CRASH COURSE ON FLOWS

Let M be a manifold.

A **vector field** X on M is a map that associates to each point $m \in M$ a tangent vector in $T_m M$, denoted $X|_m$ or $X(m)$, that is smooth in the following sense. In local coordinates $x^1, \dots, x^n$, a vector field has the form $X = \sum a^j(x) \frac{\partial}{\partial x^j}$; we require that the functions $x \mapsto a^j(x)$ be smooth.

A **flow** on M is a smooth one parameter group of diffeomorphisms $\psi_t \colon M \to M$. This means that ψ_0 =identity and $\psi_{t+s} = \psi_t \circ \psi_s$ for all t and s in $\mathbb{R}$ (so that $t \mapsto \psi_t$ is a group homomorphism from $\mathbb{R}$ to $\mathrm{Diff}(M)$, the group of diffeomorphisms of M), and that $(t, m) \mapsto \psi_t(m)$ is smooth as a map from $\mathbb{R} \times M$ to M.

Its **trajectories**, (or *flow lines*, or *integral curves*) are the curves $t \mapsto \psi_t(m)$. The manifold M decomposes into a disjoint union of trajectories. Moreover, if $\gamma_1(t)$ and $\gamma_2(t)$ are trajectories that both pass through a point p, then there exists an s such that $\gamma_2(t) = \gamma_1(t+s)$ for all $t \in \mathbb{R}$. Hence, the velocity vectors of γ_1 and γ_2 at p coincide.

Its **velocity field** is the vector field X that is tangent to the trajectories at all points. That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t_0, which is a tangent vector to M at the point $p = \psi_{t_0}(m)$, is the vector $X(p)$. We express this as

$$\frac{d}{dt}\psi_t = X \circ \psi_t.$$

Conversely, any vector field X on M generates a *local flow*. This means the following. Let X be a vector field. Then there exists an open subset $A \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map $\psi \colon A \subset \mathbb{R} \times M$ such that the following holds. Write $A = \{(t, x) \mid a_x < t < b_x\}$ and $\psi_t(x) = \psi(t, x)$.

(1) ψ_0 =identity.
(2) $\frac{d}{dt}\psi_t = X \circ \psi_t$.
(3) For each $x \in M$, if $\gamma \colon (a, b) \to M$ satisfies the differential equation $\dot{\gamma}(t) = X(\gamma(t))$ with initial condition $\gamma(0) = x$, then $(a, b) \subset (a_x, b_x)$ and $\gamma(t) = \psi_t(x)$ for all t.

Moreover, $\psi_{t+s}(x) = \psi_t(\psi_s(x))$ whenever these are defined. Finally, if X is compactly supported, then $A = \mathbb{R} \times M$, so that X generates a (globally defined) flow. Good references are chapter 8 of "Introduction to differential topology" by Bröcker and Jänich and chapter 5 of "A comprehensive introduction to differential geometry", volume I, by Michael Spivak.

A *time dependent vector field* parametrized by the interval $[0,1]$ is a family of vector fields X_t, for $t \in [0,1]$, that is smooth in the following sense. In local coordinates it has the form $X_t = \sum a^j(t,x)\frac{\partial}{\partial x^j}$; we require a^j to be smooth functions of $(t, x^1, \ldots, x^n)$.

An *isotopy* (or *time dependent flow*) of M is a family of diffeomorphisms $\psi_t \colon M \to M$, for $t \in [0,1]$, such that $\psi_0 =$identity and $(t,m) \mapsto \psi_t(m)$ is smooth as a map from $[0,1] \times M$ to M.

An isotopy ψ_t determines a unique time dependent vector field X_t such that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t. \tag{1}$$

That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t, which is a tangent vector to M at the point $p = \psi_t(m)$, is the vector $X_t(p)$.

A time dependent vector field X_t on M determines a vector field $\tilde{X}$ on $[0,1] \times M$ by $\tilde{X}(t,m) = \frac{\partial}{\partial t} \oplus X_t(m)$. In this way one can treat time dependent vector fields and flows through ordinary vector fields and flows.

In particular, a time dependent vector field X_t, $t \in [0,1]$, generates a "local isotopy" $\psi_t(x) = \psi(t,x)$. If X_t is compactly supported then $\psi_t(x)$ is defined for all $(t,x) \in [0,1] \times M$. If $X_t(m) = 0$ for all $t \in [0,1]$ then there exists an open neighborhood U of m such that $\psi_t \colon U \to M$ is defined for all $t \in [0,1]$.

The *Lie derivative* of a k-form α in the direction of a vector field X is

$$L_X\alpha = \left.\frac{d}{dt}\right|_{t=0} \psi_t^*\alpha$$

where ψ_t is the flow generated by X.

We have

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta)$$

and

$$L_X(d\alpha) = d(L_X\alpha).$$

These follow from $\psi^*(\alpha \wedge \beta) = \psi^*\alpha \wedge \psi^*\beta$ and $\psi^* d\alpha = d\psi^*\alpha$.

Cartan formula:

$$L_X\alpha = \iota_X d\alpha + d\iota_X\alpha$$

where $\iota_X \colon \Omega^k(M) \to \Omega^{k-1}(M)$ is

$$(\iota_X\alpha)(u_1, \ldots, u_{k-1}) = \alpha(X, u_1, \ldots, u_{k-1}).$$

(Outline of proof: it is true for functions. If it is true for α and β then it is true for $\alpha \wedge \beta$ and for $d\alpha$.)

Let α_t be a time dependent k-form and X_t a time dependent vector field that generates an isotopy ψ_t. Then

$$\frac{d}{dt}\psi_t^*\alpha_t = \psi_t^*(\frac{d\alpha_t}{dt} + L_{X_t}\alpha_t).$$

Outline of proof: if it is true for α and for β then it is true for $\alpha \wedge \beta$ and for $d\alpha$. Hence, it is enough to prove it for functions.

The left hand side, applied to a time dependent function f_t and evaluated at $m \in M$, is the limit as $t \to t_0$ of the difference quotient

$$\frac{f_t(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

This difference quotient is equal to

$$\left(\frac{f_t - f_{t_0}}{t - t_0}\right)(\psi_t(m)) + \frac{f_{t_0}(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

The limit as $t \to t_0$ of the first summand is

$$\left.\frac{df_t}{dt}\right|_{t=t_0}(\psi_{t_0}(m)) = \left(\psi_{t_0}^* \left.\frac{df_t}{dt}\right|_{t=t_0}\right)(m).$$

The limit as $t \to t_0$ of the second summand is the derivative of f_{t_0} along the tangent vector

$$\left.\frac{d}{dt}\right|_{t=t_0} \psi_t(m) = X_{t_0}(\psi_{t_0}(m));$$

this derivative is

$$\left(X_{t_0} f_{t_0}\right)\left(\psi_{t_0}(m)\right) = \left(\psi_{t_0}^*(L_{X_{t_0}} f_{t_0})\right)(m).$$