MAT1300 PROBLEM SET – WEEK 8

(1) Prove that every $n \times n$ matrix whose entires are all non-negative has a non-negative eigenvalue. (Hint: use the fact that $S^{n-1} \cap \mathbb{R}^n_{\geq 0}$ is homeomorphic to a closed disc and apply Brouwer's fixed point theorem to it.)

(2) Let $\varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4$ be the standard basis of $(\mathbb{R}^4)^*$. Show that $\alpha := \varepsilon^1 \wedge \varepsilon^2 + \varepsilon^3 \wedge \varepsilon^4$ is not decomposable. (Hint: if $\alpha$ were decomposable then $\alpha \wedge \alpha$ would be zero.)

(3) The $k$th de Rham cohomology of a smooth manifold $M$ is the quotient vector space

$$H^k(M) := \{\text{closed } k\text{-forms on } M\}/\{\text{exact } k\text{-forms on } M\}.$$

Show that the map

$$\cup\colon H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$$
$$[\alpha], [\beta] \mapsto [\alpha \wedge \beta]$$

is well defined.

(4) Compute the pullback of the differential form

$$\frac{1}{(x^2+y^2+z^2)^{3/2}}\,(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$$

under the map $(u, v) \mapsto (u, v, \sqrt{1-u^2-v^2})$ from the open unit disc in $\mathbb{R}^2$ to the complement of the origin in $\mathbb{R}^3$.