

# On the Monopolist's Problem Facing Consumers with Linear and Nonlinear Price Preferences

Robert J McCann

University of Toronto

*Click 'Talk 1' at [www.math.toronto.edu/mccann](http://www.math.toronto.edu/mccann)*

with Kelvin Shuangjian Zhang (ENS Paris / Waterloo / Fudan)

CPAM '19 + work in progress

30 June 2022

# Outline

- 1 Monopolist's problem
- 2 Examples and History
- 3 Hypotheses
- 4 Results
- 5 Proofs
- 6 A new duality certifying solutions
- 7 A free boundary problem hidden in Rochet-Choné's square example
- 8 Conclusions

# Monopolist's problem

Given compact sets  $X \subset \mathbf{R}^m$ ,  $Y \subset \mathbf{R}^n$ ,  $Z = [\underline{z}, \infty) \subset \mathbf{R}$ , and 'direct utility'

$G(x, y, z)$  = value of product  $y \in Y$  to buyer  $x \in X$  at price  $z \in Z$

$d\mu(x)$  = relative frequency of buyer  $x \in X$  (as compared to  $x' \in X$ )

$\pi(x, y, z)$  = value to monopolist of selling  $y$  to  $x$  at price  $z$

**Monopolist's problem:** choose price menu  $v : Y \rightarrow Z$  to maximize profits

# Monopolist's problem

Given compact sets  $X \subset \mathbf{R}^m$ ,  $Y \subset \mathbf{R}^n$ ,  $Z = [\underline{z}, \infty) \subset \mathbf{R}$ , and 'direct utility'

$G(x, y, z)$  = value of product  $y \in Y$  to buyer  $x \in X$  at price  $z \in Z$

$d\mu(x)$  = relative frequency of buyer  $x \in X$  (as compared to  $x' \in X$ )

$\pi(x, y, z)$  = value to monopolist of selling  $y$  to  $x$  at price  $z$

**Monopolist's problem:** choose price menu  $v : Y \rightarrow Z$  to maximize profits

$$\tilde{\Pi}(v) := \int_X \pi(x, y_v(x), v(y_v(x))) d\mu(x), \quad \text{where}$$

**Agent  $x$ 's problem:** choose  $y_v(x)$  to maximize

$$y_v(x) \in \arg \max_{y \in Y} G(x, y, v(y))$$

Constraints:  $v$  lower semicontinuous,  $(0, 0) \in Y \times Z$  and  $v(0) = 0$ .

# Examples

- airline ticket pricing
- insurance: monopolist's profit  $\pi(x, y, z)$  may depend strongly on buyer's identity  $x$ , even if regulation/ ignorance prohibits price  $v(y)$  from doing so
- $z$ -dependence of  $G(x, y, z)$  reflects different buyers price sensitivity / risk non-neutrality
- educational signaling
- optimal taxation: replace profit maximization with a budget constraint for providing services

Some history:  $G(x, y, z) = b(x, y) - z$

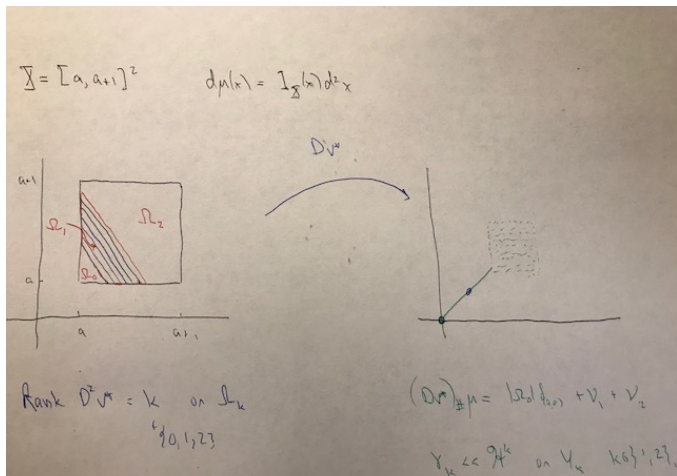
Mirrlees '71, Spence '73 ( $n = 1 = m$ ):  $\frac{\partial^2 b}{\partial x \partial y} > 0$  implies  $\frac{dy_v}{dx} \geq 0$

Rochet-Choné '98 ( $n = m > 1$ ):  $b(x, y) = x \cdot y$  bilinear implies  
 $y_v(x) = Dv^*(x)$  convex gradient; bunching

Some history:  $G(x, y, z) = b(x, y) - z$

Mirrlees '71, Spence '73 ( $n = 1 = m$ ):  $\frac{\partial^2 b}{\partial x \partial y} > 0$  implies  $\frac{dy_v}{dx} \geq 0$

Rochet-Choné '98 ( $n = m > 1$ ):  $b(x, y) = x \cdot y$  bilinear implies  $y_v(x) = Dv^*(x)$  convex gradient; bunching for  $\pi(x, y, z) = z - \frac{1}{2}|y|^2$



Carlier-Lachand-Robert '03:  $v^* \in C^1(\text{spt } \mu)$ ; Caffarelli-Lions  $v^* \in C^{1,1}$

Carlier '01:  $b(x, y)$  general implies existence of optimizer  $v = v^{b\tilde{b}}$

Chen '13:  $u \in C^1$  under Ma-Trudinger-Wang (MTW) conditions, where

$$u(x) = v^b(x) := \max_{y \in Y} b(x, y) - v(y)$$



Carlier-Lachand-Robert '03:  $v^* \in C^1(\text{spt } \mu)$ ; Caffarelli-Lions  $v^* \in C^{1,1}$

Carlier '01:  $b(x, y)$  general implies existence of optimizer  $v = v^{b\tilde{b}}$

Chen '13:  $u \in C^1$  under Ma-Trudinger-Wang (MTW) conditions, where

$$u(x) = v^b(x) := \max_{y \in Y} b(x, y) - v(y)$$

is called the 'indirect utility' to shopper  $x$

Figalli-Kim-M. '11:

convexity of principal's problem under strengthening of (MTW) on  $b(x, y)$

Noldeke-Samuelson (ECTA '18), Zhang (ET '19):

existence of maximizing  $v$  for general  $G \in C^0$

Daskalakis-Dekelbaum-Tzamos (ECTA '17), Kleiner-Manelli (ECTA '19):

duality for multigood auctions

# Hypotheses (c.f. Trudinger's generated Jacobian equations)

(G0)  $G \in C^1(X \times Y \times Z)$ ,  $m \geq n$ , and for each  $x, x_0 \in X \subset \mathbf{R}^m$ :

(G1)  $(y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z)$  is a **homeomorphism**

(G2) with **convex** range  $(Y \times Z)_x := (D_x G, G)(x, Y, Z)$  and inverse  $\bar{y}_G$ .

# Hypotheses (c.f. Trudinger's generated Jacobian equations)

(G0)  $G \in C^1(X \times Y \times Z)$ ,  $m \geq n$ , and for each  $x, x_0 \in X \subset \mathbf{R}^m$ :

(G1)  $(y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z)$  is a **homeomorphism**

(G2) with **convex** range  $(Y \times Z)_x := (D_x G, G)(x, Y, Z)$  and inverse  $\bar{y}_G$ .

DEFN:  $t \in [0, 1] \mapsto (x, y_t, z_t) \in X \times Y \times Z$  is called a **G-segment** if

$$(D_x G, G)(x, y_t, z_t) = (1 - t)(D_x G, G)(x, y_0, z_0) + t(D_x G, G)(x, y_1, z_1)$$

(G3) Assume  $t \mapsto G(x_0, y_t, z_t)$  is **convex** along **each G-segment**  $(x, y_t, z_t)$

# Hypotheses (c.f. Trudinger's generated Jacobian equations)

(G0)  $G \in C^1(X \times Y \times Z)$ ,  $m \geq n$ , and for each  $x, x_0 \in X \subset \mathbf{R}^m$ :

(G1)  $(y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z)$  is a **homeomorphism**

(G2) with **convex** range  $(Y \times Z)_x := (D_x G, G)(x, Y, Z)$  and inverse  $\bar{y}_G$ .

DEFN:  $t \in [0, 1] \mapsto (x, y_t, z_t) \in X \times Y \times Z$  is called a **G-segment** if

$$(D_x G, G)(x, y_t, z_t) = (1 - t)(D_x G, G)(x, y_0, z_0) + t(D_x G, G)(x, y_1, z_1)$$

(G3) Assume  $t \mapsto G(x_0, y_t, z_t)$  is **convex** along **each G-segment**  $(x, y_t, z_t)$

(G4)  $\frac{\partial G}{\partial z} < 0$  throughout  $X \times Y \times Z$  (i.e. buyers prefer lower prices)

(G5)  $\inf_{z \in Z} G(x, y, z) < G(x, 0, 0)$  for all  $(x, y) \in X \times Y$

(i.e. high enough prices force all buyers out of market)

(G6)  $\pi \in C^0(X \times Y \times Z)$

# Monopolists problem in terms of buyers' indirect utilities $u$

$$u(x) := v^G(x) := \max_{y \in Y} G(x, y, v(y)) \quad (1)$$

implies

$$(Du, u)(x) = (D_x G, G)(x, y_v(x), v(y_v(x)))$$

so we identify

$$(y_v(x), v(y_v(x)))$$

# Monopolists problem in terms of buyers' indirect utilities $u$

$$u(x) := v^G(x) := \max_{y \in Y} G(x, y, v(y)) \quad (1)$$

implies

$$(Du, u)(x) = (D_x G, G)(x, y_v(x), v(y_v(x)))$$

so we identify

$$(y_v(x), v(y_v(x))) = \bar{y}_G(Du(x), u(x), x)$$

and minimize

$$\begin{aligned} \tilde{\Pi}(v) &= \int_X \pi(x, \bar{y}_G(Du(x), u(x), x)) d\mu(x) \\ &=: \Pi(u) \end{aligned}$$

among  $u$  of form (1) (i.e. among so called  $G$ -convex  $u(\cdot) \geq G(\cdot, 0, 0)$ )

$$\max_{G(\cdot, 0, 0) \leq u \in \mathcal{U}} \Pi(u)$$

where

$$\mathcal{U} := \{u \mid u(\cdot) = \sup_{y \in Y} G(\cdot, y, v(y)) \text{ on } X \text{ for some } v : Y \rightarrow Z\}$$

THM 0: Given (G0-G1, G4-G6) the maximum above is attained. If  $\mu \ll \mathcal{L}^m$  the map  $x \rightarrow \bar{y}_G(Du(x), u(x), x)$  gives the consumer to (product, price) correspondence.

$$\max_{G(\cdot, 0, 0) \leq u \in \mathcal{U}} \Pi(u)$$

where

$$\mathcal{U} := \{u \mid u(\cdot) = \sup_{y \in Y} G(\cdot, y, v(y)) \text{ on } X \text{ for some } v : Y \rightarrow Z\}$$

THM 0: Given (G0-G1, G4-G6) the maximum above is attained. If  $\mu \ll \mathcal{L}^m$  the map  $x \rightarrow \bar{y}_G(Du(x), u(x), x)$  gives the consumer to (product, price) correspondence.

THM 1: If (G0-G2, G4-G5) hold then  $\mathcal{U}$  is convex if and only if (G3) holds.



$$\max_{G(\cdot, 0, 0) \leq u \in \mathcal{U}} \Pi(u)$$

where

$$\mathcal{U} := \{u \mid u(\cdot) = \sup_{y \in Y} G(\cdot, y, v(y)) \text{ on } X \text{ for some } v : Y \rightarrow Z\}$$

THM 0: Given (G0-G1, G4-G6) the maximum above is attained. If  $\mu \ll \mathcal{L}^m$  the map  $x \rightarrow \bar{y}_G(Du(x), u(x), x)$  gives the consumer to (product, price) correspondence.

THM 1: If (G0-G2, G4-G5) hold then  $\mathcal{U}$  is convex if and only if (G3) holds.

THM 2: If (G0-G6) hold then  $\Pi$  is concave on  $\mathcal{U}$  for all  $\mu \ll \mathcal{L}^m$  if and only if  $t \in [0, 1] \mapsto \pi(x, y_t, z_t)$  is concave on every G-segment  $(x, y_t, z_t)$ .

THM 2': same statement with both concaves replaced by convex.

- $\pi$  is 2-uniformly concave along all  $G$ -segments if and only if  $\Pi$  is 2-uniformly concave on  $\mathcal{U} \subset W^{1,2}(X, d\mu)$ .
  - alternately, strict concavity of  $\pi$  implies that of  $\Pi$ .
  - in either case above, when  $\mu \ll \mathcal{L}^m$  the hypotheses of THM 2 imply the principal's optimal strategy  $u$  is unique  $\mu$ -a.e. and stable:
- i.e.  $(G_i, \pi_i, \mu_i) \rightarrow (G_\infty, \pi_\infty, \mu_\infty)$  in  $C^2 \times C^0 \times (C^0)^*$  implies  $u_i \rightarrow u_\infty$  in  $L^\infty(d\mu_\infty)$

- $\pi$  is 2-uniformly concave along all  $G$ -segments if and only if  $\Pi$  is 2-uniformly concave on  $\mathcal{U} \subset W^{1,2}(X, d\mu)$ .
  - alternately, strict concavity of  $\pi$  implies that of  $\Pi$ .
  - in either case above, when  $\mu \ll \mathcal{L}^m$  the hypotheses of THM 2 imply the principal's optimal strategy  $u$  is unique  $\mu$ -a.e. and stable:
- i.e.  $(G_i, \pi_i, \mu_i) \rightarrow (G_\infty, \pi_\infty, \mu_\infty)$  in  $C^2 \times C^0 \times (C^0)^*$  implies  $u_i \rightarrow u_\infty$  in  $L^\infty(d\mu_\infty)$
- the Rochet-Choné  $G(x, y, z) = x \cdot y - z$  lies on the boundary of the set of preferences satisfying (G3)
  - if  $\|A\|_{C^1} \leq 1, \|B\|_{C^1} \leq 1$  with  $A$  convex,  $G(x, y) = x \cdot y - z - A(x)B(y)$  satisfies (G3) if and only if  $B$  is convex

# Proof of THM 1 (convexity of space $\mathcal{U}$ of utilities on $X$ )

Given  $u_0, u_1 \in \mathcal{U}$  and  $x_0 \in X$ , since  $u_0(\cdot) = \max_{y \in Y} G(\cdot, y, v_0(y))$  there exists  $(y_0, z_0) \in Y \times Z$  such that

$$u_0(\cdot) \geq G(\cdot, y_0, z_0) \quad \text{with equality at } x_0$$

# Proof of THM 1 (convexity of space $\mathcal{U}$ of utilities on $X$ )

Given  $u_0, u_1 \in \mathcal{U}$  and  $x_0 \in X$ , since  $u_0(\cdot) = \max_{y \in Y} G(\cdot, y, v_0(y))$  there exists  $(y_0, z_0) \in Y \times Z$  such that

$$u_0(\cdot) \geq G(\cdot, y_0, z_0) \quad \text{with equality at } x_0$$

Similarly

$$u_1(\cdot) \geq G(\cdot, y_1, z_1) \quad \text{with equality at } x_0$$

We'd like to deduce the same for  $\frac{1}{2}(u_0 + u_1)$ .

Adding the preceding yields

$$\begin{aligned}\frac{1}{2}(u_0 + u_1)(\cdot) &\geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\ &\geq G(\cdot, y_{\frac{1}{2}}, z_{\frac{1}{2}})\end{aligned}$$

by (G3), provided  $(y_{\frac{1}{2}}, z_{\frac{1}{2}})$

Adding the preceding yields

$$\begin{aligned}\frac{1}{2}(u_0 + u_1)(\cdot) &\geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\ &\geq G(\cdot, y_{\frac{1}{2}}, z_{\frac{1}{2}})\end{aligned}$$

by (G3), provided  $(y_{\frac{1}{2}}, z_{\frac{1}{2}})$  defined (using (G1-G2)) by

$$(D_x G, G)(x_0, y_t, z_t) := (1 - t)(D_x G, G)(x_0, y_0, z_0) + t(D_x G, G)(x_0, y_1, z_1)$$

Moreover, both inequalities are saturated at  $\cdot = x_0$ .

Adding the preceding yields

$$\begin{aligned}\frac{1}{2}(u_0 + u_1)(\cdot) &\geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\ &\geq G(\cdot, y_{\frac{1}{2}}, z_{\frac{1}{2}})\end{aligned}$$

by (G3), provided  $(y_{\frac{1}{2}}, z_{\frac{1}{2}})$  defined (using (G1-G2)) by

$$(D_x G, G)(x_0, y_t, z_t) := (1 - t)(D_x G, G)(x_0, y_0, z_0) + t(D_x G, G)(x_0, y_1, z_1)$$

Moreover, both inequalities are saturated at  $\cdot = x_0$ .

Thus  $\frac{1}{2}(u_0 + u_1) \in \mathcal{U}$ .

Conversely...







## Proof of THM 2 (concavity of $\Pi(u)$ )

Proof: For  $u_t := (1 - t)u_0 + tu_1 \in \mathcal{U}$ , we've assumed concavity (in  $t$ ) of

$$\pi(x, \bar{y}_G((1 - t)Du_0 + tDu_1, (1 - t)u_0 + tu_1, x)) \quad (2)$$

## Proof of THM 2 (concavity of $\Pi(u)$ )

Proof: For  $u_t := (1 - t)u_0 + tu_1 \in \mathcal{U}$ , we've assumed concavity (in  $t$ ) of

$$\pi(x, \bar{y}_G((1 - t)Du_0 + tDu_1, (1 - t)u_0 + tu_1, x)) \quad (2)$$

$$\Pi(u_t) := \int_{\mathcal{X}} \pi(x, \bar{y}_G(Du_t(x), u_t(x), x)) d\mu(x) \quad (3)$$

inherits this concavity.

Conversely,

## Proof of THM 2 (concavity of $\Pi(u)$ )

Proof: For  $u_t := (1 - t)u_0 + tu_1 \in \mathcal{U}$ , we've assumed concavity (in  $t$ ) of

$$\pi(x, \bar{y}_G((1 - t)Du_0 + tDu_1, (1 - t)u_0 + tu_1, x)) \quad (2)$$

$$\Pi(u_t) := \int_{\mathcal{X}} \pi(x, \bar{y}_G(Du_t(x), u_t(x), x)) d\mu(x) \quad (3)$$

inherits this concavity.

Conversely, if concavity of (2) fails for some  $t, x, u_0$  and  $u_1$ , it also fails in (3) for  $\mu$  concentrated uniformly on a small enough ball around  $x$ .  $\square$

## Differential condition for (G3)

When  $n = m$  set  $\bar{x} = (x_0, x)$ ,  $\bar{y} = (y, z)$  and  $\bar{G}(\bar{x}, \bar{y}) := x_0 G(x, y, z)$ .

Assume

(G7)  $\det D_{\bar{x}^i \bar{y}^j}^2 \bar{G}(\bar{x}, \bar{y}) \neq 0$  throughout  $\{-1\} \times X \times Y \times Z$

(G8)  $H(x, y, \cdot) = G^{-1}(x, y, \cdot)$  also satisfies hypotheses (G1-G2)

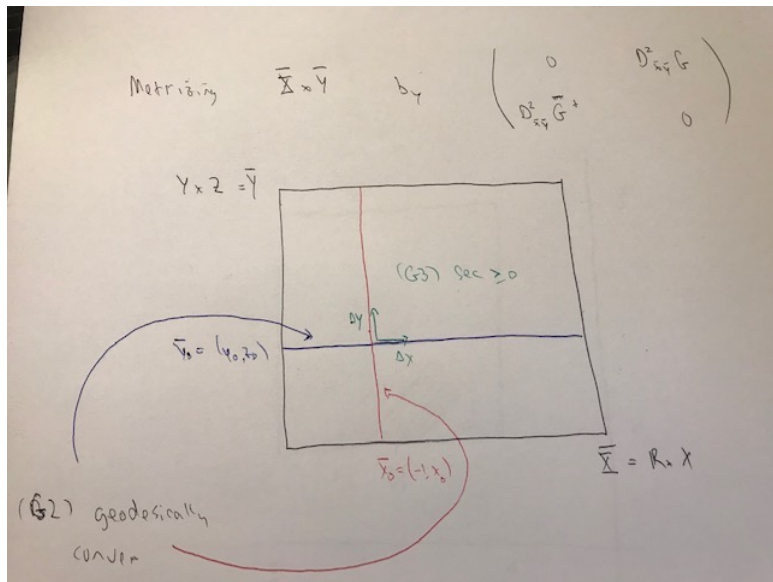
THM 3: If  $G \in C^4$  satisfies (G0-G2) and (G4-G8), then (G3) is equivalent to

$$\left. \frac{\partial^4}{\partial s^2 \partial t^2} \bar{G}(\bar{x}_s, \bar{y}_t) \right|_{(s,t)=(s_0,t_0)} \geq 0$$

holding along all  $C^2$  curves  $\bar{x}_s$  and  $\bar{y}_t$  for which  $t \in [0, 1] \rightarrow (x_{s_0}, \bar{y}_t)$  forms a **G-segment**.

Remark: (G3) is a **curvature** condition on  $(-\infty, 0) \times X \times Y \times Z$

# Pseudo-Riemannian geometry à la Kim-McCann '10



# A new duality for bilinear preferences

Following Rochet-Choné '98 choose  $G(x, y, z) = x \cdot y - z$  and  $X, Y \subset \mathbf{R}^n$  convex so

$$\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$

$$\in \mathcal{U} := \{u : X \rightarrow [0, \infty] \text{ convex} \mid Du(X) \subset Y\}$$

THM 3 (in progress):

$$\max_{u \in \mathcal{U}} \Pi(u) =$$

# A new duality for bilinear preferences

Following Rochet-Choné '98 choose  $G(x, y, z) = x \cdot y - z$  and  $X, Y \subset \mathbf{R}^n$  convex so

$$\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$

$$\in \mathcal{U} := \{u : X \rightarrow [0, \infty] \text{ convex} \mid Du(X) \subset Y\}$$

THM 3 (in progress):

$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \left\{ S : X \rightarrow \mathbf{R}^n \mid \int_X [(x - S(x)) \cdot Du - u(x)] d\mu(x) \leq 0 \right\}$$



THM 3:

$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \{S : X \rightarrow \mathbf{R}^n \mid \langle x \cdot Du(x) - u(x) \rangle_{\mu} \leq \langle S(x) \cdot Du(x) \rangle_{\mu}\}$$

In words: the **monopolists maximum profit** coincides with the **net value of a co-op** able to offer its members good  $y \in Y$  at price=cost  $c(y)$ , **minimized over possible distributions  $S_{\#}\mu$  of co-op memberships satisfying**

THM 3:

$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \{S : X \rightarrow \mathbf{R}^n \mid \langle x \cdot Du(x) - u(x) \rangle_{\mu} \leq \langle S(x) \cdot Du(x) \rangle_{\mu}\}$$

In words: the **monopolists maximum profit** coincides with the **net value of a co-op** able to offer its members good  $y \in Y$  at price=cost  $c(y)$ , **minimized over possible distributions  $S_{\#}\mu$  of co-op memberships** satisfying the strange constraint that when members whose true type is  $S(x)$  irrationally display the behaviour of  $x$  facing each monopolist price menu, the expected gross value of the resulting assignment  $Du(x)$  to those co-op members dominates the monopolist's expected gross revenue  $\langle x \cdot Du(x) - u(x) \rangle_{\mu}$ .

Proof sketch ( $\leq$ ):  $S \in \mathcal{S}$ ,  $u \in \mathcal{U}$  and the definition of  $c^*$  imply

$$\Pi(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_{\mu} \leq \langle c^* \circ S \rangle_{\mu}$$

$\geq$ : Conversely, using a convex-concave saddle argument in  $(S, u)$

$$\sup_{u \in \mathcal{U}} \langle x \cdot Du(x) - u(x) - c(Du(x)) \rangle_{\mu}$$

$$= \sup_{u \in \mathcal{U}} \inf_{T: Y \rightarrow \mathbf{R}^m} \langle x \cdot Du(x) - u(x) - T(Du(x)) \cdot Du(x) + c^*(T(Du(x))) \rangle_{\mu}$$

$$\geq \sup_{u \in \mathcal{U}} \inf_{S: X \rightarrow \mathbf{R}^m} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) + c^*(S(x)) \rangle_{\mu}$$

$\geq$ : Conversely, using a convex-concave saddle argument in  $(S, u)$

$$\begin{aligned} & \sup_{u \in \mathcal{U}} \langle x \cdot Du(x) - u(x) - c(Du(x)) \rangle_{\mu} \\ &= \sup_{u \in \mathcal{U}} \inf_{T: Y \rightarrow \mathbf{R}^m} \langle x \cdot Du(x) - u(x) - T(Du(x)) \cdot Du(x) + c^*(T(Du(x))) \rangle_{\mu} \\ &\geq \sup_{u \in \mathcal{U}} \inf_{S: X \rightarrow \mathbf{R}^m} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) + c^*(S(x)) \rangle_{\mu} \\ &= \inf_{S: X \rightarrow \mathbf{R}^m} \langle c^*(S(x)) \rangle_{\mu} + \sup_{u \in \mathcal{U}} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) \rangle_{\mu} \\ &= \inf_{S \in \mathcal{S}} \langle c^* \circ S \rangle_{\mu}. \end{aligned}$$

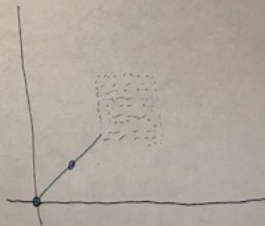
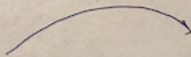
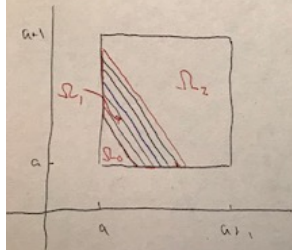
(To justify this argument rigorously requires approximating both problems before applying Fenchel-Rockafellar duality to obtain an infinite-dimensional version of the von Neumann min-max theorem.)

# Rochet-Choné's square example revisited; $c(y) = \frac{1}{2}|y|^2$

$$\mathcal{Y} = [a, a+1]^2$$

$$d\mu(x) = \mathbb{1}_{\mathcal{X}}(x) dx^2$$

$DV^*$



$$\text{Rank } D^2 V^* = k \quad \text{on } \mathcal{I}_k$$

$k \in \{0, 1, 2, 3\}$

$$(D\mu)_{\#} \mu = \mathbb{1}_{\Omega_0} \mu_{\Omega_0} + \nu_1 + \nu_2$$

$$\gamma_k < \mathcal{H}^k \quad \text{on } \mathcal{Y}_k \quad k \in \{1, 2, 3\}$$

# Variational calculus for

$$u \in \arg \max_{\text{convex } u \geq 0} \int_{[a, a+1]^2} [x \cdot Du - u(x) - \frac{1}{2}|Du(x)|^2] d\mu(x)$$

gives  $u = u_i$  on  $\Omega_i = \{x \mid \text{Rank}(D^2 u(x)) = i\}$  where

- on  $\Omega_0$  exclusion:  $u_0 = 0$

- on  $\Omega_1$ , Euler-Lagrange ODE: if  $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$  then  
 $k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + \text{const}$   
subject to boundary conditions  $u_1 = u_0$  and  $Du_1 = Du_0$  at lower boundary.

# Variational calculus for

$$u \in \arg \max_{\text{convex } u \geq 0} \int_{[a, a+1]^2} [x \cdot Du - u(x) - \frac{1}{2}|Du(x)|^2] d\mu(x)$$

gives  $u = u_i$  on  $\Omega_i = \{x \mid \text{Rank}(D^2 u(x)) = i\}$  where

- on  $\Omega_0$  exclusion:  $u_0 = 0$

- on  $\Omega_1$ , Euler-Lagrange ODE: if  $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$  then
$$k(s) = \frac{3}{4}s^2 - as - \log |s - 2a| + \text{const}$$
subject to boundary conditions  $u_1 = u_0$  and  $Du_1 = Du_0$  at lower boundary.

- on  $\Omega_2$  Euler-Lagrange PDE:  $\Delta u_2 = 3$  subject to boundary conditions

$$(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \bar{\Omega}_2$$

$$(Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad (\text{Neumann})$$

# Variational calculus for

$$u \in \arg \max_{\text{convex } u \geq 0} \int_{[a, a+1]^2} [x \cdot Du - u(x) - \frac{1}{2}|Du(x)|^2] d\mu(x)$$

gives  $u = u_i$  on  $\Omega_i = \{x \mid \text{Rank}(D^2 u(x)) = i\}$  where

- on  $\Omega_0$  exclusion:  $u_0 = 0$

- on  $\Omega_1$ , Euler-Lagrange ODE: if  $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$  then
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + \text{const}$$
subject to boundary conditions  $u_1 = u_0$  and  $Du_1 = Du_0$  at lower boundary.

- on  $\Omega_2$  Euler-Lagrange PDE:  $\Delta u_2 = 3$  subject to boundary conditions

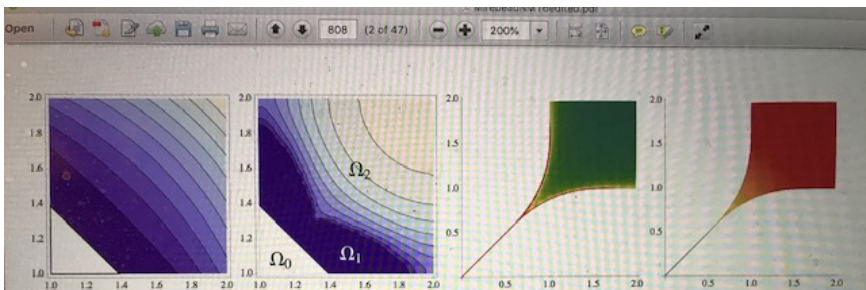
$$(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \bar{\Omega}_2$$

$$(Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad (\text{Neumann})$$

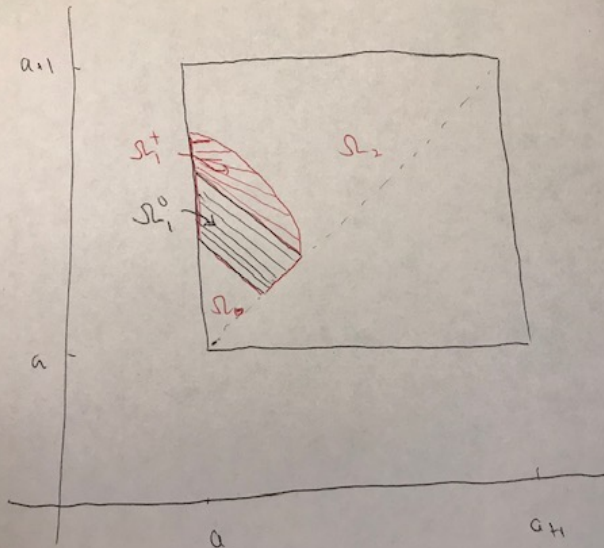
$$u_2 = u_1 \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad (\text{Dirichlet})$$

**OVERDETERMINED!**





**Fig. 1** Numerical approximation  $U$  of the solution of the classical Monopolist's problem (1), computed on a  $50 \times 50$  grid. *Left* level sets of  $U$ , with  $U = 0$  in white. *Center left* level sets of  $\det(\nabla^2 U)$  (with again  $U = 0$  in white); note the degenerate region  $\Omega_1$  where  $\det(\nabla^2 U) = 0$ . *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). Color scales on Fig. 10 (color figure online)



# Free boundary problem

$u = u_i$  on  $\Omega_i$  where

- on  $\Omega_0$  exclusion:  $u_0 = 0$
- on  $\Omega_1^0$ , **Rochet-Choné's** ODE:  $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$  where
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + \text{const}$$

subject to boundary conditions  $k = 0$  and  $k' = 0$  at **lower boundary**.

- on  $\Omega_1^+$ ,  $u_1 = u_1^+$  given by a **NEW** system of ODE (for height  $h(\cdot)$  and length  $R(\cdot)$  of isochoice segments together with profile of  $u_1^+(\cdot)$  along them), with boundary conditions

# Free boundary problem

$u = u_i$  on  $\Omega_i$  where

- on  $\Omega_0$  exclusion:  $u_0 = 0$

- on  $\Omega_1^0$ , **Rochet-Choné's** ODE:  $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$  where  
 $k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + \text{const}$

subject to boundary conditions  $k = 0$  and  $k' = 0$  at **lower boundary**.

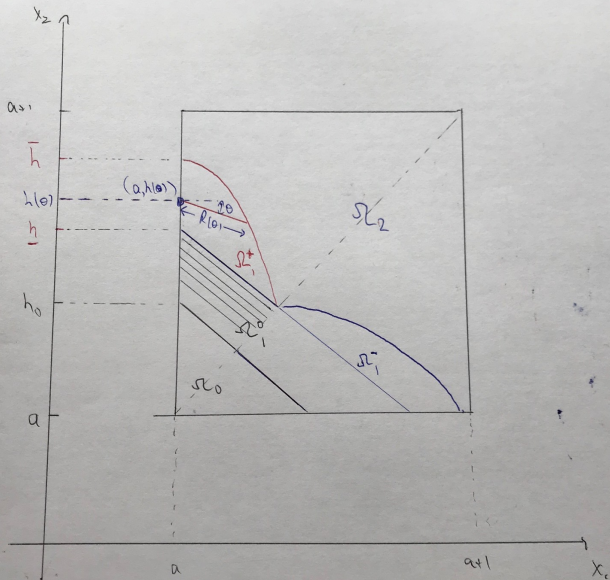
- on  $\Omega_1^+$ ,  $u_1 = u_1^+$  given by a **NEW** system of ODE (for height  $h(\cdot)$  and length  $R(\cdot)$  of isochoice segments together with profile of  $u_1^+(\cdot)$  along them), with boundary conditions  $u_1^+(x_1, x_2) = k(x_1 + x_2)$  and  
 $Du_1^+ = (k', k')$  on  $\partial\Omega_1^0 \cap \partial\Omega_1^+$

- on  $\Omega_2$ , PDE:  $\Delta u_2 = 3$  with **Rochet-Choné's overdetermined** conditions

$$(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \bar{\Omega}_2 \quad \text{and on} \quad \{x_1 = x_2\}$$

$$(Du_2 - Du_1^+) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad (\text{Neumann})$$

$$u_2 = u_1^+ \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad (\text{Dirichlet})$$



# Precise Euler-Lagrange equation in the 'missing' region $\Omega_1^+$

Index each isochoice segment in  $\Omega_1^+$  by its angle  $\theta \geq -\frac{\pi}{4}$  to horizontal. Let  $(a, h(\theta))$  denote its left-hand endpoint and parameterize the segment by distance  $r \in [0, R(\theta)]$  to  $(a, h(\theta))$ . Along this segment of length  $R(\theta)$ ,

$$u_1^+ \left( (a, h(\theta)) + r(\cos \theta, \sin \theta) \right) = m(\theta)r + b(\theta).$$

# Precise Euler-Lagrange equation in the 'missing' region $\Omega_1^+$

Index each isochoice segment in  $\Omega_1^+$  by its angle  $\theta \geq -\frac{\pi}{4}$  to horizontal. Let  $(a, h(\theta))$  denote its left-hand endpoint and parameterize the segment by distance  $r \in [0, R(\theta)]$  to  $(a, h(\theta))$ . Along this segment of length  $R(\theta)$ ,

$$u_1^+ \left( (a, h(\theta)) + r(\cos \theta, \sin \theta) \right) = m(\theta)r + b(\theta).$$

For  $\underline{h} \in [a, a+1]$ ,  $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow [0, a\sqrt{2})$  with  $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ , solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta) \sin \theta - m(\theta) \cos \theta + a) = \frac{3}{2}R^2(\theta) \cos \theta \quad (4)$$

$$m(-\frac{\pi}{4}) = 0, \quad m'(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}k'(a + \underline{h}).$$

# Precise Euler-Lagrange equation in the 'missing' region $\Omega_1^+$

Index each isochoice segment in  $\Omega_1^+$  by its angle  $\theta \geq -\frac{\pi}{4}$  to horizontal. Let  $(a, h(\theta))$  denote its left-hand endpoint and parameterize the segment by distance  $r \in [0, R(\theta)]$  to  $(a, h(\theta))$ . Along this segment of length  $R(\theta)$ ,

$$u_1^+ \left( (a, h(\theta)) + r(\cos \theta, \sin \theta) \right) = m(\theta)r + b(\theta).$$

For  $\underline{h} \in [a, a+1]$ ,  $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow [0, a\sqrt{2})$  with  $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ , solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta) \sin \theta - m(\theta) \cos \theta + a) = \frac{3}{2}R^2(\theta) \cos \theta \quad (4)$$

$$m(-\frac{\pi}{4}) = 0, \quad m'(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}k'(a + \underline{h}). \quad \text{Then set} \quad (5)$$

$$h(t) = \underline{h} + \frac{1}{3} \int_{-\pi/4}^t (m''(\theta) + m(\theta) - 2R(\theta)) \frac{d\theta}{\cos \theta}, \quad (6)$$

$$b(t) = \frac{1}{2}k(a + \underline{h}) + \int_{-\pi/4}^t (m'(\theta) \cos \theta + m(\theta) \sin \theta) h'(\theta) d\theta. \quad (7)$$



- for  $\underline{h} \in [a, a + 1]$ ,  $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow [0, a\sqrt{2})$  Lipschitz (say) and  $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$  we can solve (4)–(7) to find  $\Omega_1^+$  and  $u_+^1$ .
- we can then solve the resulting Neumann problem for  $\Delta u_2 = 3$  on  $\Omega_2$
- while it is not yet *rigorously* proved is that some choice of  $\underline{h}$  and  $R(\cdot)$  also yields  $u_1 - u_2 = \text{const}$  on  $\partial\Omega_2 \setminus \partial X$ ,

- for  $\underline{h} \in [a, a + 1]$ ,  $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow [0, a\sqrt{2})$  Lipschitz (say) and  $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$  we can solve (4)–(7) to find  $\Omega_1^+$  and  $u_+^1$ .
- we can then solve the resulting Neumann problem for  $\Delta u_2 = 3$  on  $\Omega_2$
- while it is not yet *rigorously* proved is that some choice of  $\underline{h}$  and  $R(\cdot)$  also yields  $u_1 - u_2 = \text{const}$  on  $\partial\Omega_2 \setminus \partial X$ , we hope to do this in the future
- if a choice exists such that, absorbing the constant into  $u_2$ , the resulting  $u$  given by  $u_i^{(\pm)}$  on  $\Omega_i^{(\pm)}$  for  $i \in \{0, 1, 2\}$  is in  $\mathcal{U}$ , our **new duality** can be used to certify that  $u$  is the desired **optimizer**

WHY DO WE EXPECT SUCH A CHOICE TO EXIST?

- for  $\underline{h} \in [a, a + 1]$ ,  $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow [0, a\sqrt{2})$  Lipschitz (say) and  $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$  we can solve (4)–(7) to find  $\Omega_1^+$  and  $u_+^1$ .
- we can then solve the resulting Neumann problem for  $\Delta u_2 = 3$  on  $\Omega_2$
- while it is not yet *rigorously* proved is that some choice of  $\underline{h}$  and  $R(\cdot)$  also yields  $u_1 - u_2 = \text{const}$  on  $\partial\Omega_2 \setminus \partial X$ , we hope to do this in the future
- if a choice exists such that, absorbing the constant into  $u_2$ , the resulting  $u$  given by  $u_i^{(\pm)}$  on  $\Omega_i^{(\pm)}$  for  $i \in \{0, 1, 2\}$  is in  $\mathcal{U}$ , our **new duality** can be used to certify that  $u$  is the desired **optimizer**

## WHY DO WE EXPECT SUCH A CHOICE TO EXIST?

- a unique optimizer  $\bar{u} \in \mathcal{U}$  is known to exist (Rochet-Choné) and  $\bar{u} \in C_{loc}^{1,1}(X^0)$  (Caffarelli-Lions); if the sets  $\Omega_i$  where its Hessian is rank  $i$  are **smooth enough**, and  $\Omega_1$  has the expected **3 components**, then (4)–(7) and the **overdetermined** Poisson problem  $\Delta u_2 = 3$  must be satisfied
- but maybe  $\Omega_i$  are not smooth enough, or  $\Omega_1$  is not (simply) connected and/or has more than three components (some too small for the numerics to resolve); we seriously doubt this, but can't rule it out rigorously yet...

## CONCLUSIONS

- **Concavity**, when present, is a powerful tool for optimization
- for numerics, uniqueness, stability, and characterization of optimum
- **Duality** of price menu  $v(y)$  with buyers' indirect utilities  $u(x) = v^G(x)$
- Necessary and sufficient conditions for **concavity** of monopolist's problem (as a function of  $u$ )
- Related to **curvature conditions** governing **regularity** in generated Jacobian equations (à la **Ma-Trudinger-Wang** and **Kim-McCann**) but
- adapted to payoffs  $G(x, y, z)$  which may depend **nonlinearly** on price  $z$
- **new duality** certifying solutions for  $G(x, y, z) = x \cdot y - z$
- square example requires solving an unexpected **free boundary** problem

## CONCLUSIONS

- **Concavity**, when present, is a powerful tool for optimization
- for numerics, uniqueness, stability, and characterization of optimum
- **Duality** of price menu  $v(y)$  with buyers' indirect utilities  $u(x) = v^G(x)$
- Necessary and sufficient conditions for **concavity** of monopolist's problem (as a function of  $u$ )
- Related to **curvature conditions** governing **regularity** in generated Jacobian equations (à la **Ma-Trudinger-Wang** and **Kim-McCann**) but
- adapted to payoffs  $G(x, y, z)$  which may depend **nonlinearly** on price  $z$
- **new duality** certifying solutions for  $G(x, y, z) = x \cdot y - z$
- square example requires solving an unexpected **free boundary** problem

THANK YOU!