6.4. Sobolev inequalities

Sobolev inequalities constitute another popular family of inequalities at the border between geometry and functional analysis. Whenever $n \geq 1$ is an integer and $p \geq 1$ is a real number, define the Sobolev space

$$W^{1,p}(\mathbb{R}^n) = \Big\{ f \in L^p(\mathbb{R}^n); \quad \nabla f \in L^p(\mathbb{R}^n) \Big\}.$$

When $p \in [1, n)$, define

$$p^\star = \frac{np}{n-p}. \tag{6.28}$$

Then the (critical) Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^\star}(\mathbb{R}^n)$ asserts the existence of a finite constant $S_n(p) > 0$ such that

$$\forall f \in W^{1,p}(\mathbb{R}^n), \qquad \|f\|_{L^{p^\star}} \leq S_n(p) \left(\int_{\mathbb{R}^n} |\nabla f|^p \right)^{1/p}. \tag{6.29}$$

Without loss of generality, we assume that $S_n(p)$ is the smallest admissible (or optimal) constant in this inequality.

For the great majority of applications, it is not necessary to know more about the Sobolev embedding, except maybe explicit bounds on $S_n(p)$. However, in some circumstances one is interested in the exact value of the smallest admissible constant $S_n(p)$ in (6.29). There are usually two possible motivations for this: either for the computation of the ground state energy in some physical model, or because it provides some geometrical insights. For instance, in some recent work about isoperimetry on compact manifolds [**114, 115**] it is important to know that $S_n(p) \longrightarrow S_n(1)$ as $p \to 1^+$.

Of course, the value can be deduced from the identification of **extremal functions** in (6.29). The best constant $S_n(p)$ in (6.29) for $p > 1$ was first computed in the sixties, in unpublished work by Rodemich; then independently obtained by Aubin [**23**] and Talenti [**234**]. For $p = 1$ it has been known for a very long time that (6.29) with sharp constant is equivalent to the classical Euclidean isoperimetric inequality.

Below, we shall obtain the sharp constants in these Sobolev inequalities, as a simple application of the machinery developed in the previous chapters. The proof will follow the recent work [**93**]. With respect to other existing proofs, the argument has the merit of being very elementary, and of applying to any norm (not necessary Euclidean) in $\mathbb{R}^n$. As a bonus, it exhibits an unexpected dual problem, just as Barthe's proof of Theorem 6.18.

For $1 < p < n$, we define the function h_p by

$$h_p(x) = \frac{1}{(\sigma_p + |x|^{p'})^{\frac{n-p}{p}}}, \tag{6.30}$$

where $p' = p/(p-1)$ is the dual exponent of p (not to be mistaken for $p^\star$), and σ_p is determined by the condition

$$\|h_p\|_{L^{p^\star}} = 1. \tag{6.31}$$

These functions will be the optimizers in the Sobolev inequality. Somewhat surprisingly, this property does not in fact depend on the choice of the norm. Note that h_p does not necessarily lie in L^p (which has no importance whatsoever).

Theorem 6.21 (Optimal Sobolev inequalities). *Let $p \in (1, n)$. Whenever $f, g \in L^{p^\star}(\mathbb{R}^n)$ are two functions satisfying $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}}$ and $\nabla f \in L^p(\mathbb{R}^n)$, then*

$$\frac{\displaystyle\int |g|^{p^\star(1-1/n)}}{\left(\displaystyle\int |y|^{p'} |g(y)|^{p^\star}\, dy\right)^{1/p'}} \le \frac{p(n-1)}{n(n-p)} \|\nabla f\|_{L^p}, \tag{6.32}$$

and equality holds if $f = g = h_p$.

As immediate consequences we have

(i) the duality principle

$$\sup_{\|g\|_{L^{p^\star}}=1} \frac{\displaystyle\int |g|^{p^\star(1-1/n)}}{\left(\displaystyle\int |y|^{p'} |g(y)|^{p^\star}\, dy\right)^{1/p'}} = \frac{p(n-1)}{n(n-p)} \inf_{\|f\|_{L^{p^\star}}=1} \|\nabla f\|_{L^p} \tag{6.33}$$

with h_p extremal in both variational problems;

(ii) the sharp Sobolev inequality: if $f \neq 0$ lies in $L^{p^\star}(\mathbb{R}^n)$, then

$$\frac{\|\nabla f\|_{L^p}}{\|f\|_{L^{p^\star}}} \ge \|\nabla h_p\|_{L^p}; \tag{6.34}$$

(iii) the Sobolev embedding $W^{1,p(\mathbb{R}^n)} \subset L^{p^\star}(\mathbb{R}^n)$.

Proof of Theorem 6.21. It is clear that (6.32) implies both (i) and (ii), and that the latter is sharp. It is also clear that (iii) follows from (ii), because any function $f \in W^{1,p}$ can be approximated by functions f_k in $W^{1,p} \cap L^{p^\star}$ in such a way that $\|\nabla f_k\|_{L^p}$ converge to $\|\nabla f\|_{L^p}$. So we just have to prove the inequality (6.32) for arbitrary f and g. Thanks to the identity $|\nabla f| = |\nabla |f||$, we only need to consider the case when f is nonnegative. By a density argument, it is sufficient to consider the case when f and g are smooth and compactly supported. Also, by homogeneity, we only need to consider the case $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}} = 1$.

We introduce the two probability densities

$$F(x) = f^{p^\star}(x), \qquad G(y) = g^{p^\star}(y)$$

on $\mathbb{R}^n$. By Theorem 2.12 there exists a gradient of a convex function (uniquely determined almost everywhere on the support of f) such that

$$\nabla\varphi\#(F\,dx) = G\,dy.$$

Moreover, $\mathrm{Supp}(G) = \overline{\nabla\varphi(\mathrm{Supp}(F))}$.

Recall from Chapter 5 that the functional $F \longmapsto -\int F^{1-1/n}$ is displacement convex. This can be expressed by the above-tangent formulation of Section 5.2.6; more precisely, from Theorem 5.30 we find that

$$\int G^{1-\frac{1}{n}} \le \frac{1}{n}\int F^{1-\frac{1}{n}}\Delta_A\varphi. \tag{6.35}$$

Since G is compactly supported, it follows that $\nabla\varphi$ is bounded, and φ can be extended into a convex function on the whole of $\mathbb{R}^n$ (exercise). Then, since F is smooth and compactly supported, we can write

$$\frac{1}{n}\int F^{1-\frac{1}{n}}\,\Delta_A\varphi \le \frac{1}{n}\int F^{1-\frac{1}{n}}\,\Delta_{\mathcal{D}'}\varphi = -\frac{1}{n}\int \nabla(F^{1-\frac{1}{n}})\cdot\nabla\varphi. \tag{6.36}$$

Returning to our original notation $F = f^{p^\star}$ and $G = g^{p^\star}$, we have just shown, combining (6.35) and (6.36), that

$$\int g^{\frac{p(n-1)}{n-p}} \le -\frac{p(n-1)}{n(n-p)}\int f^{\frac{n(p-1)}{n-p}}\nabla f\cdot\nabla\varphi = -\frac{p(n-1)}{n(n-p)}\int f^{p^\star/p'}\nabla f\cdot\nabla\varphi. \tag{6.37}$$

By **Hölder's inequality** (in its vector-valued version),

$$-\int f^{p^\star/p'}\nabla f\cdot\nabla\varphi \le \|\nabla f\|_{L^p}\left(\int f^{p^\star}|\nabla\varphi|^{p'}\right)^{1/p'}. \tag{6.38}$$

But, by the definition of push-forward, $\displaystyle\int f^{p^\star}|\nabla\varphi|^{p'} = \int |y|^{p'} g^{p^\star}(y)\,dy.$ Therefore the combination of (6.37) and (6.38) concludes the proof of inequality (6.32).

Let us now choose $f = g = h_p$, and check that equality holds at all the steps of the proof, and therefore in (6.32). Of course this function is not compactly supported, but in this particular case the Brenier map reduces to the identity map $\nabla\varphi(x) = x$, and all the steps can be checked explicitly. Indeed, $\nabla\varphi(x) = x$ leads to an equality in (6.35) and in (6.36) (via integration by parts). Then one can also check that there is equality in (6.38). This ends the proof of Theorem 6.21. □

Remarks 6.22. (i) The choice $f = g = h_p$ is not mysterious: it can be guessed by looking at equality cases in Hölder's inequality. In fact, equality in (6.38) implies $\|\nabla f(x)\|^p = k f^{p^\star}(x)\|\nabla\varphi(x)\|^{p'}$ for almost all $x \in \mathbb{R}^n$. If we now assume $\nabla\varphi(x) = x$, and look for radially symmetric minimizers, we arrive at h_p.

(ii) In the present case, inequality (6.35) can be proven in a more direct way without invoking Theorem 5.30: using the definition of push-forward and the Monge-Ampère equation (4.10), one can write

$$\int G(y)\, G(y)^{-1/n}\, dy = \int F(x)\, G(\nabla\varphi(x))^{-1/n}\, dx$$
$$= \int F(x) F(x)^{-1/n} [\det D_A^2 \varphi(x)]^{1/n}\, dx;$$

then (6.35) follows by the inequality $(\det D_A^2\varphi)^{1/n} \le (\Delta_A \varphi)/n$, which is another instance of the arithmetic-geometric inequality (Lemma 5.23).

(iii) The very same proof works, mutatis mutandis, for arbitrary norms on $\mathbb{R}^n$. Letting $\|\cdot\|$ be a norm on $\mathbb{R}^n$, we define its dual norm by

$$\|X\|_* = \sup_{\|Y\| \le 1} X \cdot Y.$$

Then the Sobolev norm is defined by $\left(\int \|\nabla f\|_*^p\right)^{1/p}$, and the minimizers by $(\sigma_p + \|x\|^{p'})^{-\frac{n-p}{p}}$.

(iv) It is possible to establish (6.37) directly, even when F and G are neither smooth, nor compactly supported, assuming only that $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in L^{p^\star}(\mathbb{R}^n)$ are two nonnegative functions such that $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}} = 1$ and $\displaystyle\int g^{p^\star}(y)\, |y|^{p'}\, dy < +\infty$. A proof is given in [**93**].

(v) By tracing cases of equality in the inequalities used above, one can prove

Theorem 6.23 (Cases of equality in the Sobolev inequality). *A function $f \in L^{p^\star}(\mathbb{R}^n)$ is optimal in the Sobolev inequality* (6.34) *if and only if there exist $C \in \mathbb{R}$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$ such that*

$$f(x) = C\, h_p\big(\lambda(x - x_0)\big). \tag{6.39}$$

Again, the theorem applies to general norms in $\mathbb{R}^n$. The proof based on the strategy above is rather technical, see [**93**]; it is however somewhat simpler than the "classical" proof, based on rearrangement inequality and reduction to a one-dimensional problem (the inequality $\|\nabla f^*\|_{L^p} \le \|\nabla f\|_{L^p}$, where f^* is the monotone radially symmetric rearrangement of f, is known as the **Pólya-Szegö principle**; see Lieb [**176**] for a short proof based on the Riesz rearrangement inequality in the case $p = 2$).

In [**93**] it is also shown how to obtain by the same method the optimal Gagliardo-Nirenberg inequalities proven by Dolbeault and Del Pino [**104**],