CRASH COURSE ON MANIFOLDS

A **manifold** is a (Hausdorff, second countable) topological space M equipped with an equivalence class of **atlases**.

An **atlas** is an open covering $M = \bigcup_i U_i$ and homeomorphisms $\varphi_i \colon U_i \to \Omega_i$ where $\Omega_i \subset \mathbb{R}^n$ is open, such that the transition maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth (that is, are of type C^∞: all partial derivatives of all orders exist and are continuous).

Two atlases $\{\varphi_i \colon U_i \to \Omega_i\}$ and $\{\tilde{\varphi}_j \colon \tilde{U}_j \to \tilde{\Omega}_j\}$ are **equivalent** if their union is an atlas, that is, if $\tilde{\varphi}_j \varphi_i^{-1}$ and $\varphi_i \tilde{\varphi}_j^{-1}$ are smooth for all i, j.

$\varphi_i \colon U_i \to \Omega_i$ is called a **coordinate chart**.

$\varphi_i^{-1} \colon \Omega_i \to U_i$ is called a **parametrization**.

One can write $\varphi_i = (x^1, \ldots, x^n)$. $\quad x^j \colon U_i \to \mathbb{R}$ are **coordinates**.

Let $X \subset \mathbb{R}^N$ be any subset and $\Omega \subset \mathbb{R}^n$ open. A continuous map $\varphi \colon X \to \Omega$ is called **smooth** if every point in X is contained in an open subset $V \subset \mathbb{R}^N$ such that there exists a smooth function $\tilde{\varphi} \colon V \to \Omega$ with $\tilde{\varphi}|_{X \cap V} = \varphi$. A continuous map $\psi \colon \Omega \to X$ is called **smooth** if it is smooth as a map to $\mathbb{R}^N$. A **diffeomorphism** $\varphi \colon X \to \Omega$ is a homeomorphism such that both φ and φ^{-1} are smooth.

Theorem. Let $M \subset \mathbb{R}^N$ be a subset that is "locally diffeomorphic to $\mathbb{R}^n$": for every point in M there exists a neighborhood $U \subset M$ and there exists an open subset $\Omega \subset \mathbb{R}^n$ and there exists a diffeomorphism $\varphi \colon U \to \Omega$. Then M is a manifold with atlas $\{\varphi \colon U \to \Omega\}$. (Exercise: the transition maps are automatically smooth.) Such an M is called an **embedded submanifold of $\mathbb{R}^N$.**

Example. $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. For instance, x, y are coordinates on the upper hemisphere.

A continuous function $f \colon M \to \mathbb{R}$ is **smooth** if $f \circ \varphi_i^{-1} \colon \Omega_i \to \mathbb{R}$ is smooth for all i (as a function of n variables).

$C^\infty(M) := \{$ the smooth functions $f \colon M \to \mathbb{R}$ $\}$.

A (continuous) curve $\gamma \colon \mathbb{R} \to M$ is **smooth** if $\varphi_i \circ \gamma$ is smooth for all i.

We will define the **tangent space** T_mM = "directions along M at the initial point m".

A smooth curve $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = m$ defines "differentiation along the curve", which is the linear functional $C^\infty(M) \to \mathbb{R}$,

$$D_\gamma\colon f \mapsto \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

We define an equivalence of such curves by $\gamma \sim \tilde{\gamma}$ if $D_\gamma = D_{\tilde{\gamma}}$.

This means that γ and $\tilde{\gamma}$ have the **same direction** at the point $m = \gamma(0) = \tilde{\gamma}(0)$, that is, they are **tangent** to each other at this point.

Geometric definition of the tangent space:

$T_mM = \{$ the equivalence classes of curves in M through $m.\}$

Leibnitz property: $D_\gamma(fg) = (D_\gamma f)g(m) + f(m)(D_\gamma g)$.

Definition. A **derivation at** m is a linear functional $D\colon C^\infty(M) \to \mathbb{R}$ that satisfies the Leibnitz property.

Theorem. The derivations at m form a linear vector space: if D_1, D_2 are derivations and $a, b \in \mathbb{R}$ then $aD_1 + bD_2$ is a derivation.

Theorem. If $x^1, \ldots, x^n$ are coordinates near m then every derivation is a linear combination of $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. (The proof uses Hadamard's lemma: for any $f \in C^\infty(\mathbb{R}^n)$ there exist $f_i \in C^\infty(\mathbb{R}^n)$ such that $f(x) = f(0) + \sum x_i f_i(x)$. Proof of the lemma: $f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)dt = \sum x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$ by the chain rule.)

Corollary. For every derivation D there exists a curve γ such that $D = D_\gamma$. T_mM is a linear vector space (identified with the space of derivations at m). If $x^1, \ldots, x^n$ are coordinates near m then $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ is a basis of T_mM.

Differential of a function: $df|_m \in T_m^*M = (T_mM)^*$ is given by

$$df|_m(v) = vf,$$

the derivative of f in the direction of $v \in T_mM$.

If $x^1, \ldots, x^n$ are coordinates then

$$dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial x^i}{\partial x^j} = \delta_{ij},$$

so $dx^1, \ldots, dx^n$ is the basis of T_m^*M which is dual to the basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ of T_mM.

In coordinates, $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$.

A **differential form of degree 0** is a smooth function.

A **differential form of degree 1**, $\alpha \in \Omega^1(M)$, associates to each $m \in M$ a linear functional $\alpha_m \in T^*_m M$. In coordinates: $\alpha = \sum_i c_i(x) dx^i$. We require that the coefficients $c_i(x)$ be smooth functions of $x = (x^1, \dots, x^n)$.

A **differential form of degree 2**, $\alpha \in \Omega^2(M)$, associates to each $m \in M$ an alternating (i.e., anti-symmetric) bilinear form $\alpha_m \colon T_m M \times T_m M \to \mathbb{R}$. In coordinates: $\alpha = \sum\limits_{i,j} c_{ij}(x) dx^i \wedge dx^j$ (where

$$dx^i \wedge dx^j \colon (u, v) \mapsto \det \begin{bmatrix} u^i & v^i \\ u^j & v^j \end{bmatrix}$$

if $u = \sum u^k \frac{\partial}{\partial x^k}$ and $v = \sum v^k \frac{\partial}{\partial x^k}$).

A **differential form of degree k**:

$$\alpha = \sum_{i_1, \dots, i_k} c_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

(where $dx^{i_1} \wedge \dots \wedge dx^{i_i}$ is similarly given by a $k \times k$ determinant).

Exterior derivative:

$$d\alpha = \sum_{i_1, \dots, i_k, j} \frac{\partial c_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

α is **closed** if $d\alpha = 0$; α is **exact** if there exists β such that $\alpha = d\beta$.

De Rham cohomology: $H^k_{dR}(M) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\}$.

An **oriented manifold** is a manifold equipped with an equivalence class of oriented atlases. (Jacobians of transition maps must have positive determinants.)

Integration: Let M be an oriented manifold of dimension n. For an n-form with support in a coordinate neighborhood U_i: write it as $f(x) dx^1 \wedge \dots \wedge dx^n$ where $x^1, \dots, x^n$ are (oriented) coordinates and take the Riemann integral of f on $\mathbb{R}^n$. For an arbitrary compactly supported form α: choose a **partition of unity** $\rho_i \colon M \to \mathbb{R}$, $\sum \rho_i = 1$, $\operatorname{supp} \rho_i \subset U_i$, and define

$$\int_M \alpha = \sum_i \int (\rho_i \alpha).$$

Pullback: $f: M \to N$ induces $f^*: \Omega^k(N) \to \Omega^k(M)$. This enables us to integrate a k-form over an oriented k-submanifold.

A **manifold with boundary** is defined like a manifold except that the Ω's are open subsets of the upper half space. Its boundary ∂M is well defined and is a manifold of one dimension less.

Stokes's theorem: $\int_M d\alpha = \int_{\partial M} \alpha$.

$\alpha \in \Omega^k(M)$ is closed iff $\int_N \alpha = 0$ whenever N is the boundary of a compact oriented submanifold-with-boundary of M. If α is exact, $\int_N \alpha = 0$ for every compact oriented submanifold $N \subset M$. (α is exact iff $\int_N \alpha = 0$ for every smooth cycle $N \subset M$.) If the integral of a closed form on N is nonzero, we can think that N "wraps around a hole in M".

Theorem: if M is oriented and compact n-manifold then $\alpha \mapsto \int_M \alpha$ induces an isomorphism $H^n_{\mathrm{dR}}(M) \to \mathbb{R}$.

Multiplicative structure: $[\alpha] \cdot [\beta] = [\alpha \wedge \beta]$ is a well defined ring structure on $H^*_{dR}(M)$. $f: M \to N$ induces a ring homomorphism $f^*: H^*_{dR}(N) \to H^*_{dR}(M)$.