SYMPLECTIC LEFSCHETZ FIBRATIONS

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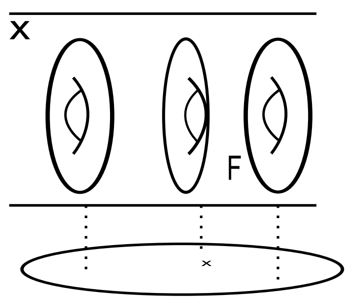
1. **Introduction**

A Lefschetz pencil is a construction that comes from algebraic geometry, but it is closely related with symplectic geometry. Indeed, as shown by Gompf and Donaldson, a four dimensional manifold has the structure of a Lefschetz pencil if and only if it admits a symplectic form. In this short summary, I present a brief introduction to this theory and show some of the connections between the Letschetz fibrations theory and symplectic geometry. For simplicity, I will work the four dimensional case, but most of the statements can be generalized with suitable hypotheses.

2. **Symplectic Lefschetz Fibrations**

Definition 2.1. *A Lefschetz fibration on a 4-manifold X is a map $\pi : X \to \Sigma$, where Σ is a closed 2-manifold, such that (i) The critical points of π are isolated, (ii) If $p \in X$ is a critical point of π then there are local coordiantes (z_1, z_2) on X and z on Σ with $p = (0, 0)$ and such that in these coordinates π is given by the complex map $z = \pi(z_1, z_2) = z_1^2 + z_2^2$.*



Theorem 1 (**Gompf** [4]**).** *Assume that a closed 4-manifold X admits a Lefschetz fibration $\pi : X \to \Sigma$, and let $[F]$ denote the homology class of the fiber. Then X admits a symplectic structure with symplectic fibers if $[F] \neq 0$ in $H_2(X; \mathbb{R})$. If $e_1, \cdots, e_n$ is a finite set of sections of the Lefschetz fibration, the symplectic form ω can be chosen in such a way that all these sections are symplectic.*

Proof. (Sketch) If $[F] \neq 0$ then there is some $c \in H^2(X, \mathbb{R})$ with $\int_F c > 0$. It is enough to build a closed form $\alpha \in \Omega^2(X)$, such that $[\alpha] = c$ and whose restriction to any fiber $\pi : X \to \Sigma$ is symplectic. Indeed, given such α, let $\omega = \alpha + K\pi^*\omega_\Sigma$ where $K >> 0$ and ω_Σ is an area form for Σ. Then ω is closed, $\omega|_{\text{fiber}} = \alpha|_{\text{fiber}}$, and ω is symplectic for K large enough.

Let $\eta_0 \in \Omega^2(X)$ be any closed 2-form which represents the class c, i.e., $[\eta_0] = c$. Near a smooth fiber $F_p = \pi^{-1}(p)$, trivialize a neighborhood so we have that $\pi^{-1}(U_p) \simeq F \times U_p$, where $U_p \subset \Sigma$ is a disc at p and consider an area form σ_p on F_p such that $[\sigma_p] = \iota_p^* c$, where $\iota_p : F_p \hookrightarrow X$ is the natural inclusion. Then $\alpha_p = pr_1^*(\sigma_p)$ is a 2-form on $\pi^{-1}(U_p)$. The form α_p is symplectic restricted on fibers and $[\alpha_p] = c|_{\pi^{-1}(U_p)}$.

Near a singular fiber, let $U \subset X$ be a neighborhood near a critical point of π so that there are local coordinates in which $\pi(z_1, z_2) = z_1^2 + z_2^2$. We have the standard symplectic structure ω_U on U. The map π is holomorphic, so if $p \in \Sigma$ then $\pi^{-1}(p) \cap U$ is a holomorphic curve, so that it is ω_U-symplectic. For fixed p extend the symplectic structure on $\pi^{-1}(p) \cap U$ to a symplectic structure α_p to a neighborhood of the rest of the fiber $\pi^{-1}(p)$. We can rescale α_p to make $\int_F \alpha_p = \int_F c$ so that $[\alpha_p] = c|_{\pi^{-1}(U_p)}$.

Let $\{U_p\}_p$ be an appropriate open covering of Σ, and let α_p be as constructed above. If the open sets U_p in the cover are chosen to be contractible then the forms $\alpha_p - \eta_0 \in \Omega^2(\pi^{-1}(U_p))$ are exact. Choose a collection of 1-forms $\lambda_p \in \Omega^1(\pi^{-1}(U_p))$ such that

$$\alpha_p - \eta_0 = d\lambda_p.$$

Now choose a partition of unity $\rho_p : \Sigma \to [0, 1]$ which is subordinate to the cover $\{U_p\}_p$ and define $\alpha \in \Omega^2(X)$ by

$$\alpha = \eta_0 + \sum d((\rho_p \circ \pi)\lambda_p).$$

The 1-form $d(\rho_p \circ \pi)$ vanishes on vectors tangent to the fibre and hence

$$\iota_b^*\alpha = \iota_b^*\eta_0 + \sum(\rho_p \circ \pi)\iota_b^* d\lambda_p = \sum(\rho_p \circ \pi)\iota_b^*(\eta_0 + d\lambda_p) = \sum(\rho_p \circ \pi)\iota_b^*\alpha_p$$

We have constructed a closed 2-form $\alpha \in \Omega^2(M)$, with $[\alpha] = c$, whose restriction to any fiber of $\pi : X \to \Sigma$ is symplectic, as we wanted. □

Corollary 2.2 (**Thurston**). *If $\Sigma_g \to X \to \Sigma_h$ is a surface bundle with fiber non-torsion in homology, then X is symplectic*

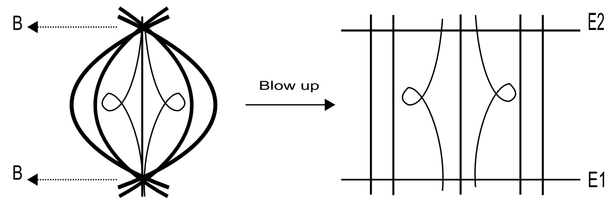
3. Lefschetz pencils

3.1. **Blow-up.** Let $L = \{(l, p) \in \mathbb{CP}^1 \times \mathbb{C}^2 : p \in l\}$. The projection $pr_1 : L \to \mathbb{CP}^1$ gives a complex line bundle structure to L. This fibration is called the tautological bundle over $\mathbb{CP}^1$. The projection $pr_2 : L \to \mathbb{C}^2$ to the second factor has the following property that for a point $p \in \mathbb{C}^2$ the inverse image $pr_2^{-1}(p)$ is a single point if $p \neq 0$, and $pr_2^{-1}(0) = \mathbb{CP}^1$. Moreover the map pr_2 is a biholomorphism between $L - pr_2^{-1}(0)$ and $\mathbb{C}^2 - \{0\}$. Thus we may think of L as obtained from $\mathbb{C}^2$ by replacing the origin by the space of all lines through the origin. If S is a complex surface with $P \in S$ and a neighborhood $U \subset S$ of P which is biholomorphic to an open subset V of $\mathbb{C}^2$ (with P mapped to $0 \in \mathbb{C}^2$), then by removing U and replacing it with $pr_2^{-1}(V) \subset L$, we get a new complex manifold S' called the blow-up of S at P. Extending pr_2 to S', one obtains a map $pr : S' \to S$ which is a biholomorphism between $S' - pr^{-1}(P)$ and $S - \{P\}$, and $pr^{-1}(P)$ is biholomorphic to $\mathbb{CP}^1$. The subset $pr^{-1}(P)$ is called the exceptional sphere. As a smooth manifold, S' is diffeomorphic to $S\sharp\overline{\mathbb{CP}^2}$. In general, for a smooth, oriented four manifold X, the connected sum $X' = X\sharp\overline{\mathbb{CP}^2}$ is called the blow-up of X. The sphere $\overline{\mathbb{CP}^1}$ in the $\overline{\mathbb{CP}^2}$ is called an exceptional sphere. The blow-up operation can be performed symplectically for a symplectic manifold (X, ω) and if $\Sigma \simeq \mathbb{CP}^1$ is a symplectically embedded 2-sphere with self intersection number -1, we can symplectically blow-down the manifold along this sphere (see [5] for more details).

3.2. Symplectic Lefschetz pencils.

Definition 3.1. *A Lefschetz pencil on a 4-manifold X is a finite base locus $B \subset X$ and a map $\pi : X - B \to \mathbb{CP}^1$ such that, (i) Each $b \in B$ has an orientation preserving local coordinate map to $(\mathbb{C}^2, 0)$ under which π corresponds to projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$, and (ii) Each critical point of f has an orientation-preserving local coordinate chart in which $\pi(z_1, z_2) = z_1^2 + z_2^2$ for some holomorphic local chart in $\mathbb{CP}^1$.*

By the definition of a Lefschetz pencil, if we blow up X at the base locus, we get a new four manifold $pr : X' \to X$, and π extends over all of X' and gives a Lefschetz fibration $\pi' : X' \to \mathbb{CP}^1$ with distinguished sections $E_1, \cdots, E_n$ (the exceptional curves of the blow-ups). Conversely given a Lefschetz fibration with section of selfintersection -1, we can blow down and get a Lefschetz pencil.



Example 3.2 (**Lefschetz Pencil of cubics).** *Let p_0 and p_1 be two cubics intersecting in nine points $P_1, \cdots, P_9$. For $Q \in \mathbb{CP}^2 - \{P_1, \cdots, P_9\}$ take the unique cubic $p_{[t_0:t_1]} = t_0 p_0 + t_1 p_1$ which passes through Q, and then define $\pi : \mathbb{CP}^2 - \{P_1, \cdots, P_9\}$ by $\pi(Q) = [t_0 : t_1] \in \mathbb{CP}^1$. By blowing up $\mathbb{CP}^2$ at $P_1, \cdots, P_9$ we extend π to a Lefschetz fibration $\pi' : \mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}$ whose fibers are cubic curves and the generic fiber is a elliptic curve, i.e., a torus. This is an Elliptic surface usually denoted as $E(1)$.*

Example 3.3 (**Fiber Sum of Lefschetz fibrations).** *Let $\pi_1 : X_1 \to \Sigma_1$ and $\pi_2 : X_2 \to \Sigma_2$ be two Lefschetz fibrations whose generic fibers have the same genus. One begins with neighborhoods ν_i of generic fibers F_i of π_i. These are diffeomorphic to $D^2 \times \Sigma_g$ where Σ_g is the Riemann surface with the same genus as the F_i. One then picks an orientation-reversing diffeomorphism $\phi : S^1 \times \Sigma_g$ of the boundaries of $X_i - \nu_i$ and identifies them via ϕ. We obtain a new Lefeschetz fibration $\pi : X_1 \sharp_\phi X_2 \to \Sigma_1 \sharp \Sigma_2$. As one special case, we can form the elliptic surface $E(n) = E(n-1)\sharp_\phi E(1)$*

Example 3.4 (**Lefschetz Pencil of Complex projective surfaces).** *Let X be a complex submanifold of $\mathbb{CP}^N$. Let $A \subset \mathbb{CP}^N$ be a generic linear subspace of complex codimension 2, so it is copy of $\mathbb{CP}^{N-2}$ cut out by two homogeneous linear equations $p_0(z) = p_1(z) = 0$. The set of all hyperplanes through A is parametrized by $\mathbb{CP}^1$. They are given by the equations $y_0 p_0(z) + y_1 p_1(z) = 0$, for $(y_0, y_1) \in \mathbb{C}^2 \backslash \{0\}$ up to scale. These hyperplanes intersect X in a family of (possibly singular) complex curves $\{F_y : y \in \mathbb{CP}^1\}$. Since the hyperplanes fill $\mathbb{CP}^N$, we have $\bigcup_{y \in \mathbb{CP}^1} F_y = X$. Let $B = X \cap A$. The canonical map $\mathbb{CP}^N - A \to \mathbb{CP}^1$ induced by the hyperplanes restricts to $X - B$ and gives to X the structure of a Lefschetz pencil (see* [4] *for further details).*

Theorem 2 (**Gompf).** *If a 4-manifold X admits a Lefschetz pencil, then it has a symplectic structure*

Proof. (*Sketch*) By blowing up X in the n points of the base locus, we get a Lefschetz fibration $X' = X \sharp n\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ whose fibers are non-trivial in homology. The blow-up manifold admits a symplectic structure for which the exceptional spheres (a finite set of sections) are symplectic. Now symplectically blowing down the exceptional spheres results in a symplectic structure on the manifold X. □

Theorem 3 (**Donaldson** [3]**).** *Any symplectic 4-manifold X admits a Lefschetz pencil.*

4. **Monodromy**

Let $\pi : X \to \mathbb{CP}^1$ be a Lefschetz fibration with a symplectic form ω on the total space which restricts to a symplectic form on the fibres and with symplectic fiber (F, σ). We can define a symplectic orthogonal complement to $T_x \pi^{-1}(p)$ inside $T_x X$. This is a 2-real dimensional subspace projecting isomorphically to $T_p \mathbb{CP}^1$ along $d\pi$ and we can use it as a connetion on the symplectic fibration $X - \pi^{-1}(\mathfrak{crit})$.

Proposition 4.1. *Parallel transport along a path $\gamma : [0,1] \to \mathbb{CP}^1 - \mathfrak{crit}$ by using this connection gives a symplectomorphism $P_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$*

Proof. Let v' denote the horizontal lift of a vector field v from the base $\mathbb{CP}^1 - \mathfrak{crit}$. Define $\alpha = \iota_{v'}\omega$ and notice that by definition α vanishes on vertical vectors. The derivative of ω under parallel transport along v' is

$$\mathcal{L}_{v'}\omega = d\iota_{v'}\omega + \iota_{v'}d\omega = d\alpha$$

Now let's take a single point and pick coordinates x_i centred at that point such that ∂_1, ∂_2 are vertical and ∂_3, ∂_4 are horizontal at that point (can't do it in a neighborhood because connection could be curved). Since α vanishes on vertical vectors, $d\alpha(\partial_1', \partial_2') = \partial_1'\alpha(\partial_2') - \partial_2'\alpha(\partial_1') - \alpha([\partial_1', \partial_2']) = 0$, where ∂_1, ∂_2 are extended to vertical vector fields ∂_1', ∂_2' respectively. Then $d\alpha$ applied to two vertical vectors must clearly vanish. Since this is measuring the derivative along v' of ω restricted to a fibre, we see that parallel transport preserves the symplectic form on fibres. $\square$

Lemma 4.2. *If ψ_t is a path of symplectomorphisms with $\psi_0 = id$ then the flux is the cohomology class*

$$\mathfrak{flux}(\psi_t)_0^T = \left[\int_0^T \iota_{X_t}\omega dt\right] \in H^1(X; \mathbb{R}) \quad \textit{where } \dot{\psi}_t = X_t \circ \psi_t.$$

If $\mathfrak{flux}(\psi_t)_0^1 = 0$, then ψ_t is a isotopic with fixed endpoints to a Hamiltonian isotopy.

Proposition 4.3. *With the same assumptions that were made in the previous Proposition, if γ is a nullhomotopic loop then P_γ is a Hamiltonian symplectomorphism of $\pi^{-1}(\gamma(0))$*

Proof. Let $\gamma : S^1 \to \mathbb{CP}^1 - \mathfrak{crit}$ and $h : D^2 \to \mathbb{CP}^1 - \mathfrak{crit}$ be a nullhomotopy. Let $z = x + iy$ denote the coordinate on the unit disc D^2. Pullback the fibration along h, since a bundle over the disc is trivialisable we may pick a trivialization $\tau : D^2 \times F \to h^*X$ which is symplectic in the sense that $\tau : (\{p\} \times F, \sigma) \to (F_p, \omega|_{F_p})$ is a symplectomorphism for all $p \in D^2$. The pull-back of the form ω is given by

$$\tau^*\omega = \sigma + \alpha \wedge dx + \beta \wedge dy + f dx \wedge dy$$

where $\alpha(z), \beta(z) \in \Omega^1(F)$ and $f(z) \in \Omega^0(F)$ for $z \in D$. Since ω is closed and $\sigma(z) = \sigma$ for all $z \in D$, we have

$$d\alpha = d\beta = 0, \qquad df = \partial_x\beta - \partial_y\alpha$$

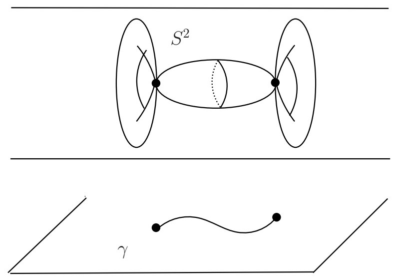
Now the holonomy of the connection form $\tau^*\omega$ around the loop $z(t) = e^{2\pi i t} = x(t) + iy(t)$ is the path of symplectomorphisms $\Psi_t : F \to F$ given by $\dot{\Psi}_t = X_t \circ \Psi_t$, and $\iota_{X_t}\sigma := \alpha_t = \alpha(z)\dot{x} + \beta(z)\dot{y}$

The formula $df = \partial_x\beta - \partial_y\alpha$ shows that the differential of the 1-form $\alpha dx + \beta dy \in \Omega^1(D, \Omega^1(F))$ is given by $df dx \wedge dy \in \Omega^2(D, \Omega^1(F))$. Hence the 1-form

$$\int_0^1 \alpha_t dt = d\int_D f(x,y)dxdy$$

is exact, and this implies that the flux $\mathfrak{flux}(\Psi_t)_0^1 = \int_0^1 [\alpha_t]dt$ is zero and $\Psi_1 : F \to F$ is a Hamiltonian symplectomorphism. $\square$

4.1. **Vanishing cycles.** Let's consider a path $\gamma : [0,1] \to \mathbb{CP}^1$ with $\gamma(t) \in \mathbb{CP}^1 - \mathfrak{crit}$ for $t < 1$ and $\gamma(1) = y \in \mathfrak{crit}$. We'll write $\gamma(0) = x$ and $\star$ for the critical point in $\pi^{-1}(y)$. The vanishing thimble D_γ associated with the path γ is the set of points $v \in \pi^{-1}\gamma$ such that $P_\gamma(t)(v) \to \star$ as $t \to 1$. A vanishing cycle is the intersection of the vanishing thimble with a fibre. The thimble D_γ is an embedded Lagrangian disc whose boundary S_γ (a vanishing cycle) is an embedded Lagrangian "sphere" S_γ in the fiber $\pi^{-1}(x)$. If we now consider an arc γ joining two critical values y_1, y_2 of π passing through y with corresponding Lagrangian thimbles D_1 and D_2 such that the boundary of the discs D_1 and D_2 coincide, i.e, the same cycle degenerates at each of the paths, then $D_1 \cup D_2$ is a Lagrangian sphere embedded in X. The arc γ is called a matching path in the Lefschetz fibration π. Matching paths are an important source of Lagrangian spheres and more generally of embedded Lagrangian submanifolds.



Theorem 4. *If $p \in \mathfrak{crit}, \gamma \subset \mathbb{CP}^1 - \mathfrak{crit}$ is a small loop encircling p and no other critical point and δ is a path from p to $\gamma(0)$, then the symplectomorphism $P_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(0))$ is a Dehn twist in the vanishing circle associated to δ*

Proof. Let's work in the local model and consider the singular symplectic fibration $\pi : \mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \to z_1^2 + z_2^2$ equipped with the standard symplectic form ω on $\mathbb{C}^2$. We will compute the holonomy around the loop $t \to e^{2\pi i t}$ in the base. Let $F = \pi^{-1}(1) = \{x + iy \in \mathbb{C}^2 : ||x||^2 - ||y||^2 = 1, \langle x, y\rangle = 0\}$.

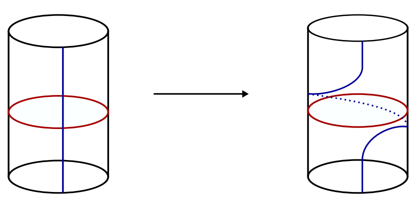
The horizontal subspace at $z \in \mathbb{C}^2$ is given by $Hor_z = \mathbb{C} \cdot \bar{z}$. It is not difficult to see that the horizontal lifts of $t \to e^{2\pi i t}$ satisfy the differential equation

$$\dot{z} = \frac{\pi i e^{2\pi i t}}{||z||^2} \bar{z}$$

After solving this differential equation, the holonomy of the fibration $\mathbb{C}^2 \to \mathbb{C}$ around the loop $[0, 1] \to \mathbb{C} : t \to e^{2\pi i t}$ is the symplectomorphim $\psi : F \to F$ given by $\psi(x_0, y_0) = (x_1, y_1)$ where

$$\frac{x_1}{||x_1||} + i\frac{y_1}{||y_1||} = -\exp\Big(\frac{2\pi i ||x_0|| ||y_0||}{\sqrt{1 + 4||x_0||^2||y_0||^2}}\Big)\Big(\frac{x_0}{||x_0||} + i\frac{y_0}{||y_0||}\Big)$$

Note that ψ is close to the identity when $||y_0||$ is very large and is equal to the antipodal map for $y_0 = 0$, i.e., $\psi(x_0, 0) = (-x_0, 0)$.



$\square$

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