ON THE EQUIVALENCE OF TWO DEFINITIONS OF SMOOTH EMBEDDINGS

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Theorem. *A map $F : M \to N$ is a diffeomorphism of M with an embedded submanifold of N if and only if it is an immersion and a homeomorphism with its image.*

Proof. In this proof we will denote the image of M under F by M', that is $M' := F(M)$.

First assume that $F : M \to N$ is a diffeomorphism of M with an embedded submanifold of N, namely M'. It is immediate that F is a homeomorphism with its image (since it is in fact a diffeomorphism), thus it remains to show that F is an immersion. That is, we wish to show that for each $p \in M$

$$F_* : T_pM \to T_{F(p)}N$$

is an injection. Fix an arbitrary $p \in M$. We know that $\widehat{F} : M \to M'$, defined by $\widehat{F}(m) = F(m)$ for each $m \in M$, is a diffeomorphism by assumption. Therefore, $\widehat{F}_* : T_pM \to T_{\widehat{F}(p)}M'$ is a linear isomorphism. Furthermore, if $i : M' \hookrightarrow N$ is an inclusion map, then $i_* : T_{\widehat{F}(p)}M' \to T_{F(p)}N$ is one-to-one. Since $F = i \circ \widehat{F}$, by properties of pushforward we have

$$F_* = i_* \circ \widehat{F}_* : T_pM \to T_{F(p)}N$$

and we conclude that F_* is an injection as a composition of two injective maps. This completes the "only if" part of the proof.

Now we prove the reverse implication. Assume that $F : M \to N$ is an immersion and a homeomorphism of M with M'. We wish to show that M' is an embedded submanifold of N and that F is diffeomorphism of M with M'. We will prove this statement in two ways using two different notions of embedded submanifold.

Method 1: For this method we will use the definition of the embedded submanifold in terms of slices. First we show that M' is indeed an embedded m-submanifold of N, where m is the dimension of M. That is, we would like to show that for each point $y \in M'$ there is a chart (V, ψ) for N near y such that

$$\psi(V \cap M') = \{(x^1, \dots, x^m, c^1, \dots c^{n-m}) \in \psi(V)\}$$

for some fixed $c^1, \dots, c^{n-m}$. Fix a point $y \in M'$. Since F is a homeomorphism with its image, $y = F(x)$ for some $x \in M$. By immersion theorem, there exist charts $\varphi = (x^1, \dots, x^m) : U \to \widehat{U}$ for M near x and a chart $\psi = (y^1, \dots, y^n) : V \to \widehat{V}$ for N near y such that $\psi \circ F \circ \varphi^{-1} : (x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$ and $F(U) \subset V$. Note that in this case

$$\psi(F(U)) = \{(x^1, \dots, x^m, 0, \dots, 0) \in \psi(V)\}$$

Now we would like to shrink V to some neighbourhood $\widetilde{V}$ so that $\widetilde{V} \cap M' = F(U)$ and then restrict ψ to $\widetilde{V}$ to get the desired conclusion. We will use an approach similar to the one used in the proof of the equivalence of the two definitions of embedded submanifold. Since F is a homeomorphism, $F(U)$ is open in M' equipped with subspace topology. Therefore, there exists a V' open in N such that $V' \cap M' = F(U)$. We let $\widetilde{V} = V \cap V'$ and we observe that $\widetilde{V} \cap M' = V \cap V' \cap M' = F(U)$ as required.

Finally, we prove that F is a diffeomorphism of M with M'. Since F is a homeomorphism, we know that F is a bijection and it suffices to show that F is a local diffeomorphism. Fix $p \in M$. Since $T_{F(p)}M'$ is a subspace of $T_{F(p)}N$ and F is an immersion, we know that $F_* : T_pM \to T_{F(p)}M'$ is one-to-one. Moreover, we know that T_pM and $T_{F(p)}M'$ have the same dimension since both M and M' are m-dimensional manifolds. Thus, F_* is in fact a linear isomorphism. Therefore, by Inverse Function Theorem, there exist neighbourhood U in M near p and V in M' near $F(p)$ such that $F : U \to V$ is a diffeomorphism. This shows that F is a local diffeomorphism and the proof is now complete.

Method 2: Now we use the other notion of the embedded submanifold. We will first prove that F is a diffeomorphism with its image. We already know that F is a smooth map from M to N and we also know that it is a bijection (since it is a homeomorphism). Therefore, it suffices to prove that $F^{-1} : M' \to M$ is smooth. That is, we would like to show that for every $y \in M'$ there is neighbourhood V in N near y and a smooth map $G : V \to M$ such that $G|_{V \cap M'} = F^{-1}$.

Fix an arbitrary point $y \in M'$. Then there is $x \in M$ such that $y = F(x)$ since F is a bijection. By immersion theorem, there are coordinate charts $\varphi = (x^1, \ldots, x^m) : U \to \widehat{U}$ for M near x and a chart $\psi = (y^1, \ldots, y^n) : V \to \widehat{V}$ for N near y such that $\psi \circ F \circ \varphi^{-1} : (x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$ and $F(U) \subset V$. Note that

$$\psi(F(U)) = \varphi(U) \times \{0\}^{n-m}$$

Now we would like to define a map G as a composition $\varphi^{-1} \circ \pi \circ \psi$, where π is a projection map from $\mathbb{R}^n$ to $\mathbb{R}^m$. However, we need to correctly restrict the domain of this map to some neighbourhood V', in order for it to be well-defined. In particular, we would like to have $\pi \circ \psi(V') \subset \varphi(U)$, which is equivalent to requiring $\psi(V') \subset \varphi(U) \times \mathbb{R}^{n-m}$. Notice that $\varphi(U) \times \mathbb{R}^{n-m}$ is open in $\mathbb{R}^n$ as a Cartesian product of two open sets and $(\varphi(U) \times \mathbb{R}^{n-m}) \cap \widehat{V}$ is open in $\mathbb{R}^n$ as an intersection of two open sets. Therefore, we may set

$$V' = \psi^{-1}((\varphi(U) \times \mathbb{R}^{n-m}) \cap \widehat{V})$$

which is open in N since ψ is a homeomorphism. We shrink V' to W as in our previous argument, so that $W \cap M' = F(U)$. Now $G : W \to M$ is well-defined and smooth as a composition of smooth maps. Moreover, a straightforward computation verifies that $G|_{W \cap M'} = F^{-1}$, which proves that F^{-1} is also smooth. Therefore, we conclude that F is a diffeomorphism.

It remains to verify that M' is an embedded submainfold, that is for every point in M' there exists a relative neighbourhood $W \subset M'$ and a diffeomorphism φ from W to an open subset $\widehat{W}$ in $\mathbb{R}^m$. For an arbitrary $y = F(x) \in M'$ we consider a coordinate chart $\psi : U \to \widehat{U}$ near x and we let $W = F(U)$ which is already known to be open in M'. Then $\varphi \circ F^{-1} : W \to \widehat{U}$ is the desired diffeomorphism. □