $u_{x_2}(x''') - u_{x_2}(x'') \ge (a+1) - (a + \frac{1}{10}) = \frac{9}{10}$.

This contradiction shows the C^1 differentiability of the maximizer up to the boundary is inconsistent with the convexity of Rochet and Choné (1998)'s alleged solution, in which $\Omega_1^{\pm}$ are empty. $\square$

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