SYMPLECTIC STRATIFIED SPACES AND REDUCTION

PETER CROOKS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO

Given a Hamiltonian G-space $(M, \omega, \mathcal{A}, \mu)$, let us consider the topological subspace $\mu^{-1}(0)$ of M. Since $0 \in \mathfrak{g}^*$ is a fixed point of the coadjoint representation, and since μ is G-equivariant, it follows that $\mathcal{A}$ restricts to a G-action on $\mu^{-1}(0)$. Accordingly, we may consider the quotient topological space $M_0 := \mu^{-1}(0)/G$, called the reduced space of $(M, \omega, \mathcal{A}, \mu)$.

In the presence of certain additional hypotheses, M_0 is naturally a symplectic manifold[1]. However, this will not hold for the general Hamiltonian G-space. Nevertheless, if one requires G to be compact, then M_0 will have intriguing topological properties. In particular, there is a partition of M_0 into symplectic manifolds fitting together in some desirable ways. This partition realizes M_0 as a so-called symplectic stratified space. We will develop the notions necessary to formulate a precise definition of this object, and we will subsequently exhibit M_0 as a symplectic stratified space.

Definition 0.1. *Let X be a paracompact Hausdorff topological space, and I a partially ordered set. An I-decomposition of X is a disjoint locally finite cover, $\{S_i\}_{i \in I}$, of X by locally closed subsets[2], satisfying the below two properties.*

(i) *For each $i \in I$, the subspace S_i is a topological manifold.*
(ii) *If $(i, j) \in I \times I$, then $S_i \cap \overline{S_j} \neq \emptyset \iff S_i \subseteq \overline{S_j} \iff i \leq j$.*[3]

A decomposed space is a paracompact Hausdorff topological space X, together with a distinguished I-decomposition $\{S_i\}_{i \in I}$ of X for some partially ordered set I. We shall call the partially ordered set I the index set of the decomposed space, and the subspaces S_i the pieces of the space.

Example 0.1. *Let X be a topological space. Recall that the cone over X, CX, is defined to be the quotient of $X \times [0, \infty)$ obtained by identifying the points in $X \times \{0\}$. If $(X, \{S_i\}_{i \in I})$ is a decomposed space, then there is a canonical realization of CX as a decomposed space. Precisely, one defines the set $J = I \sqcup \{0\}$, and augments it with the partial order coinciding with that on $I \subseteq J$, such that $0 \leq i$ for all $i \in I$. Now, for each $i \in I$, define $\tilde{S}_i$ to be the image of $S_i \times (0, \infty)$ under the quotient map $X \times [0, \infty) \to CX$. Also, let $\tilde{S}_0$ be the image of $X \times \{0\}$ under the quotient map. We note that $\{\tilde{S}_j\}_{j \in J}$ is a J-decomposition of CX, as desired.*

Definition 0.2. *Let X be a decomposed space, and $S \subseteq X$ a piece. We define an S-chain in X of length $n \geq 0$ to be a sequence, $S = A_0, A_1, \ldots, A_n$, of $n+1$ pieces, with the property that if $i, j \in \{0, \ldots, n\}$ and $j = i + 1$, then $A_i \neq A_j$ and $A_i \subseteq \overline{A_j}$. The depth of S, $depth_X(S)$, is defined to be*

$$depth_X(S) = \sup\{n \geq 0 : \exists \text{ an } S\text{-chain of length } n\}.$$

[1]Indeed, if G is compact and acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)$ is an embedded submanifold of M, there exists a unique smooth manifold structure on M_0 for which the quotient map $\pi : \mu^{-1}(0) \to M_0$ is a submersion, and there is a unique symplectic form ω_0 on the smooth manifold M_0 for which $\pi^*(\omega_0)$ is the restriction of ω to $\mu^{-1}(0)$. This is a statement of the Marsden-Weinstein-Meyer Theorem.

[2]A subset of a topological space is called locally closed if it is open with respect to the subspace topology of its closure.

[3]One calls this the Frontier Condition.

Definition 0.3. *Let X be a decomposed space with non-empty index set I and pieces $\{S_i\}_{i\in I}$. The depth of X, $depth(X)$, is defined by*

$$depth(X) = \sup_{i\in I} depth_X(S_i).$$

Remark 0.1. *In the interest of our being able to define the depth of an arbitrary decomposed space, we shall require that each of our decomposed spaces come equipped with a non-empty index set.*

Definition 0.4. *A 0-stratified space is a decomposed space X of depth 0. An n-stratified space, $n \geq 1$, is a decomposed space $(X, \{S_i\}_{i\in I})$ of depth n, with the property that for each piece S of X and point $x \in S$, there exist an open neighbourhood $U(x)$ of x in X, an open coordinate ball $B(x)$ of x in S, an m-stratified stratified space $(L, \{P_j\}_{j\in J})$ with $m < n$, and a homeomorphism $\varphi_x : B(x) \times CL \to U(x)$, such that for each piece of $B(x) \times CL$, φ_x restricts to a homeomorphism of that piece with a piece of $U(x)$[4]. We shall refer to the pieces of a stratified space as strata.*

Definition 0.5. *A smooth stratified space consists of a stratified space X, together with the below data.*

(i) *a smooth manifold structure for each stratum of X*
(ii) *a distinguished subalgebra, $C^\infty(X)$, of the $\mathbb{R}$-algebra $C^0(X)$ of continuous maps $X \to \mathbb{R}$, with the property that $f|_S \in C^\infty(S)$ for all strata S of X and for all $f \in C^\infty(X)$*

Definition 0.6. *A symplectic stratified space consists of a smooth stratified space X, augmented with the below additional data.*

(i) *a symplectic form, $\omega_S \in \Omega^2(S)$, for each stratum S of X*
(ii) *a Poisson algebraic structure[5], $\{,\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ on $C^\infty(X)$, for which the restriction maps to strata, $i_S^* : C^\infty(X) \to C^\infty(S)$, are Poisson algebra morphisms[6]*

Let G be a group and M a set with a left G-action. We wish to associate a canonical partially ordered set to this action. To this end, denote by G^S the collection of those subgroups of G with the property of being conjugate in G to the stabilizer subgroup of a point in M. More succinctly,

$$G^S := \{H \leq G : \exists p \in M,\ g \in G \text{ such that } gHg^{-1} = Stab_G(p)\}.$$

Identifying conjugate subgroups of G^S, we obtain an equivalence relation. Let I denote the resulting quotient space. We define a partial order, $\leq$, on I by $[H] \leq [K]$ if and only if K is contained in a conjugate of H in G. Well-definedness follows from the observation that K is contained in a conjugate of H if and only if for every conjugate H' of H and K' of K, K' is contained in a conjugate of H'.

For each $\alpha \in I$, consider the set $M_\alpha := \{p \in M : [Stab_G(p)] = \alpha\}$.

Let us specialize to the case in which G is a compact Lie group with Lie algebra $\mathfrak{g}$, and $(M, \omega, \mathcal{A}, \mu)$ is a Hamiltonian G-space. For future reference, we shall let $Z := \mu^{-1}(0)$, the zero-level set of the moment map. Consider the quotient map $\pi : Z \to M_0$, and for each $\alpha \in I$, set $(M_0)_\alpha := \pi(M_\alpha \cap Z)$. We observe that if $\alpha, \beta \in I$ and $(M_0)_\alpha \cap (M_0)_\beta \neq \emptyset$, then we may choose $p \in M_\alpha \cap Z$ and $q \in M_\beta \cap Z$, such that $\pi(p) = \pi(q)$. By the definition of our quotient space M_0, it follows that p and q lie in the same G-orbit, and hence $Stab_G(p)$ and $Stab_G(q)$ are conjugate in G. Therefore, $[Stab_G(p)] = [Stab_G(q)]$ in I. However, $p \in M_\alpha$ and $q \in M_\beta$, implying that $\alpha = [Stab_G(p)]$ and $\beta = [Stab_G(q)]$. It follows that $\alpha = \beta$, and we conclude that the sets $\{(M_0)_i : i \in I\}$ are disjoint. Furthermore, the sets M_i cover M, meaning that the sets $M_i \cap Z$ cover Z, and hence that the sets $(M_0)_i$ cover M_0 (as π is surjective).

In light of our determinations, it perhaps seems sensible to regard the $(M_0)_\alpha$'s as candidates for strata of the reduced space M_0. However, there is an example of a reduced space in which one of these

[4]The decomposed space structures of $B(x)\times CL$ and $U(x)$ are canonically induced by those of CL and X, respectively. Specifically, the pieces of $B(x) \times CL$ are $\{B(x) \times \tilde{P}_j\}_{j\in J\sqcup\{0\}}$, while those of $U(x)$ are $\{U(x) \cap S_i\}_{i\in I}$

[5]A Poisson algebra over a field K is an associative K-algebra A, together with a Lie bracket on A that is simultaneously a derivation of A.

[6]We view $C^\infty(S)$ as the Poisson algebra canonically induced by the symplectic form ω_S.

candidate strata has connected components of different dimensions (meaning that this stratum is not a topological manifold). Fortunately, some semblance of a resolution is obtained via partitioning the candidate strata into connected components.

Theorem 0.1. *The reduced space M_0 is a disjoint union of the subspaces $\{(M_0)_\alpha : \alpha \in I\}$. This decomposition has the below properties.*

(i) *If $\alpha \in I$, then each connected component of $(M_0)_\alpha$ is a topological manifold.*
(ii) *If $(\alpha, \beta) \in I \times I$, then $\alpha \leq \beta \Leftrightarrow (M_0)_\alpha \cap \overline{(M_0)_\beta} \neq \emptyset \Leftrightarrow (M_0)_\alpha \cap \overline{(M_0)_\beta} \neq \emptyset$ and every connected component of $(M_0)_\alpha$ intersecting $\overline{(M_0)_\beta}$ non-trivially belongs to $\overline{(M_0)_\beta}$.*
(iii) *There is a canonical realization of M_0 as a symplectic stratified space with strata the connected components of the $(M_0)_\alpha$'s.*[7]

Claim 0.1. *If $\alpha \in I$ and $p \in M_\alpha \cap Z$, then there is an open subset $U \subseteq (M_0)_\alpha$ containing $[p]$, and a realization of the subspace U as a symplectic manifold.*

Given $\alpha \in I$ and $p \in M_\alpha \cap Z$, let $\mathcal{O}_p$ denote the G-orbit of p in M. Since G is compact, $\mathcal{O}_p$ is an embedded submanifold of M. More intriguingly, perhaps, this embedding is isotropic (the proof of which was given in the presentation).

Lemma 0.1. *The embedding $i : \mathcal{O}_p \hookrightarrow M$ is isotropic.*

Theorem 0.2. *(Weinstein's Equivariant Isotropic Embedding Theorem) Let K be a compact Lie group, B a smooth K-manifold, and (E, ω), (E', ω') symplectic manifolds, each augmented with a K-action by symplectic automorphisms. Suppose that $i : B \hookrightarrow E$ and $i' : B \hookrightarrow E'$ are K-equivariant isotropic embeddings with isomorphic symplectic normal bundles*[8]*. Then, there exist K-invariant open neighbourhoods, U and U', of $i(B)$ in E and $i'(B)$ in E', respectively, and a K-equivariant symplectomorphism, $\varphi : U \to U'$, such that*

$$\varphi \circ i = i'$$

as maps $B \to E'$

With the Equivariant Isotropic Embedding Theorem in mind, we observe that the inclusion $\mathcal{O}_p \hookrightarrow M$ is a G-equivariant isotropic embedding of $\mathcal{O}_p$ into a symplectic manifold. Seeking to apply our theorem, we will G-equivariantly and isotropically embed $\mathcal{O}_p$ into another symplectic G-manifold, such that the associated symplectic normal bundle is isomorphic to that of the embedding $\mathcal{O}_p \hookrightarrow M$.

To this end, consider the fibre $V := (N^\omega \mathcal{O}_p)_p$ of the symplectic normal bundle of $\mathcal{O}_p \hookrightarrow M$. It is easily verified that $\omega(p)$ descends to a symplectic form on V. Setting $H := Stab_G(p)$, we note that H acts on V by symplectic vector space automorphisms. Now, let $\mathfrak{h} = Lie(H) \subseteq \mathfrak{g}$, noting that H is a closed subgroup (hence an embedded submanifold) of G. Note that $\mathfrak{h}$ is an invariant subspace of the restricted adjoint representation $H \to Aut(\mathfrak{g})$, allowing for us to induce an H-representation on $\mathfrak{g}/\mathfrak{h}$. Of course, one then has the canonical dual representation on $(\mathfrak{g}/\mathfrak{h})^*$. Furthermore, we may consider the direct sum $(\mathfrak{g}/\mathfrak{h})^* \oplus V$ of linear H-representations.

Now, consider the principal H-bundle $G \to \mathcal{O}_p$, $g \mapsto g \cdot p$, and form the so-called associated bundle $Y := G \times_H ((\mathfrak{g}/\mathfrak{h})^* \oplus V)$. Recall that Y is the product manifold $G \times ((\mathfrak{g}/\mathfrak{h})^* \oplus V)$, modulo the free left H-action $h \cdot (g, v) = (gh^{-1}, h \cdot v)$. It is natural, then, to consider the map $\pi : Y \to \mathcal{O}_p$ given by $[(g, v)] \mapsto g \cdot p$. This constitutes a vector bundle with total space Y and base space $\mathcal{O}_p$. Accordingly,

[7]Since we do not claim to have exhibited M_0 as a decomposed space in the sense of Definition 1.2, we must specify precisely what is meant by item (iii). To this end, we mean that each of our advertised strata has a canonical symplectic manifold structure, and that M_0 has a canonical $C^0(M_0)$-subalgebra, $C^\infty(M_0)$, with a Poisson bracket for which the restriction maps to strata S define Poisson algebra morphisms $C^\infty(M_0) \to C^\infty(S)$.

[8]Recall that the symplectic perpendicular of the embedding $i : B \hookrightarrow E$, $T^\omega B$, is the subbundle of the restricted tangent bundle $TE|_B$ with fibres $(T^\omega B)_p = \{v \in T_pE : \omega(p)(v, w) = 0 \ \forall w \in T_pB\}$, $p \in B$. The symplectic normal bundle of the isotropic embedding, $N^\omega(B)$, is then defined to be the quotient bundle $N^\omega(B) := T^\omega B/TB$, noting that our embedding induces an inclusion $TB \subseteq T^\omega(B) \subseteq TE|_B$.

we consider the zero-section embedding $s : \mathcal{O}_p \hookrightarrow Y$, $g \cdot p \mapsto [(g, 0)]$. If one endows Y with the left G-action $g \cdot [(g', v)] = [(gg', v)]$, then s becomes a G-equivariant embedding. It therefore remains to exhibit Y as a symplectic manifold, such that G acts on Y by symplectic automorphisms, and such that s is an isotropic embedding whose symplectic normal bundle is isomorphic to that of $\mathcal{O}_p \hookrightarrow M$.

Consider the trivialization $\Psi : G \times \mathfrak{g}^* \to T^*G$ of the cotangent bundle of G defined by $\Psi(g, \theta) = (g, \theta \circ dL_{g^{-1}}(g))$. One then considers the G-action on $G \times \mathfrak{g}^*$ given by $g \cdot (g', \theta) = (g'g^{-1}, Ad^*(g)(\theta))$, where $Ad^* : G \to Aut(g^*)$ is the coadjoint representation of G. Deploying our trivialization, we obtain a Hamiltonian G-action on T^*G (where we are regarding T^*G as augmented with its canonical symplectic form). This restricts to a Hamiltonian H-action, as an associated moment map is obtained by composing the previous moment map with the projection $\mathfrak{g}^* \to \mathfrak{h}^*$.

Now, recall that H acts on V by symplectic vector space automorphisms. Indeed, this action is actually Hamiltonian. Accordingly, it will be advantageous to consider the Hamiltonian H-space $T^*G \times V$. To see that this H-action is free, suppose $h \in H$ fixes $((g, \theta), v) \in (G \times \mathfrak{g}^*) \times V \cong T^*G \times V$. By definition, $((gh^{-1}, Ad^*(h)(\theta)), h \cdot v) = ((g, \theta), v)$. In particular, $g = gh^{-1}$, meaning that $h = e$. Note also that H is a compact Lie group by virtue of being a closed subspace of the compact Lie group G. The Marsden-Weinstein-Meyer Theorem therefore gives a canonical symplectic manifold structure on the reduced space $\Phi^{-1}(0)/H$, where $\Phi : T^*G \times V \to \mathfrak{h}^*$ is the moment map.

Next, one constructs an H-equivariant diffeomorphism, $G \times ((\mathfrak{g}/\mathfrak{h})^* \oplus V) \to \Phi^{-1}(0)$, and obtains an induced diffeomorphism $Y = G \times ((\mathfrak{g}/\mathfrak{h})^* \oplus V)/H \to \Phi^{-1}(0)/H$. Hence, we endow Y with the symplectic manifold structure for which this diffeomorphism is a symplectomorphism. We leave it to the interested reader to verify that G acts on Y by symplectomorphisms, and that $s : \mathcal{O}_p \hookrightarrow Y$ is an isotropic embedding with symplectic normal bundle isomorphic to that of $\mathcal{O}_p \hookrightarrow M$.

By Theorem 1.1, we may choose G-invariant open submanifolds U and U' of $\mathcal{O}_p$ in M and of the zero-section in Y, respectively, and a G-equivariant symplectomorphism $\varphi : U \to U'$ respecting the embeddings $\mathcal{O}_p \hookrightarrow M$ and $\mathcal{O}_p \hookrightarrow Y$. The G-action on Y is incidentally Hamiltonian, with a moment map $J : Y \to \mathfrak{g}^*$ explicitly constructed in [3]. Hence, $J|_{U'} \circ \varphi : U \to \mathfrak{g}^*$ is a moment map of the Hamiltonian G-action on U, meaning that $\mu|_U = J|_{U'} \circ \varphi + f$ for some constant map $f : U \to \mathfrak{g}^*$. Since $\mu(p) = 0$, it follows that $f = -J(\varphi(p))$. Because φ respects the embeddings $\mathcal{O}_p \hookrightarrow M$ and $\mathcal{O}_p \hookrightarrow Y$, $\varphi(p)$ belongs to the zero-section of the vector bundle $Y \to \mathcal{O}_p$. However, the moment map J vanishes on the zero-section, meaning that $f = -J(\varphi(p)) = 0$. It follows that $J|_{U'} \circ \varphi = \mu|_U$. Therefore, φ is an isomorphism of the Hamiltonian G-spaces $(U, \mu|_U)$ and $(U', J|_{U'})$. In particular, for a given $\alpha \in I$, φ must therefore induce an identification of the quotients $(U_\alpha \cap \mu^{-1}(0))/G = (M_\alpha \cap U \cap \mu^{-1}(0))/G$ and $(U'_\alpha \cap J^{-1}(0))/G = (Y_\alpha \cap U' \cap J^{-1}(0))/G$. We will realize $(Y_\alpha \cap U' \cap J^{-1}(0))/G$ as a symplectic manifold and our identification will then induce a symplectic manifold structure on $(M_\alpha \cap U \cap \mu^{-1}(0))/G$. Since the quotient projection $M_\alpha \cap \mu^{-1}(0) \to (M_\alpha \cap \mu^{-1}(0))/G = (M_0)_\alpha$ is an open map, $(M_\alpha) \cap U \cap \mu^{-1}(0))/G$ is an open subset of $(M_0)_\alpha$, and we will therefore have realized an open neighbourhood of an arbitrary point of $(M_0)_\alpha$ as a symplectic manifold.

Since the quotient projection $Y_\alpha \cap J^{-1}(0) \to (Y_\alpha \cap J^{-1}(0))/G$ is also an open map, it follows that $(Y_\alpha \cap U' \cap J^{-1}(0))/G$ is an open subset of $(Y_\alpha \cap J^{-1}(0))/G$. Accordingly, it will suffice to exhibit $(Y_\alpha \cap J^{-1}(0))/G$ as a symplectic manifold, as one will then obtain an induced symplectic structure on the open submanifold $(Y_\alpha \cap U' \cap J^{-1}(0))/G$.

Now, consider the linear subspace $V_H := \{v \in V : h \cdot v = v \ \forall h \in H\}$ of V. It is easily established that the restriction of the symplectic form on V to V_H yields a symplectic form on V_H. This realizes V_H as a symplectic manifold. Furthermore, the authors in [3] use properties of the moment map J to identify the quotient $(Y_\alpha \cap J^{-1}(0))/G$ with V_H, and in so doing, they endow this quotient with the structure of a symplectic manifold (as desired). We have thus outlined the proof of our claim.

Let us briefly address the symplectic structure on M_0. To this end, let $\pi : Z \to M_0$ be the quotient map, and define $f \in C^0(M_0)$ to be an element of $C^\infty(M_0)$ if $f \circ \pi = F|_Z$ for some $F \in C^\infty(M)^G$. The Poisson bracket, $\{f, g\}_{M_0}$, of $f, g \in C^\infty(M_0)$ is given by $\{f, g\}_{M_0}(p) = \{f|_S, g|_S\}_S(p)$, where $p \in M_0$,

S is the stratum of M_0 containing p, and $\{,\}_S : C^\infty(S) \times C^\infty(S) \to C^\infty(S)$ is the Poisson bracket on $C^\infty(S)$.

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