REVIEW OF HOMOLOGY

Homology.

Singular homology:

$$H_k(M, \mathbb{Z}) = \{k\text{-cycles}\}/\{k\text{-boundaries}\}.$$

A k-chain is a formal integer combination

$$\sum m_\sigma[\sigma\colon \Delta^k \to M]$$

where Δ^k is the standard k-simplex. The boundary map takes k-chains to $(k-1)$-chains for each k. Cycles are chains with zero boundary.

Example: a finite triangulation of a closed oriented embedded k-submanifold of M gives a k-cycle; its homology class is independent of the triangulation. Similarly, a continuous map from a closed oriented manifold to M represents a homology class.

Functoriality: $f\colon M \to M'$ gives $f_*\colon H_k(M) \to H_k(M')$.
– $(g \circ f)_* = g_* \circ f_*$;
– if $f = \text{identity}_M$ then $f_* = \text{identity}_{H_k(M)}$;
– if f is a diffeomorphism then f_* is an isomorphism.

If M is a closed, oriented, n-dimensional manifold then $H_n(M) \cong \mathbb{Z}$, generated by the *fundamental class* $[M]$.

Integration. $\alpha \in H^k_{\text{dR}}$ and $A \in H_k(M)$ give

$$\int_A \alpha \;\in\; \mathbb{R}.$$

(Represent A by a smooth cycle.)

If M, M' are closed oriented manifolds of the same dimension and $\Psi\colon M \to M'$, we have $\Psi_*[M] = d[M']$ for $d \in \mathbb{Z}$, the degree of Ψ. If Ψ is an orientation preserving diffeomorphism then $d = 1$; if Ψ is an orientation reversing diffeomorphism then $d = -1$.

For $f\colon M \to M'$ we have $\int_{f_* A} \alpha = \int_A f^*\alpha$. If f is a diffeomorphism then $\int_M f^*\alpha = \pm \int_M \alpha$. Functoriality: $(f \circ g)^* = g^* \circ f^*$. Also, $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$.

Example.

$$\begin{aligned} H_2(S^2 \times S^2) &= \mathbb{Z}[S^2 \times \text{point}] + \mathbb{Z}[\text{point} \times S^2] \\ H_4(S^2 \times S^2) &= \mathbb{Z}[S^2 \times S^2]. \end{aligned}$$

$$\dim H^k(S^2 \times S^2) = \begin{cases} 1 & k = 0 \\ 2 & k = 2 \text{ (by Künneth)} \\ 1 & k = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Denote $\omega_{a,b} = a\omega_{S^2} \oplus b\omega_{S^2}$.

$$H^2(S^2 \times S^2) = \{[\omega_{a,b}] \mid a, b \in \mathbb{R}\} \xrightarrow{\cong} \mathbb{R}^2$$

via $[\omega_{a,b}] \mapsto (a, b)$.

$$a = \frac{1}{4\pi} \int_{S^2 \times \text{point}} [\omega_{a,b}] \quad ; \quad b = \frac{1}{4\pi} \int_{\text{point} \times S^2} [\omega_{a,b}] \ .$$

A diffeomorphism of $S^2 \times S^2$ induces a linear isomorphism of $H^2(S^2 \times S^2) \cong \mathbb{R}^2$ that is represented by $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{G}L(2, \mathbb{Z})$, i.e., α, β, γ, $\delta \in \mathbb{Z}$, and $\alpha\delta - \beta\gamma = \pm 1$.

Example.

$\mathbb{C}^2 \hookrightarrow \mathbb{C}^{n+1}$ gives $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$. For $\omega =$ Fubini Study, $\int_{\mathbb{CP}^1} \omega = \pi$.

$$H_2(\mathbb{CP}^n) = \mathbb{Z}[\mathbb{CP}^1] \quad , \quad H_{2n}(\mathbb{CP}^n) = \mathbb{Z}[\mathbb{CP}^n].$$

$$\dim H^k(\mathbb{CP}^n) = \begin{cases} 1 & k = 0, 2, 4, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\quad 1, [\omega], [\omega^2], \ldots, [\omega^n] \quad$ are a basis for $H^*(\mathbb{CP}^n)$.