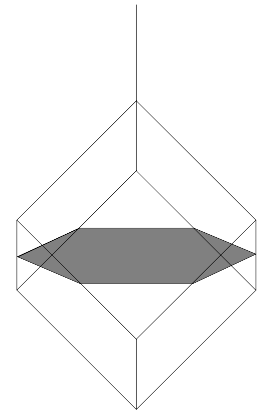
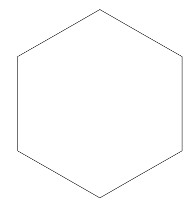
Cut cube



Take a cube and suspend it by one corner and then slice it in half with a horizontal plane through its centre. The cross-section will be a two-dimensional shape. What does it look like?

We ask the class to guess and the most popular conjecture is a square. Then we have them construct cubes out of straws and pipe cleaners, and they see that it's actually a hexagon.

In fact, using ideas of symmetry we can argue that a hexagon is the only possibility without even looking at a model. Indeed we will find in this problem that we can squeeze a lot of power out of symmetry.



Here's the argument . Of the six faces of the cube, three are adjacent to the top vertex and three to the bottom. By symmetry of the three top faces the cutting plane will have to cut those three in the same way, and the same with the three bottom faces. Then by symmetry between the top and bottom vertex, the intersection with all six faces has to be the same. Now the intersection (of a square with a plane) in a line, so we have a six sided figure. By symmetry of all six faces, the sides will all be the same length, so the intersection is a regular hexagon.

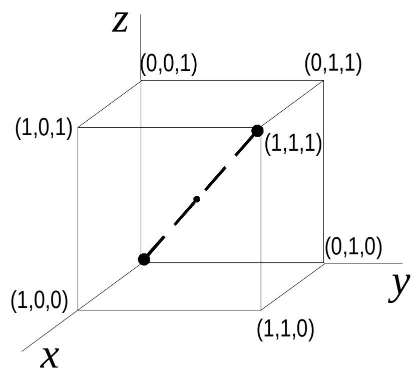
Into hyperspace.
Now for the real problem. Suspend a tesseract (a four dimensional hypercube) by one corner and again slice it in half with a horizontal (3-dimensional) hyperplane. What is the shape of the cross-section?

This is an awesome problem. More than any I have recently seen it illustrates the power of a simple idea, if used creatively, to circumvent a difficult piece of analysis. In this case, the idea is that of symmetry.

A lot of our thinking for this problem is done by analogy, going from 3 up to 4. In three dimensions, the intersection was a two-dimensional polygon, so in four dimensions it would appear to be a *three*-dimensional *polyhedron*—something you could hold in your hands! But what would it look like? How many faces would it have and what shape would they be? At first we do not have a lot of intuition for the answer. But the problem is nevertheless quite enticing.

Even with the analogy working for us, we will probably find geometric arguments difficult. This looks like a place where coordinate analysis will come in useful. We start by documenting the 3-dimensional case.

Position the cube in the positive octant with one vertex at the origin and the opposite vertex at (1,1,1). Then the six faces are the planes *x*=0, *x*=1, *y*=0, *y*=1, *z*=0 and *z*=1, and the eight vertices are the points with coordinates either 0 or 1. And the inside of the cube is the set of all points whose coordinates are all between 0 and 1.



Now we have to hang the cube by a vertex. But which vertex should we take? In many ways (1,1,1) is the obvious choice for the top and its opposite (0,0,0) would be the bottom. There might be some nice symmetry advantages to this choice—the coordinates of both are symmetric in *x*, *y* and *z*. More of this to come!

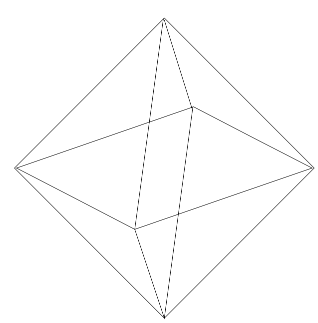
Then the three faces adjacent to the top vertex (1,1,1) are the planes *x*=1, *y*=1, and *z*=1, and the three faces adjacent to the bottom vertex (0,0,0) are the planes *x*=0, *y*=0, and *z*=0.

Now let's try the same thing in 4 dimensions. To describe the hypercube we need four coordinates: *w*, *x, y* and z. We hang it from the vertex (1,1,1,1) and the bottom vertex will be (0,0,0,0). We can't visualize it, but we know that the "faces" (which will be three dimensional) will be the hyperplanes *x*=0, *x*=1, *y*=0, *y*=1, *z*=0, *z*=1, *w*=0 and *w*=1and thus there are 8 of them, four adjacent to the top vertex and four adjacent to the bottom. And there will be 2^4 = 16 vertices, being all the points with coordinates either 0 or 1.

What does a face look like? It's a three-dimensional cube. This is easiest to see algebraically. For example the face *w*=0 is the set of all *x*, *y*, *z* coordinates which are between 0 and 1, and that's just the 3-D cube!

Now let's think about the intersection of the cutting hyperplane and the hypercube. As before, we observe that there's a symmetry among the top four faces and among the bottom four and also between the top and the bottom, and we conclude that the cutting plane must intersect each of the 8 faces in exactly the same way, and thus the intersection polyhedron has eight identical faces, each one a subset of one of the eight faces of the hypercube.

This already tells us a lot. How many polyhedra are there with 8 identical faces?—only one and that's the regular octagon which has 8 triangular faces. So at this early stage, with very little hard analysis, we have a strong conjecture about the shape of the intersection.



But are the faces triangular? Let's try a little verification. How might we show that the faces of the intersection polyhedron were indeed triangles? For example, consider the face w=1 of the hypercube. We want to show that the cutting hyperplane plane intersects this in a triangle.

We tackle this by finding an algebraic description of the cutting hyperplane. And to give the analogy a chance to work for us, let's drop down to 3 dimensions.

What is the equation of the cutting plane in 3-space? Well it's a plane so it will have an equation of the form

$$ax + by + cz = d.$$

Now one thing we know, since the top and bottom vertex are symmetric in x, y and z, that the equation of cutting plane must be symmetric in these variables, and therefore a, b and c must all be the same. We can normalize by taking them all to be 1, and hence the equation has the form:

$$x + y + z = d$$

for some d. To find d note that that the plane must pass through the very centre of the cube and that will be the point (½,½,½). [*Why?—because it has to be halfway between the top (1,1,1) and the bottom (0,0,0).*] Putting x=y=z=1/2 we get d=3/2 and the plane is

$$x + y + z = 3/2.$$

If you know a bit of vector algebra, you can also find the equation of this plane using the fact that it must be perpendicular to the "vertical," that is to the line joining the origin to the point (1,1,1). But it's nice to see that we can go a long way simply using the symmetry of the variables.

And indeed that same symmetry easily gives us the 4-D result. The hyperplane must have an equation of the form:

$$x+y+z+w = d$$

and when we plug in the coordinates of the centre (½,½,½,½) we get d=2, which gives us the equation

$$x+y+z+w = 2.$$

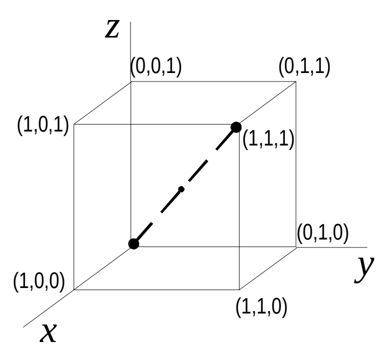
Now let's try to show that the intersection of this hyperplane with any face of the hypercube must be a triangle. To be specific, take the face w=1. Then the intersection will be described algebraically by substituting w=1 into the above equation, giving us

$$x+y+z = 1.$$

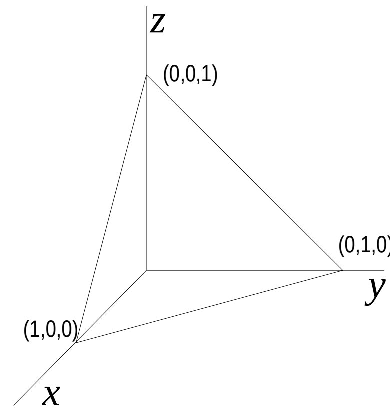
We only need x, y and z to describe this as w is fixed at 1.

The Platonic solids

A polyhedron is called regular if it has identical faces which are all regular polygons (all sides the same length). There are exactly five regular polyhedra, the "Platonic solids" and they are the tetrahedron (4 triangular faces), the cube (6 square faces), the octagon(8 triangular faces), the dodecahedron (12 pentagonal faces) and the icosahedron (20 triangular faces).

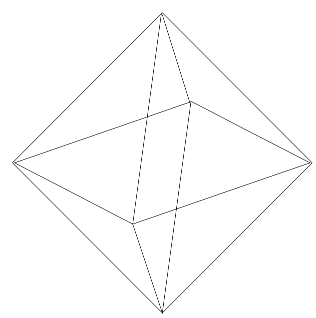


Now what is this?—it's the equation of a plane in 3-space. The set we are interested in is found by restricting x, y and z to lie between 0 and 1. It's enough actually to require all three coordinates to be ≥ 0, as then none of them can exceed 1.



So we want the intersection of the plane $x+y+z = 1$ with the positive octant, and this is easily seen to be a triangle with vertices (1,0,0), (0,1,0) and (0,0,1) on the three axes. And of course by symmetry, the triangle is equilateral.

One can of course go on and do additional checking and verification, but our accomplishments so far are quite impressive. We have a figure with 8 identical equilateral triangular faces. And that's an octahedron.



Note added after class: Ben made the observation that vertices of the 4D-cube with two 1's and two 0's, such as (0,1,1,0) must (by symmetry) lie half-way between (1,1,1,1) and (0,0,0,0) and therefore ought to be on the cutting hyper-plane. Indeed we can be sure of this: the equation of this plane is $x+y+z+w = 2$ and such points satisfy the equation! How many such vertices are there?—six! (the number of ways of choosing the 2 slots (out of 4) to put the 1's. So these must be the six vertices of the octahedron. That's interesting and unexpected—the vertices of the octahedron were already vertices of the 4D-cube.

Problems

1. Take the octahedron we found above and mark the correct 4-D coordinates on its 6 vertices. Of course there will be many right answers depending on how you decide to orient of the octahedron, but we have to be careful to get internal consistency at each triangular face. Once you have the picture labelled, label each of the 8 triangles with the correct face of the hypercube.

2.(a) Think about the 3-dimensional case in which the cutting plane passed through the centre of the hanging cube giving us a hexagonal cross-section. Suppose we slowly raise the cutting plane. Then we expect the shape of the intersection to change. Finally, when the cutting plane is raised all the way to the top, the intersection will be the single point (1,1,1). Describe how the intersection changes as the plane is lifted.

(b) Now, using the analogy between 3 and 4 dimensions, what can you say about how the intersection changes in the 4-dimensional case?