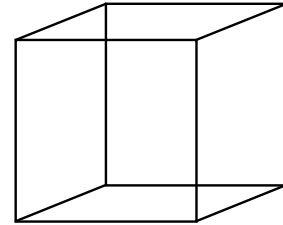


Cube face

How many faces does a cube have?—the answer is 6.

Okay, I'm going to restate the question: how many 2-dimensional faces does a 3-dimensional cube have?

The answer hasn't changed—it's still 6. But now we are aware of more general questions. There are different dimensions of cubes, and different dimensions of faces.



Let's stick for a moment with a 3-D cube, which is what we normally call simply a cube. It has faces of dimension 2 (squares) and 1 (edges) and 0 (vertices). There are 6 2-D faces, 12 1-D faces, and 8 0-D faces. And while we're at it, there is 1 3-D face (the whole cube).

I don't give a formal definition of face at this point, and it doesn't seem to be necessary. Its meaning will be clear from the first few examples. And at the end, we get a nice way to provide a definition.

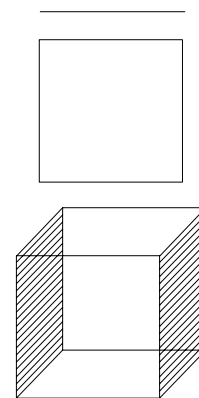
We are generalizing the notion of "cube" to other dimensions. For example, a 2-D cube is what we normally call a square, a 1-D cube is what we call a line segment. And a 0-D cube is a point. And if we try to count up the faces of different dimensions, we get the following table.

<i>dimension of cube</i>	# of Faces					
	0-D	1-D	2-D	3-D	4-D	5-D
0 (point)	1					
1 (segment)	2	1				
2 (square)	4	4	1			
3 (cube)	8	12	6	1		
4 (hypercube)	?	?	?	?	1	

What I call the hypercube is the 4-D generalization of the cube. I'm not sure quite what that is yet, but whatever it is, I expect it will have faces of all dimensions between 0 and 4, and the single face of dimension 4 will be itself. The problem, of course, is to finish the table.

Let's start by asking what the hypercube should be. The right approach is to ask how each previous cube is related to the cube one dimension lower, and see if that tells us how to construct the hypercube out of the cube.

Well, the 1-D cube is essentially two copies of the 0-D cube, on opposite ends of a line segment. Can we describe the 2-D cube in the same way? Yes we can--it's two copies of the 1-D cube (two segments) with line segments joining both pairs of corresponding vertices. Now let's try the 3-D cube. Yes it's the same thing--two copies of the 2-D cube (two squares) with line segments joining all four pairs of corresponding vertices. So that's what the 4-D cube ought to be--two copies of the 3-D cube (two cubes) with line segments joining all eight pairs of corresponding vertices. It's hard of course for us poor 3-D creatures to visualize but maybe we can "see" enough to fill in the table.



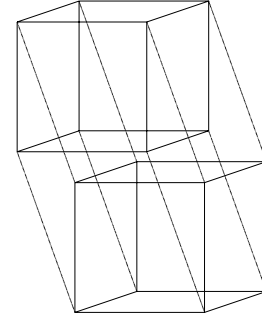
Okay: the faces of the 4-D cube.

0-D faces: easy--they're just the vertices, and there are 8 of them on each "end" for a total of 16.

1-D faces: not hard either. We have the edges of the two end cubes--that's 24 of those, and the 8 edges joining the 8 pairs of opposite vertices, for a total of $24 + 8 = 32$.

2-D faces: a bit more care is needed here. There are the 2-D faces of the two end cubes—6 on each for a total of 12. In addition, there'll be some created by the joins—one for every opposite pair of edges, which means there's as many of these as there are edges in a single cube: 12. The total is $12 + 12 = 24$.

3-D faces. Each of the ends is such a face—that's 2. In addition, there'll be some created by the joins—one for every opposite pair of 2-D faces, which means there's as many of these as there are 2-D faces in a single cube: 6. The total is $2 + 6 = 8$.



Finally we have a single 4-D face--the whole thing. And the table looks like this:

<i>dimension of cube</i>	# of Faces					
	0-D	1-D	2-D	3-D	4-D	5-D
0 (point)	1					
1 (segment)	2	1				
2 (square)	4	4	1			
3 (cube)	8	12	6	1		
4 (hypercube)	16	32	24	8	1	

In fact, the structure of our argument gives us a compelling arithmetic rule for getting the last row from the one above: the first entry is twice the one above, and the last entry is 1, and the other entries are each double the one above plus the one above to the left. And that also works for the other rows, and the same geometric thinking we used for the 4-D case established this. And it seems reasonable to extend this to higher dimensions too.

We can write a formal recursion. I'm going to use let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the number of k -dimensional faces of the n -cube. Then the above rule can be written:

$$\begin{bmatrix} n \\ k \end{bmatrix} = 2 \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

A solution to this recursion would give us an explicit formula for the entries of our table.

Now, an interesting question arises—can we find a "solution" to this recursion? Because, if so, we would have an explicit formula for the number of k -dimensional faces of the n -cube.

When I collect ideas about this, I find that someone often notices the resemblance to the recursion for the combinatorial coefficients. Surprise, surprise, as I guess my notation gave the game away right at the start.

Recall (see **Pascal**) that $\binom{n}{k}$ denotes the number of ways of choosing k objects from n . Then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and the solution to this recursion (with the appropriate initial conditions) is Pascal's Triangle:

				1										
			1	2	1									
		1	3	6	10	6	3	1						
	1	4	10	20	35	35	21	10	4	1				
1	1	6	15	35	70	105	126	126	105	70	35	15	6	1

In fact that suggests we try to write the above face-count table in the Pascal triangle form. I have done this directly below Pascal's triangle.

				1																												
			2	4	6	8	10	12	14	16	18	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100

To get each entry you add the two entries above it (like Pascal), except you first double the entry on the right.

Can we see how to get a formula for $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ in terms of the Pascal entries $\binom{n}{k}$? If we compare the triangles,

we notice a striking pattern. Note the diagonals on the modified triangle going down to the right. The first is all 1's and that's the same as the corresponding Pascal diagonal. The next is 2,4,6,8,10... and that's twice the Pascal diagonal. The next is 4,12,24,40... and that's four times the Pascal diagonal, etc. And so forth.

Well, that should give us a formula. Let's see—the top diagonal are the $k = n$ terms, the next diagonal are the $k=n-1$ terms, etc. We get:

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = 2^{n-k} \binom{n}{k}.$$

This is a lovely formula, and we can establish it rigorously by showing directly that it satisfies the recursion (Prob. 1)

Problems

1. Verify directly that $\begin{bmatrix} n \\ k \end{bmatrix} = 2^{n-k} \binom{n}{k}$ satisfies the recursion

$$\begin{bmatrix} n \\ k \end{bmatrix} = 2 \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

2. Our purpose here is to find a direct argument, using coordinate geometry, that the above formula for $\begin{bmatrix} n \\ k \end{bmatrix}$ must hold. I'll get you started by looking at the 1-D faces of the 3-D cube. First position the cube in the positive octant of x - y - z space, so that it is described algebraically as the set $0 \leq x, y, z \leq 1$. Think of the 12 edges (the 1-faces). Each such edge is parallel to one of the coordinate axes (x , y and z), and there are four edges parallel to each axis. In this way, 12 can be regarded as the product of 4 and 3. Now how does the 4 arise? Consider, for example, the 4 edges parallel to the z -axis. To specify any one of these, we need its x and y coordinates, and these can be either 0 or 1, giving 4 possibilities.

- (a) Produce a similar description for the 6 2-D faces of the cube, and for the 24 2-D faces of the 4-D cube.
 (b) Use this method to obtain a general derivation of the formula

$$\begin{bmatrix} n \\ k \end{bmatrix} = 2^{n-k} \binom{n}{k}$$

3. I'm going to define a sequence of figures called *simplexes*, one in each dimension. The 0-D simplex is a point. The 1-D simplex is a line segment. The 2-D simplex is a triangle. The 3-D simplex is a tetrahedron, drawn at the right. Observe that it is obtained from the triangle by taking a fourth vertex in a new dimension, and joining to it all three vertices of the triangle. In fact, note also that the same description will produce the triangle from the segment, and the segment from the point. If we perform the same operation on the tetrahedron, using the 4th dimension, we get the 4-D simplex, and then from that we can get the 5-D simplex, etc. The problem (of course) is this: how many k -dimensional faces does the n -D simplex have?

