MAT1300 PROBLEM SET 9 (WEEK 10) – QUESTION 1:

Let $x^1, \dots, x^n$ be the standard coordinates on $\mathbb{R}^n$.

(a) Let $y^1, \dots, y^n$ be the coordinates on $\mathbb{R}^n$ with respect to another oriented orthonormal basis.
  (i) Show: $dy^1 \wedge \dots \wedge dy^n = dx^1 \wedge \dots \wedge dx^n$.
  (ii) Show: $\sum y^i \frac{\partial}{\partial y^i} = \sum x^i \frac{\partial}{\partial x^i}$. (This is sometimes called the *Euler vector field*. In fact, we don't really need the second basis to be oriented nor orthonormal.) Hint: for $v \in \mathbb{R}^n$, compute in coordinates the velocity vector at $t = 0$ of the curve $t \mapsto e^t v$.

(b) Consider the $(n-1)$ form

$$\omega := \frac{1}{\|x\|^n} \sum (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

on $S^{n-1} \subset \mathbb{R}^n$.
  (i) Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a rotation (represented by a matrix in $\mathrm{SO}(n)$). Show that $A^*\omega = \omega$. Hint: use parts (a) and (b).
  (ii) Show that ω is a volume form on S^{n-1}.