CONVEX GEOMETRY

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1942. Volume XXXVI, № 7

MATHEMATICS

SMOOTHNESS OF THE CONVEX SURFACE OF BOUNDED GAUSSIAN CURVATURE

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(Communicated by I. M. Vinogradow, Member of the Academy, 25. VI. 1942)

Let F be a convex surface, i. e. a domain on the boundary of an arbitrary convex body. We take a point O of the surface F and a domain E on F containing the point O. Let further $\omega(E)$ denote the area of the spheric image* of the domain E, and $S(E)$ the area of E. We say that the Gaussian curvature of the surface F in the point O is bounded if the quotient

$$\frac{\omega(E)}{S(E)}$$

is bounded, when the domain E tends to the point O in an arbitrary manner (i. e. when the diameter of E tends to zero). The boundedness of the above quotient does evidently not imply the existence of the limit of the quotient, which is the Gaussian curvature. But if the Gaussian curvature exists and is bounded, then the above quotient is bounded.

The aim of the present note is to outline the proof of the following

Theorem 1. *If the Gaussian curvature of the surface F in the point O is bounded, then either there exists the tangent plane to F at the point O, or there exists a rectilinear edge along which the surface is broken, passing through O (O being not the end point of the edge). No other singularities are possible.*

Since every rectilinear segment belonging to a closed surface possesses an end point, Theorem 1 implies

Theorem 2. *The closed convex surface with everywhere bounded Gaussian curvature is smooth.*

Under the conditions of Theorem 1 it is possible that there exists a rectilinear edge passing through the point O. Such is, e. g., the plane broken along a straight line. As to the rectilinear edge, we have

Theorem 3. *If the point O of a convex surface F lies on a rectilinear segment L belonging to F (L may be an edge or not), then for any $\varepsilon > 0$ there exists an arbitrarily small domain E containing O such that*

$$\frac{\omega(E)}{S(E)} < \varepsilon.$$

* The spheric image of E is the set of end points of outer normals to all supporting planes of F at the points of E, these normals being laid off from the centre of a unit sphere. It is important that this definition be valid for any convex surface.

The following theorem follows from Theorems 1 and 3.

T h e o r e m 4. *If for a convex surface F the Gaussian curvature or the more general quotient* $\frac{\omega(E)}{S(E)}$ *is bounded by two positive numbers, then F is a smooth surface that does not contain any rectilinear segments.*

Theorems 1—4 are of interest not so much by themselves, as in connexion with the fact that the area of the spheric image $\omega(E)$ and the area $S(E)$ of the domain E are determined by the inner metric of the surface. Namely, as I have proved, the area of the spheric image of any domain on an arbitrary convex surface is determined by the inner metric of the surface, i. e. by the distance between its points* (the distance between any two points of the surface is defined as the greatest lower bound of the lengths of curves lying on the surface and joining the given points). Our theorems establish thus that the smoothness of the convex surface depends on the inner metric only.

For want of space we omit here the proof of Theorem 3 which will be published elsewhere.

The following theorem that follows from Theorem 1 gives the solution of a problem suggested by Cohn-Vossen (2,3).

T h e o r e m 5. *The regular surface of positive curvature cannot be bent without self-intersections in such a manner that one of its points should become an isolated singular point, i. e. that the surface should have no tangent plane at this point, being smooth in the neighbourhood of it.*

As it has been proved by Cohn-Vossen(2), if for a surface of positive curvature with the analytical metric ds^2 there exists a tangent plane at a point O and at every point of the neighbourhood of O, then the surface is twice differentiable in the point O, i. e. it possesses the finite elliptic indicatrix of Dupin.

We shall now outline the proof of Theorem 1. This theorem may evidently assume the following form as well.

If the surface F has no tangent plane at the point O and there is no rectilinear edge on F passing through O, then the Gaussian curvature in O is not bounded.

If the tangent plane at the point O does not exist and O does not lie on a rectilinear edge of the surface F, then, as it is known, three cases are possible:

I. O is a conic point, i. e. there exists an infinity of supporting planes passing through O and bounding a convex cone that is not reduced to a dihedral angle.

II. O is «edge-point», i. e. O does not belong to any rectilinear edge, but the supporting planes at O bound a dihedral angle.

III. O is an end point of a rectilinear edge.

Each of these cases must be considered separately.

1. If O is a conic point, then the area of the spheric image of O itself is positive, consequently, the Gaussian curvature in O is infinite.

2. Suppose that O is an «edge-point», P_1 and P_2 are supporting semi-planes that are the sides of the dihedral angle to which the tangent cone at the «edge-point» is reduced. Let the angle between these planes be $\pi-2\theta$, so that 2θ is the angle between the normals to them. Denote by T the intersection line of T_1 and T_2.

We take the plane Q perpendicular to the bisector of the angle formed by the normals to P_1 and P_2 at a distance z from O, so that Q cuts off a piece

* This result has been published without proof in (1). It is not necessary for the proof of Theorems 1—4.

E from the surface F*. We also suppose that the distance z is sufficiently small for the angles between Q, and the supporting planes to E to be less than $\frac{\pi}{2}$.

We take the planes Q_1 and Q_2 perpendicular to T in such a manner that the piece E of the surface cut off by Q lie between Q_1 and Q_2, having points in common both with Q_1 and Q_2. Let x_1 and x_2 be distances of the point O from the planes Q_1 and Q_2, respectively, so that $x=x_1+x_2$ is the distance between Q_1 and Q_2, or the length of the segment of T bounded by Q_1 and Q_2.

The planes P_1, P_2, Q_1, Q_2 and Q bound a trihedral prism containing E. The planes Q_1 and Q_2 do not cross E, since otherwise E would have two supporting planes perpendicular to Q, which is impossible by the hypothesis.

It is obvious that the area of the surface of this prism is not less than that of E.

The height of the prism is x, the distance of its edge lying on the straight line T from the plane Q is z, and the angle at this edge is equal to $\pi-2\theta$. Thus, the area of the surface of the prism is

$$2x\frac{z}{\sin\theta}+2xz\operatorname{ctg}\theta+z^2\operatorname{ctg}\theta.$$

Consequently,

$$S(E)\leqslant 2x\frac{z}{\sin\theta}+2xz\operatorname{ctg}\theta+z^2\operatorname{ctg}\theta,$$

whence, strengthening the inequality,

$$S(E)<\frac{2z(2x+z)}{\sin\theta}. \tag{1}$$

Take now the cone K projecting from the point O the basis of E. Any supporting plane P to K either is a supporting plane to E, or it cuts off a piece from E, at the «summit» of which there exists a supporting plane parallel to P. Hence follows that the spheric image of the cone K is included in the spheric image of E and, consequently, the area of the former does not exceed that of the latter, i. e.

$$\omega(K)\leqslant\omega(E). \tag{2}$$

We shall denote the outer normals to the supporting planes by the same letters as the corresponding points of the spheric image.

Let N_0 be the outer normal to the supporting plane at the point O which is the bisector of the angle between the normals N_1 and N_2 to P_1 and P_2. The points N_0, N_1, N_2 belong to the spheric image of the cone K, and N_0 is the middle point of the arc N_1N_2.

We take further the supporting planes to the cone K at those boundary points of E that belong to the planes Q_1 and Q_2. If M_1 and M_2 are the outer normals to these planes, φ_1 and φ_2 are angles between M_1, N_0 and M_2, N_0, respectively, then, as it is easily seen,

$$\operatorname{tg}\varphi_1=\frac{z}{x_1},\quad \operatorname{tg}\varphi_2=\frac{z}{x_2}, \tag{3}$$

where x_1 and x_2 are distances from the point O to the planes Q_1 and Q_2, z is the distance from O to the plane Q.

As the cone K is convex, its spheric image is convex, too. Besides, the latter contains the points N_1, N_2 and M_1, M_2, and consequently it

* Such a plane exists according to the hypothesis that O does not belong to a rectilinear segment lying on F.

contains the whole spheric quadrangle $N_1M_1N_2M_2$. It is easy to see that the diagonals of this quadrangle are perpendicular and intersect in the point N_0. The arcs N_0N_1 and N_0N_2 are equal to θ, and N_0M_1, N_0M_2 are equal to φ_1 and φ_2, respectively. It is evidently possible to choose such a positive constant a that the area of the quadrangle considered be no less than $a\theta(\varphi_1+\varphi_2)$. Since the quadrangle is included in the spheric image of the cone K, we shall have

$$\omega(K) \geqslant a\theta(\varphi_1+\varphi_2). \tag{4}$$

(Here a is an absolute constant. If φ_1 and φ_2 are very small, then the quadrangle $N_1M_1N_2M_2$ can be developed almost isometrically into a plane, so that its area be equal to $\theta(\varphi_1+\varphi_2)$ up to the small quantities of higher orders).

If we take the plane Q sufficiently close to the point O, then the angles φ_1 and φ_2 will be less than a certain $\alpha<\frac{\pi}{2}$, and we shall have

$$\varphi_1 > \frac{\alpha}{\operatorname{tg}\alpha}\operatorname{tg}\varphi_1, \quad \varphi_2 > \frac{\alpha}{\operatorname{tg}\alpha}\operatorname{tg}\varphi_2. \tag{5}$$

On the other hand, since x_1 and x_2 are less than $x=x_1+x_2$, it follows from (3) that

$$\operatorname{tg}\varphi_1 > \frac{z}{x}, \quad \operatorname{tg}\varphi_2 > \frac{z}{x}. \tag{6}$$

Using (5) and (6) and putting $\frac{a\alpha}{\operatorname{tg}\alpha}=C$, we shall obtain from (4)

$$\omega(K) > 2C\theta\frac{z}{x}, \tag{7}$$

whence by inequality (2)

$$\omega(E) > 2C\theta\frac{z}{x}. \tag{8}$$

Now from this inequality and from (1) follows

$$\frac{\omega(E)}{S(E)} > \frac{C\theta}{\sin\theta}\cdot\frac{1}{x(2x+z)}. \tag{9}$$

If the plane Q cutting off E is drawn nearer and nearer to O, i. e. if $z\to 0$, then $x\to 0$. Consequently, the curvature in the point O is not bounded.

3. Let O be an end point of a rectilinear edge L, being at the same time no conic point. Let P_1 and P_2 be the extreme supporting planes passing through L. We take the plane Q perpendicular to the bisector of the angle between the normals to P_1 and P_2 and crossing the surface F. Then we take the plane Q_1 passing through the point O perpendicular to P_1 and P_2. The planes Q and Q_1 cut off a piece E from the surface F. To this piece we can apply the same reasoning as to the case of the «edge-point». Considering the spheric image of the cone K projecting the boundary of E from the point O, we must take only those supporting planes to K that do not cross the boundary of E. Otherwise E may contain no points with a supporting plane parallel to the given supporting plane to the cone K. It is easy to show that the supporting plane P to the cone K does not cross the boundary of E, if the angle ψ between the plane Q_1 and the intersection of planes P and Q satisfies the condition

$$\operatorname{ctg}\psi > \frac{z}{x}\operatorname{ctg}\theta,$$

where z is the distance from O to the plane Q; x is the maximum distance from a point of the boundary of E to the plane Q_1; $(\pi-2\theta)$ is the angle formed by the planes P_1 and P_2.

Using the above condition, it is easy to obtain the evaluation for the area of the spheric image of E that differs from the evaluation (8) by a constant factor only. This enables us to conclude in an analogous way to the preceding case that the Gaussian curvature in the point O is not bounded.

This completes the proof of the theorem.

We shall now sketch the proof of Theorem 5.

Let F be a surface without self-intersections with everywhere finite continuous positive curvature. We shall suppose F to be regular everywhere, except, perhaps, the single point O. We intend to prove that F has a tangent plane at the point O, too. According to Theorem 1, it is sufficient to prove that the surface F is convex in the neighbourhood of the point O.

We take the point A on F, different from O, and the tangent plane P_0 to F at this point. Les us take the plane P parallel to P_0 and move it from P_0 to the point O in the direction of the inner normal to F. As it was shown by Cohn-Vossen([4]), the piece E of F cut off by the plane P will be convex until the plane reaches the position P_1, at which the boundary of E contains either the boundary points of F or the singular point O. If the boundary of E contains the points of the boundary of F, however near the point A to the point O may be, then, as it is easily verified, F contains a rectilinear segment joining the point A with the boundary of F. This is impossible, because F has the positive curvature and is regular everywhere, except the point O.

Therefore, as soon as the point A is sufficiently close to O, the plane P_1 passes through O and cuts off a convex piece E from the surface F. Consequently, the neighbourhood of the point O in F consists of such convex pieces. If we subject F to an infinite similitude transformation with the centre O, it will be carried into a cone K consisting of pieces of convex cones, into which the above convex pieces are taken. We may conclude that if K has no self-intersections, K is convex. Hence it is possible to deduce that the neighbourhood of the point O consisting of convex pieces is convex.

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Received
25. VI. 1942.

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