[In this note, references are still missing]

1. Discussion

In the literature there are many versions of the notion of "submanifold". Lee in his textbook defines an "immersed submanifold" of M to be a subset S, equipped with a topology with which S is a topological manifold, and equipped with a smooth structure such that the inclusion map into M is an immersion. (Given the topology, if such a smooth structure exists then it is unique.) Thus, an immersed submanifold is exactly the image of an injective immersion, equipped with the topology with respect to which the immersion is a homeomorphism. The figure eight has two different topologies with respect to which it is an immersed submanifold of $\mathbb{R}^2$. An irrational line in a 2-torus is an immersed submanifold in a unique way.

"Immersion" is undoubtedly an important notion. But I personally am not aware of a usage of the general notion of "immersed submanifold" beyond that of the notion of an "immersion", especially in situations like the figure eight.

On the other hand, there is another notion – a "diffeological submanifold" – that I find useful. A diffeological submanifold of M is a subset S of M that satisfies a certain property; it does not require an additional structure on S (such as a topology).

A diffeological subset of M inherits from M a smooth manifold structure (in particular a topology) in a sense that we will describe soon. This smooth manifold structure, if it exists, then it is unique. In particular the topology on S that is induced in this way (and which is not necessarily the same as the subset topology) is unique.

With this induced manifold structure on S, the inclusion map of S into M is an injective immersion. Thus, a diffeological submanifold is a special case of an "immersed submanifold" in the above sense. An example of a diffeological submanifold is an irrational line in a 2-torus.

2. Definition

The notion of "diffeology" is due to Souriau.

Let X be a set. A *diffeology* on X is a set of maps from open subsets of Euclidean spaces to X that satisfies three axioms. The maps in this set are called *plots*. The axioms are

(1) A constant map is a plot.
(2) Let U be an open subset of some $\mathbb{R}^k$ and let $p\colon U \to X$ be a map. Suppose that every point in U has a neighbourhood $V \subset U$ such that $p|_V\colon V \to X$ is a plot. Then p is a plot.

(3) Let U be an open subset of $\mathbb{R}^k$, let $p\colon U \to X$ be a plot, let V be an open subset of $\mathbb{R}^\ell$, and let $f\colon V \to U$ be a smooth (C^∞) map. Then the composition $p \circ f\colon V \to X$ is a plot.

A *diffeological space* is a set equipped with a diffeology.

A map $X \to Y$ between diffeological spaces is *smooth* if its precomposition with every plot of X is a plot of Y. It is a *diffeomorphism* if it is a bijection and both it and its inverse are smooth.

Every manifold M is naturally a diffeological space: a map from an open subset U of some $\mathbb{R}^k$ to M is a plot exactly if it is smooth (C^∞) in the usual sense. A map between manifolds is smooth in the usual sense if and only if it is smooth in the diffeological sense; it is a diffeomorphism in the usual sense if and only if it is a diffeomorphism in the diffeological sense.

We equip a diffeological space with the biggest topology with respect to which all the plots are continuous. Then "manifold" can be redefined as a (Hausdorff, second countable) diffeological space in which every point has an open neighbourhood that is diffeomorphic to an open subset of a Euclidean space.

If M is a manifold (or more generally a diffeological space) then every subset S of M inherits a diffeology: a map $p\colon U \to S$ from an open subset U of some $\mathbb{R}^k$ to M is a plot of S if and only if it is a plot of M. This is called the *subset diffeology* of S.

Definition 2.1. The subset S is a *diffeological submanifold* of M if, equipped with the subset diffeology, S is a manifold.

I think that this is equivalent to the following property of S, although I didn't check this: for every point p of S there exists a neighbourhood U of p in M and a chart $\varphi\colon U \to \widetilde{U} \subset \mathbb{R}^a \times \mathbb{R}^b$ that carries the connected component of p in $U \cap S$ to the slice $\widetilde{U} \cap (\mathbb{R}^a \times \{0\})$.