**Two ants, revisited**

September 25, 2008

Dear Students,

I left tonight's class feeling like we did not arrive at a clear and convincing argument that the 40 foot path which we found from the spider to the ant was really the shortest route. So I decided to try to articulate such an argument here, as a model:

1) Suppose the spider is on the South wall of the room, and the ant is on the North wall. Colour the edges of the south wall red, and the edges of the north wall green.

2) Assume at least one shortest path exists from the spider to the ant, and imagine what happens as the spider traverses it.

3) Immediately after leaving the South wall of the box, the spider will be moving in a northward direction possibly at some angle. Whether or not he crosses from one of the long walls to another, his path cannot curve. It must be a straight line locally, near any point on the unfolded box, or else it would not be a shortest path. So he will continue to move in the northward direction at the same angle until he reaches the green edge of the north wall.

4) From red boundary to green boundary the spider will have traversed a distance of at least 30 feet. His shortest path cannot traverse this distance more than once, because we know that there is a path of length 40 which leads him to the ant. Therefore the spider traverses the distance from the red boundary to the green boundary exactly once.

5) To get from his initial position to the red boundary, the spider must follow a straight line path. If he were to exit his own square on any edge except the bottom one, he would traverse a distance of at least six feet before exiting his square. Similarly, if he were to enter the ant's square on any edge except the top one, he would traverse a distance of at least six feet from the green edge to the ant. Thus he must either exit his own square from the bottom, or he must enter the ant's square from the top, or both. Otherwise he would traverse a distance of at least $6+30+6 = 42$, which is not minimal.

6) First suppose the spider exit's his own square from the bottom. Then we might as well cut the other three red edges of the spider's square, leaving it attached to the box only along the bottom edge.

7) We can now unroll the box as we did in class, and Monica's argument becomes conclusive: using the notation T=Top, R=Right, B=Bottom, L=left, for the four long sides of the box, S=South for the spider's side, and n,),u or c for the ant's side, depending on its orientation. (The ant sits at the middle of the gap in the three sides of the letter... thus at the bottom of the letter n, the top of the letter u, and the right hand side of the letter c, and the left hand side of the parenthesis ).

$$
\begin{array}{cccccccccccccccccccccc}
 & & n & ) & u & c & n & ) & u & c & N & ) & u & c & N & ) & u & c & n & ) & u & ) & n & \\
\ldots & T & R & B & L & T & R & B & L & T & R & B & L & T & R & B & L & T & R & B & L & T & \ldots \\
 & & & s & & & & s & & & & S & & & & s & & & & s & & &
\end{array}
$$

8) Although this pattern repeats infinitely often, by symmetry I can focus on any one of the 's' sides I choose; they repeat periodically. Let's focus on the one that I capitalized.

9) If the spider goes from side $S$ to side $c$, it will either go to the $c$ immediately above $S$ and one place to the right, or immediately above $S$ and three places to the left. All the other $c$'s are further away. Similarly if the spider goes from the side $S$ to the side ), it will go the side ) immediately above and one place to the left, or else three places to the right. But the route to the right ) is further than the route to the right $c$, and the route to the left $c$ is further away than to the left ), which is the same distance from $S$ as to the right $c$: namely $\sqrt{(17)^2 + (37)^2} = \sqrt{1608}$, which we calculated in class and is larger than the $\sqrt{1600} = 40$ feet separating $S$ from either of the capitalized $N$'s, which are both the same distance from $S$.

10) Similarly, we showed in class that the distance from $S$ to the $u$ immediately above $S$ was 42 feet.

11) The schematic figure above makes it clear that the distance from $S$ to all of the other $u$'s is further that the distance from $S$ to the $u$ immediately above $S$, and that the distance from $S$ to all of the other $n$'s is further than the distance from $S$ to the capitalized $N$'s

12) So, assuming 6) is true, we can be confident that the 40 foot path from $S$ to $N$ is shorter than the path from $S$ to any $c$, $u$ or $n$, hence it is the shortest of all paths.

13) In case 6) is false, we argued in 5) that the spider enter's the ant's square from the top side. But this situation is precisely symmetrical to the one analyzed in (6)-(11) above, except with the role of the spider and the ant interchanged. So the same analysis, with North and South interchanged, yields the same conclusion: the shortest path in this case is also 40 feet.

14) QED (= quad est demonstratus).

This is an example of an optimization problem of a kind for which very few useful techniques are known: it is made complicated by the fact that it has locally minimizing solutions that are not globally minimizing, and moreover the global minimum of 40 feet is attained by more than one different path (corresponding to the two different $N$'s).

Still, to solve the problem we ended up needing to compute the precise lengths 40, $\sqrt{1608}$, and 42 of only three different paths. All other paths are ruled out by conceptual rather than computational arguments, such as the estimates at the end of 5) and in 4), 9) and 11), the schematic representation of the unfolded box using the letters T,R,B,L,S and $N$, its periodically repeating symmetry, or the insight exploited in 3) and again to compute these three lengths: namely that shortest paths must be straight lines locally on some unfolding of the box.