Smooth functions on embedded submanifolds

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be arbitrary subsets. A map $f\colon A \to B$ is *smooth* if for every point in A there exists an open neighbourhood U in $\mathbb{R}^n$ and a smooth map $F\colon U \to \mathbb{R}^m$ such that $F|_{U\cap A} = f|_{U\cap A}$. A map $A \to B$ is a *diffeomorphism* if it is a bijection and both it and its inverse are smooth. If A is open, smoothness in this sense coincides with smoothness in the usual sense. Moreover, the composition of maps that are smooth in this sense is also smooth in this sense.

Let $M \subset \mathbb{R}^N$ be an embedded submanifold, according to the definition that we gave in class: for every point in M there exists a relative neighbourhood in M that is diffeomorphic (in the above sense) to an open subset of $\mathbb{R}^n$. We proved that the set of such diffeomorphisms is a maximal atlas on M.

Now we have two notions of smoothness for a function $f\colon M \to \mathbb{R}$: the function can be smooth on M as a manifold, or it can be smooth on M as a subset of $\mathbb{R}^N$. The purpose of this note is to show that these two notions of smoothness are equivalent.

One direction is easy: suppose that $f\colon M \to \mathbb{R}$ is smooth on M as a subset of $\mathbb{R}^N$. Let $\varphi\colon U \to \widetilde{U}$ be a chart on M, that is, a diffeomorphism (in the above sense) from a relatively open subset U of M to an open subset $\widetilde{U}$ of $\mathbb{R}^N$. Consider the real valued function $f \circ \varphi^{-1}$ on the open subset $\widetilde{U}$ of $\mathbb{R}^n$. This function is a composition of smooth maps between subsets of Euclidean spaces, so it is smooth. This shows that the function f is smooth on M as a manifold.

For the other direction, we will use the following lemma.

Lemma. *For every point in M there exists an open neighbourhood W in $\mathbb{R}^N$ and a smooth map $\psi\colon W \to \mathbb{R}^n$ such that $\psi(W)$ is open, $\psi(W \cap M) = \psi(W)$, and $\psi|_{W\cap M}\colon W \cap M \to \psi(W \cap M)$ is a diffeomorphism.*

Proof. Fix a point p in M. By the definition of embedded submanifold, let U be a relative neighbourhood of p in M, and $\widetilde{U}$ an open subset of $\mathbb{R}^n$, and $\varphi\colon U \to \widetilde{U}$ a diffeomorphism.

By the definition of the relative topology, write $U = W' \cap M$ where W' is a neighbourhood of p in $\mathbb{R}^N$.

By the definition of smooth map on a subset of $\mathbb{R}^N$, let W'' be a neighbourhood of p in $\mathbb{R}^N$ and $\psi\colon W'' \to \mathbb{R}^n$ such that $\psi|_{W''\cap U} = \varphi|_{W''\cap U}$.

Let $W = W' \cap W'' \cap \psi^{-1}(\psi(W' \cap W'' \cap M))$. Clearly, $W \cap M = W' \cap W'' \cap M$, and $\psi(W) = \psi(W \cap M)$.

We have $W' \cap W'' \cap M = W'' \cap (W' \cap M) = W'' \cap U$. On this set, ψ coincides with φ, which is a homeomorphism. So $\psi(W' \cap W'' \cap M)$, being equal to $\varphi(W'' \cap U)$, is open. From the definition of W, this implies that W is open. Also, because $\psi(W' \cap W'' \cap M) = \psi(W \cap M) = \psi(W)$, this shows that $\psi(W)$ is open. Finally, because on the set $W \cap M = W' \cap W'' \cap M$ the restriction of ψ coincides with the restriction of φ, this restriction is a diffeomorphism with its image. $\square$

By the lemma, let $\{W_i\}_{i \in I}$ be a collection of open subsets of $\mathbb{R}^N$ whose union covers M, and, for each $i \in I$, let $\psi_i \colon W_i \to \mathbb{R}^n$ be a smooth map such that $\psi_i(W_i)$ is open, $\psi_i(W_i \cap M) = \psi_i(W_i)$, and $\varphi_i := \psi_i|_{W_i \cap M} \colon W_i \cap M \to \psi_i(W_i \cap M)$ is a diffeomorphism. Thus, the φ_i are an atlas on M.

Suppose that $f \colon M \to \mathbb{R}$ is smooth on M as a manifold. This means that, for each $i \in I$, the composition $f \circ \varphi_i^{-1}$ is a smooth real valued function on the open subset $\psi_i(W_i \cap M)$ of $\mathbb{R}^n$. Consider the composition

$$W_i \xrightarrow{\psi_i} \psi_i(W_i) = \psi_i(W_i \cap M) \xrightarrow{f \circ \varphi_i^{-1}} \mathbb{R}.$$

Being a composition of smooth functions, it is smooth. And its restriction to $W_i \cap M$ coincides with f. This shows that the function f is smooth on M as a subset of $\mathbb{R}^N$.