way that $k_1+k_2+\ldots+k_n \leqslant 2\pi$. Let, further, a polyhedral angle V^* be given with the area of the spherical image equal to $k_1+k_2+\ldots+k_n$ and so situated that any perpendicular to T either has no points in common with V, or its whole semi-line lies within it. There exists then an infinite convex polyhedron with n vertices projecting into $A_1, A_2, \ldots, A_n$ and having the areas of their spherical images equal to $k_1, k_2, \ldots, k_n$.

We take the $(n-1)$-dimensional manifold P consisting of all convex polyhedrons with a given limit cone V, one of the vertices of which is A_1 and the projections of the others are $A_2, \ldots, A_n$, and the $(n-1)$-dimensional manifold Q consisting of complexes of positive numbers $k_1, k_2, \ldots, k_n$ such that their sum is equal to the area of the spherical image of V. We have a natural mapping of P onto Q and by means of Brower's theorem of invariance of the domain we prove that it is a mapping of P onto Q.

The theorem, being proved for the case of polyhedrons, can be extended on the case of surfaces by means of a limit process, for which the following lemma is important:

If the sequence of ICCS's F_i converges to F, then $k_i(E)$, the corresponding integral curvatures reduced to T, weakly converge to $k(E)$, the integral curvature of F reduced to T; in other words, for any bounded continuous function $f(x)$ on T

$$\lim_{i\to\infty} \int_T f(x)\, k_i(dE) = \int_T f(x)\, k(dE).$$

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MATHEMATICS

EXISTENCE AND UNIQUENESS OF A CONVEX SURFACE WITH A GIVEN INTEGRAL CURVATURE

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Let F be a convex surface, i. e. a domain on the boundary of a convex body (finite or infinite). Let E be a subset of F. Consider the supporting planes to F at every point of E and the outer normals to these supporting planes. The set of the end-points of the normals, the latter being considered as the radii of a fixed unit sphere, is called the spherical image of E, and its area, the integral curvature of E. It is easy to see that the integral curvature is a non-negative completely additive set function defined on the totality of all Borel sets of the convex surface ([1]).

Our purpose will be 1°. to obtain necessary and sufficient conditions for a set function to represent the integral curvature of a convex surface, and 2°. to find out in how far the convex surface is determined by its integral curvature considered as a set function. We shall investigate only complete convex surfaces, i. e. complete boundaries of convex bodies; closed and open infinite surfaces will be studied separately. Our problem is yet quite indefinite, since the integral curvatures of different surfaces are defined on different systems of sets and therefore cannot be compared. To avoid this difficulty we must define the integral curvature of the surfaces on one and the same system of sets. The statements and proofs of the following theorems are given for the three-dimensional space; they hold, however, for n-dimensional space ($n \geqslant 2$). The only difference between our case and the general one is that one must take the areas of the unit sphere and the semi-sphere in the n-dimensional space instead of 4π and 2π.

Consider a unit sphere S with centre O and a closed convex surface F such that O be within F. The projection from the centre O determines a homeomorphism between F and S. If E_F and E_S are subsets of F and S, respectively, corresponding to each other under this homeomorphism, then, assigning to E_S the value of the integral curvature of the set E_F, we obtain a function defined on the subsets of S. We call this function the **integral curvature of F reduced to the sphere S**.

Theorem 1. *In order that a non-negative completely additive set function $k(E)$ defined on the Borel sets of the sphere S be the integral curvature reduced to S of a closed convex surface, it is necessary and sufficient that: 1) $k(S)=4\pi$, 2) for any convex subset E of S $k(E) < 4\pi-\varphi$, where φ is the area of the spherical image of the cone projecting E from the centre of S.*

This theorem has already been proved by the author (²). The necessity of the above conditions being evident, it is, properly speaking, a theorem of existence of a surface with a given integral curvature. The corresponding theorem of uniqueness can be formulated as follows:

T h e o r e m 2. *Let F_1 and F_2 be closed convex surfaces and O a point lying within each of them. Suppose that for every pair of Borel sets on F_1 and F_2 corresponding to each other when projected from O, the areas of their spherical images are equal. Then F_1 and F_2 are similar, O being the centre of similitude.*

P r o o f. Let F_1, F_2 and O have the meaning just defined. The points ying on the same ray drawn out of O we shall call corresponding points.

It is easy to prove that if the tangents to F_1 and F_2 at any corresponding points of smoothness* of F_1 and F_2 are parallel, then F_1 and F_2 are similar, O being the centre of similitude**.

Assume now that the tangent planes to F_1 and F_2 at the respective points of smoothness x_1 and x_2 are not parallel. By means of a transformation of similitude of F_2 with a centre O we make point x_2 coincide with x_1. The transformed surface will also be denoted by F_2. The identity of areas of the spherical images of the corresponding subsets of F_1 and F_2 is evidently invariant under this transformation.

F_1 and F_2 intersect now at the point $x=x_1=x_2$. Let F_{11} be the part of F_1 lying outside F_2; F_{21}, the respective part of F_2; F_{12}, the part of F_1 lying within F_2; F_{22}, the respective part of F_2 lying outside F_1; and finally, let $F_{13}=F_{23}$ be the common part of F_1 and F_2.

Denote by $\omega(E)$ the spherical image of the set E. We shall prove that for a given situation of the surfaces F_1 and F_2 the area of $\omega(F_{21})$ is less than that of $\omega(F_{11})$. According to the conditions imposed, these areas must be equal. This contradiction will lead to the proof of our theorem.

1) $\omega(F_{11}) \supset \omega(F_{21})$. In fact, the supporting plane cuts off, at every point of F_{21}, a convex piece of F_{11} at the «summit» of which the supporting plane is parallel to the given one.

2) $F_{12}+F_{13}$ and $F_{21}+F_{23}$ are closed, consequently $\omega(F_{12}$ and $F_{13})$ and $\omega(F_{21}+F_{23})$ are closed, too; $\Omega-\omega(F_{12}+F_{13})$ and $\Omega-\omega(F_{21}+F_{23})$, where Ω is the whole sphere, are open. At the same time

$$\Omega-\omega(F_{12}+F_{13}) \supset \omega(F_{11})$$

and

$$[\Omega-\omega(F_{21}+F_{23})]\,\omega(F_{21})=0.$$

Thus, the set

$$\omega=[\Omega-\omega(F_{12}+F_{13})]\,[\Omega-\omega(F_{21}+F_{23})]$$

is open, contained in $\omega(F_{11})$ and not intersecting $\omega(F_{21})$.

3) In order to prove that the area of $\omega(F_{11})$ exceeds that of $\omega(F_{21})$ it suffices to show that ω is not void. Let us draw the bisector plane P between the tangent planes P_1, P_2 to F_1 and F_2 at the point $x=x_1=x_2$ (so that the normal to P be the bisector of the angle formed by the outer normals to P_1 and P_2). The plane P cuts off two pieces of F_{11} and F_{21} at the «summits» of which the supporting planes are parallel to P. The spherical image of these supporting planes belongs neither to $\omega(F_{12}+F_{13})$ nor to $\omega(F_{21}+F_{23})$***. Consequently, it belongs to ω, q. e. d.

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* Point of smoothness of the surface is such a point at which there exists a tangent plane.

** It suffices to remark that almost every point of a convex surface is a point of smoothness.

*** It is quite evident, if we consider the projections of F_1, F_2, P_1, P_2 and P in the direction of the intersection line of planes P_1 and P_2.

We shall now turn to infinite complete convex surfaces, abbreviated: ICCS's. In what follows we exclude cylinders from the class of ICCS's (which, their integral curvature being zero, are of no interest). At the same time we refer to this class the rays and the infinite plane domains, except semi-planes and regions bounded by parallel straight lines, which should be considered as limit cases of cylinders. Such domains are limit cases of ICCS's. In the sequel the term ICCS will be understood to have these qualifications.

Let F be an ICCS and T a plane situated so that the intersection of any normal to T with a body bounded by F (including F) is either void, or is a semi-line. Consider the orthogonal projection of F onto T. Let E_T be a Borel set on T, and E_F—its complete φ—original, which is a Borel set, too. Assigning to E_T the value of the integral curvature of the set E_F*, we define a set function on T and call it the integral curvature of F reduced to the plane T. It is non-negative, completely additive and defined on all Borel sets of T.

T h e o r e m 3. *A non-negative completely additive set function $k(E)$ defined on all Borel sets of the plane T is an integral curvature reduced to T of an ICCS if and only if its value for the whole plane T is positive and does not exceed 2π, i. e. $0<k(T)\leqslant 2\pi$.*

It is easy to prove that the spherical image of every ICCS is a convex subset of the sphere and, consequently, contained in the closed semi-sphere, so that its area does not exceed 2π. The necessity of the condition is thus established. As to the sufficiency, we can obtain a more precise result by introducing the following new notion:

The convex cone, whose spherical image coincides with the closure of the spherical image of the ICCS F, will be called the limit cone of F. The limit cone of an ICCS always exists, in virtue of the convexity of the latter. It can be represented as the intersection of semi-spaces bounded by the planes parallel to the supporting planes to F and passing through a fixed point.

T h e o r e m 4. *Let $k(E)$ be a non-negative completely additive set function defined on Borel sets of the plane T and such that $k(T)\leqslant 2\pi$. Let, further, V be a convex cone**, the area of the spherical image of which is equal to $k(T)$, situated so that any perpendicular to T either has no points in common with T or its whole semi-line lies within T. There exists then a unique (up to a translation in the direction normal to T) ICCS such that $k(T)$ is its integral curvature reduced to T and V is its limit cone.*

The required uniqueness may be proved in the same way as Theorem 2, except that (1) the transformation of similitude must now be replaced by the translation in the direction perpendicular to T (which, of course, may be considered as a transformation of similitude with the centre at infinity); (2) the various parts of F_1 and F_2 (F_{11}, F_{12} a. o.) may be infinite; (3) the spherical images of F_1 and F_2 should be considered up to the points belonging to their common boundary***, because the fact that they do not necessarily coincide there can present certain difficulties; this does not affect the areas, since $\omega(F_1)$ and $\omega(F_2)$ are convex and their closures coincide.

As to the existence of ICCS's stated in the theorem, it can be proved essentially in the same way as Theorem 1: supposing the uniqueness to have been established and applying the principle of invariance of the domain, we shall first prove existence theorem for the case of polyhedrons.

T h e o r e m 5. *Let n points A_1, A_2, ..., A_n in the plane T be given. Suppose that a positive number k_i is assigned to each of these points A_i in such*

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* If E_F is void, then the corresponding value of the integral curvature is zero.

** Rays and convex plane angles are also considered as convex cones.

*** $\omega(F_1)$ and $\omega(F_2)$ have the common boundary, since they are convex and their closures coincide with the spherical image of the limit cone V of F_1 and F_2.