Nonholonomic Hamilton-Jacobi Theory via Chaplygin Hamiltonization

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1. Introduction

This document is a brief overview of the Hamilton-Jacobi theory of Chaplygin systems based on [**1**]. In this paper, after reducing Chaplygin systems, Ohsawa *et al.* use a technique that they call *Chaplygin Hamiltonization* to turn the reduced Chaplygin systems into Hamiltonian systems. This method was first introduced in a paper by Chaplygin in 1911 where he reduced some nonholonomic systems by the action of $\mathbb{R}^k$, for some k, and turned the corresponding dynamical equations into the Hamilton's equations. In [**1**] Ohsawa *et al.* take another step forward and formulate the conventional Hamilton-Jacobi equation, which they name *Chaplygin Hamilton-Jacobi equation*, in order to integrate Chaplygin systems. They also establish the link between this approach and the direct approach of extending Hamilton-Jacobi equation to the nonholonomic systems [**2**], which is called *nonholonomic Hamilton-Jacobi equation*.

Consider a conserved nonholonomic system with the constant energy E, the configuration manifold Q equipped with a non-involutive distribution $\mathcal{D} \subset TQ$ (defined by the nonholonomic constraints), and a Hamiltonian $H : T^*Q \to \mathbb{R}$, the corresponding nonholonomic Hamilton-Jacobi equation can be written as

$$H \circ \gamma = E, \tag{1.1}$$

for a one-form γ on Q, along with the conditions that $\forall q \in Q \quad \gamma(q) \in \mathcal{M} \subset T^*Q$ and $d\gamma|_{\mathcal{D}\times\mathcal{D}} = 0$. The codistribution $\mathcal{M} := \mathbb{F}L(\mathcal{D})$ is the constrained momentum space corresponding to $\mathcal{D}$, where $\mathbb{F}L$ is the Legendre transformation. On the other hand, the Chaplygin Hamilton-Jacobi equation is a partial differential equation for a function $\bar{W} : Q/G \to \mathbb{R}$, i.e., $\bar{H}_C \circ d\bar{W} = E$, where $\bar{H}_C : T^*(Q/G) \to \mathbb{R}$ is the reduced Hamiltonian and G is the symmetry group associated with the Chaplygin system. A comparison of these two formulations reveals that as opposed to the nonholonomic Hamilton-Jacobi equation, which involves a non-closed one form, in Chaplygin Hamilton-Jacobi equation we seek the exact one-form $d\bar{W}$. Moreover, notice the difference between the spaces on which these two equations are formulated.

In the following sections, first Chaplygin systems are defined. Next, the Chaplygin Hamiltonization is introduced and necessary and sufficient conditions for a Chaplygin system to be Hamiltonizable are derived in Section 3. This result leads to the Hamilton-Jacobi theory for Hamiltonizable Chaplygin systems. Then the relationship between the nonholonomic Hamilton-Jacobi equation and the Chaplygin Hamilton-Jacobi equation is formalized in Section 4.

2. Chaplygin Systems

Consider an n-dimensional nonholonomic system with the constraint distribution $\mathcal{D} \subset TQ$ that can be written in terms of the annihilators of a set of constraint one forms $\{\omega^s\}_{s=1}^m$, i.e., $\mathcal{D} := \{v \in TQ | \omega^s(v) = 0, s = 1, ..., m\}$. The Lagrangian of the system is a function $L : TQ \to \mathbb{R}$ such that $L(v_q) = \frac{1}{2} g_q(v_q, v_q) - V(q) \quad \forall v_q \in T_qQ$, where g is the kinetic energy metric and $V : Q \to \mathbb{R}$ is a potential energy function. Define the Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ by $\langle \mathbb{F}L(v_q), w_q \rangle := g_q(v_q, w_q)$ and accordingly the Hamiltonian function is $H(p_q) := \langle p_q, \mathbb{F}L^{-1}(p_q) \rangle - L(\mathbb{F}L^{-1}(p_q))$. Therefore, Hamilton's equation for the nonholonomic system is $\iota_X \Omega = dH - \lambda_s \pi_Q^* \omega^s$ (Einstein summation convention is used) along with the constraint that $\omega^s(T\pi_Q(X)) = 0$ for $s = 1, ..., m$, where λ_s's are the Lagrange multipliers, Ω is the tautological closed two form associated with T^*Q, and $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection.

Definition 2.1. A nonholonomic system with Hamiltonian H and distribution $\mathcal{D}$ is called a Chaplygin system if there exists a Lie group G and a free and proper group action of it on Q, say $\Phi_h : Q \to Q \quad \forall h \in G$, such that

(i) H is invariant under the G-action; and,
(ii) $\forall q \in Q$, $T_qQ = \mathcal{D} \oplus T_q\mathcal{O}_q$, where $\mathcal{O}_q$ is the orbit of the group action through q, i.e., $\mathcal{O}_q := \{\Phi_h(q) \in Q | h \in G\}$.

Therefore, a Chaplygin system gives rise to the principal bundle $\pi : Q \to Q/G =: \bar{Q}$ and the connection $\mathcal{A} : TQ \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, such that $\ker \mathcal{A}_q = \mathcal{D}_q$. Therefore, one has $T_qQ = \ker \mathcal{A}_q \oplus \ker T_q\pi$ and the map $T_q\pi|_{\mathcal{D}_q} : \mathcal{D}_q \to T_{\bar{q}}\bar{Q}$ is a linear isomorphism, where $\forall q \in Q$ $\bar{q} := \pi(q) \in \bar{Q}$. Define the horizontal lift map as $hl^{\mathcal{D}}_q : T_{\bar{q}}\bar{Q} \to \mathcal{D}_q$ such that $v_{\bar{q}} \mapsto (T_q\pi|_{\mathcal{D}_q})^{-1}(v_{\bar{q}}) =: v^h_q$. Consequently, the reduced Lagrangian $\bar{L} : T\bar{Q} \to \mathbb{R}$ may be defined as $\bar{L} := L \circ hl^{\mathcal{D}}$. Considering the metric $\bar{g}$ on $\bar{Q}$ induced by g, $\mathbb{F}\bar{L} : T\bar{Q} \to T^*\bar{Q}$ can be defined as $\langle \mathbb{F}\bar{L}(v_{\bar{q}}), w_{\bar{q}} \rangle := \bar{g}_{\bar{q}}(v_{\bar{q}}, w_{\bar{q}}) \quad \forall v_{\bar{q}}, w_{\bar{q}} \in T_{\bar{q}}\bar{Q}$. The same geometric structure can be carried over to T^*Q by defining the horizontal lift $hl^{\mathcal{M}}_q : T^*_{\bar{q}}\bar{Q} \to \mathcal{M}_q$ by $hl^{\mathcal{M}}_q(\alpha_{\bar{q}}) := \mathbb{F}L_q \circ hl^{\mathcal{D}}_q \circ (\mathbb{F}\bar{L}_{\bar{q}})^{-1}(\alpha_{\bar{q}}) =: (\alpha^h_q)$. The reduced Hamiltonian can also be defined as $\bar{H} := H \circ hl^{\mathcal{M}}$. Based on nonholonomic reduction performed by Koiller [**3**], the reduced Hamilton's equation for Chaplygin systems is

$$\iota_{\bar{X}}\bar{\Omega}^{nh} = d\bar{H}, \tag{2.2}$$

where $\bar{X}$ is a vector field on $T^*\bar{Q}$, and where $\bar{\Omega}^{nh}$ is the almost symplectic form $\bar{\Omega} - \Xi$, and $\bar{\Omega}$ is the tautological two form on $T^*\bar{Q}$. For any $\alpha_{\bar{q}} \in T^*_{\bar{q}}\bar{Q}$ and $\mathcal{Y}_{\alpha_{\bar{q}}}, \mathcal{Z}_{\alpha_{\bar{q}}} \in T_{\alpha_{\bar{q}}}T^*\bar{Q}$, $\Xi_{\alpha_{\bar{q}}}(\mathcal{Y}_{\alpha_{\bar{q}}}, \mathcal{Z}_{\alpha_{\bar{q}}}) := \langle \mathbf{J}(\alpha^h_q), \mathcal{B}_q(Y^h_q, Z^h_q) \rangle$, where $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is the momentum map associated with the G-action, $\mathcal{B} = d\mathcal{A}$ is the curvature two form of $\mathcal{A}$, $Y_{\bar{q}} := T\pi_{\bar{Q}}(\mathcal{Y}_{\alpha_{\bar{q}}})$ and $Z_{\bar{q}} := T\pi_{\bar{Q}}(\mathcal{Z}_{\alpha_{\bar{q}}})$, and $\pi_{\bar{Q}} : T^*\bar{Q} \to \bar{Q}$.

3. Chaplygin Hamiltonization

Hamiltonization of a Chaplygin system is the process of transforming (2.2) to the Hamilton's equation $\iota_{\bar{X}_C}\bar{\Omega} = d\bar{H}_C$ for a Hamiltonian $\bar{H}_C : T^*\bar{Q} \to \mathbb{R}$. This process is closely linked to the existence of an invariant measure for the Chaplygin system. In sequel, a constructive approach is taken to the Hamiltonization of a Chaplygin system.

Let $f : T^*\bar{Q} \to \mathbb{R}$ be a smooth nowhere vanishing function (at least on an open subset $U \subset T^*\bar{Q}$) that is constant on each fibre, i.e., $f(\alpha_{\bar{q}}) = f(\beta_{\bar{q}}), \quad \forall \alpha_{\bar{q}}, \beta_{\bar{q}} \in T^*_{\bar{q}}\bar{Q}$. Consider the vector field $\bar{X}/f \in \mathfrak{X}(T^*\bar{Q})$ and its flow $\phi^{\bar{X}/f}_t : T^*\bar{Q} \to T^*\bar{Q}$. Define $\psi_f : T^*\bar{Q} \to T^*\bar{Q}$ to be $\psi_f : \alpha \mapsto f\alpha$. Now, let $\phi^{\bar{X}_C}_t := \psi_f \circ \phi^{\bar{X}/f}_t \circ \psi_{1/f}$. Hence, $\bar{X}/f$ and $\bar{X}_C$ are ψ_f-related, i.e., $T\psi_f \circ (\bar{X}/f) = \bar{X}_C \circ \psi_f$.

Theorem 3.1. *If $\bar{X}_C \in \mathfrak{X}(T^*\bar{Q})$ is symplectic, i.e., $\mathcal{L}_{\bar{X}_C}\bar{\Omega} = 0$, then the reduced system in (2.2) has the invariant measure $f^{\bar{n}-1}\bar{\Lambda}$, i.e., $\mathcal{L}_{\bar{X}}(f^{\bar{n}-1}\bar{\Lambda}) = 0$. Here, $\bar{n} := \dim \bar{Q}$ and $\bar{\Lambda}$ is the Liouville volume form.*

Proof. The proof relies on the following two lemmas.

Lemma 3.2. *Let f be a nowhere vanishing smooth function on $T^*\bar{Q}$ that is fibre-wise constant. Then,* $\underbrace{(\psi^*_f\bar{\Omega}) \wedge ... \wedge (\psi^*_f\bar{\Omega})}_{\bar{n}-times} = f^{\bar{n}} \underbrace{\bar{\Omega} \wedge ... \wedge \bar{\Omega}}_{\bar{n}-times}$.

Proof. This result follows from the facts that for the tautological one-form $\bar{\Theta}$ on $T^*\bar{Q}$,

$$(\psi^*_f\bar{\Theta})_\alpha = f\bar{\Theta}_\alpha, \tag{3.3}$$

and f is fibre-wise constant. □

Lemma 3.3. *For a volume form μ and a smooth vector field X on an orientable manifold M, define $\operatorname{div}_\mu(X)\mu := \mathcal{L}_X\mu$. Let f be a nowhere vanishing smooth function on M. Then, $\operatorname{div}_\mu(fX) = f\operatorname{div}_{f\mu}(X)$.*

Now, going back to the proof of the theorem, since $\bar{X}/f$ and $\bar{X}_C$ are ψ_f-related and by the hypothesis $\mathcal{L}_{\bar{X}_C}\bar{\Omega} = 0$,

$$\mathcal{L}_{\bar{X}/f}(\psi_f^*\bar{\Omega}) = \psi_f^*\mathcal{L}_{\bar{X}_C}\bar{\Omega} = 0.$$

Thus, based on Lemma 3.2, $\mathcal{L}_{\bar{X}/f}[\overbrace{(\psi_f^*\bar{\Omega}) \wedge ... \wedge (\psi_f^*\bar{\Omega})}^{\bar{n}-times}] = \mathcal{L}_{\bar{X}/f}[f^{\bar{n}}\overbrace{\bar{\Omega} \wedge ... \wedge \bar{\Omega}}^{\bar{n}-times}] = 0$ and hence $\mathcal{L}_{\bar{X}/f}(f^{\bar{n}}\bar{\Lambda}) = \mathrm{div}_{f^{\bar{n}}\bar{\Lambda}}(\bar{X}/f)(f^{\bar{n}}\bar{\Lambda}) = 0$. Then, by Lemma 3.3, $\mathrm{div}_{f^{\bar{n}-1}\bar{\Lambda}}(\bar{X}) = f\mathrm{div}_{f^{\bar{n}}\bar{\Lambda}}(\bar{X}/f) = 0$, which implies $\mathcal{L}_{\bar{X}}(f^{\bar{n}-1}\bar{\Lambda}) = \mathrm{div}_{f^{\bar{n}-1}\bar{\Lambda}}(\bar{X})(f^{\bar{n}-1}\bar{\Lambda}) = 0$. □

In order to find necessary and sufficient conditions for the existence of the Chaplygin Hamiltonization of (2.2), first the equation that $\bar{X}_C$ satisfies should be identified.

Lemma 3.4. *The vector field $\bar{X}_C \in \mathfrak{X}(T^*\bar{Q})$ satisfies the following equation:*

$$\iota_{\bar{X}_C}\left(\bar{\Omega} + \frac{1}{f}(df \wedge \bar{\Theta} - f\Xi)\right) = d\bar{H}_C,$$

where $\bar{H}_C := \bar{H} \circ \psi_{1/f}$.

Proof. Since the vector fields $\bar{X}/f$ and $\bar{X}_C$ are ψ_f-related, $\psi_f^*\iota_{\bar{X}_C}\bar{\Omega} = \iota_{\bar{X}/f}\psi_f^*\bar{\Omega}$. By straightforward calculation and using (2.2) and (3.3), one can show that $\psi_f^*\iota_{\bar{X}_C}\bar{\Omega} = \iota_{\bar{X}/f}\psi_f^*\bar{\Omega} = d\bar{H} - \iota_{\bar{X}/f}(df \wedge \bar{\Theta} - f\Xi)$. By applying $\psi_{1/f}^*$ to both sides, $\iota_{\bar{X}_C}\bar{\Omega} + \psi_{1/f}^*\iota_{\bar{X}/f}(df \wedge \bar{\Theta} - f\Xi) = d\bar{H}_C$. Using (3.3), the definition of Ξ, and the facts that the vector fields $\bar{X}/f$ and $\bar{X}_C$ are ψ_f-related and f is fibre-wise constant, $\psi_{1/f}^*\iota_{\bar{X}/f}(df \wedge \bar{\Theta} - f\Xi) = \iota_{\bar{X}_C}\left(\frac{1}{f}(df \wedge \bar{\Theta} - f\Xi)\right)$. This completes the proof. □

Therefore, necessary and sufficient condition for the vector field $\bar{X}_C$ to satisfy the Hamilton's equation for the Hamiltonian $\bar{H}_C$ is that the one-form $\iota_{\bar{X}_C}(df \wedge \bar{\Theta} - f\Xi)$ vanishes. Now, one can define the *Chaplygin Hamiltonization* as the process of finding an f satisfying the above condition. The resulting Hamiltonian equation is called the *Hamiltonized system* and $\bar{H}_C$ is the *Chaplygin Hamiltonian.*

Theorem 3.5. *Suppose that there exists a nowhere vanishing, fibre-wise constant, smooth function $f : T^*\bar{Q} \to \mathbb{R}$ that satisfies the equation*

$$df \wedge \bar{\Theta} = f\Xi. \tag{3.4}$$

Then, the vector field $\bar{X}_C \in \mathfrak{X}(T^*\bar{Q})$ satisfies the Hamilton's equation for the Hamiltonian $\bar{H}_C$, i.e.,

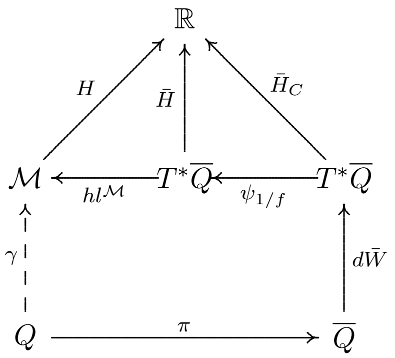
$$\iota_{\bar{X}_C}\bar{\Omega} = d\bar{H}_C, \tag{3.5}$$

and the reduced nonholonomic dynamics in (2.2) has the invariant measure $f^{\bar{n}-1}\bar{\Lambda}$.

4. Chaplygin Hamilton-Jacobi Theory

Since the Hamiltonized Chaplygin system (3.5) is a canonical Hamiltonian system on $T^*\bar{Q}$, the conventional Hamilton-Jacobi theory for an unknown function $\bar{W} : \bar{Q} \to \mathbb{R}$ may be employed to obtain the Hamilton-Jacobi equation $\bar{H}_C \circ d\bar{W} = E$, where $\bar{H}_C$ is the Chaplygin Hamiltonian and E is the constant total energy of the system. This equation is called the *Chaplygin Hamilton-Jacobi* equation whose solution plays a central role in recovering the dynamics of the nonholonomic system on Q. In the following, first the relationship between this equation and (1.1) is summarized in a commutative diagram, and subsequently a set of solutions to (1.1) is found.

(4.6)



According to the diagram (4.6) one can integrate the full dynamics of the system by solving the Chaplygin Hamilton-Jacobi equation for the reduced system, as it is summarized in the following theorem.

THEOREM 4.1. *Suppose that there exists a nowhere vanishing, fibre-wise constant function $f : T^*\bar{Q} \to \mathbb{R}$ that satisfies (3.4). Let $\bar{W} : \bar{Q} \to \mathbb{R}$ be a solution of the Chaplygin Hamilton-Jacobi equation and define γ by*

$$\gamma(q) := hl_q^{\mathcal{M}} \circ \psi_{1/f} \circ d\bar{W} \circ \pi(q) = hl_q^{\mathcal{M}} \left(\frac{1}{f(\bar{q}}) d\bar{W}(\bar{q}) \right), \tag{4.7}$$

Then γ satisfies the nonholonomic Hamilton-Jacobi equation (1.1) with its conditions.

PROOF. The fact that the one form γ satisfies (1.1) follows immediately from the diagram (4.6). The only thing left to show is that it also satisfies the required conditions for the equation (1.1), i.e., $d\gamma(Y^h, Z^h) = 0$ for arbitrary horizontal vector fields. Starting from the identity $d\gamma(Y^h, Z^h) = \mathcal{L}_{Y^h}(\gamma(Z^h)) - \mathcal{L}_{Z^h}(\gamma(Y^h)) - \gamma([Y^h, Z^h])$, and calculating the right-hand-side terms at an arbitrary point $q \in Q$ would lead to the desired result. Some of the lengthy calculations are included in the sequel. Let $Z_{\bar{q}} := T_q\pi_Q(Z_q^h)$ and $Y_{\bar{q}} := T_q\pi_Q(Y_q^h)$; hence $Z_q^h = hl_q^{\mathcal{D}}(Z_{\bar{q}})$ and $Y_q^h = hl_q^{\mathcal{D}}(Y_{\bar{q}})$. Then the identities

$$\gamma(Z^h)(q) = \frac{1}{f(\bar{q})} d\bar{W}(Z)(\bar{q}),$$

and

$$\mathcal{L}_{Y^h}(\gamma(Z^h))(q) = \left(\frac{1}{f} \mathcal{L}_Y \mathcal{L}_Z \bar{W} - \frac{1}{f^2} df(Y) d\bar{W}(Z) \right) (\bar{q}),$$

result in

$$\mathcal{L}_{Y^h}(\gamma(Z^h)) - \mathcal{L}_{Z^h}(\gamma(Y^h)) = \frac{1}{f} d\bar{W}([Y, Z]) - \frac{1}{f^2} df(Y) \wedge d\bar{W}(Y, Z).$$

For the last term on the right-hand-side, first $[Y^h, Z^h]_q$ is decomposed into the horizontal and vertical components, i.e., $[Y^h, Z^h]_q = hl_q^{\mathcal{D}}([Y^h, Z^h]_{\bar{q}}) - (\mathcal{B}_q(Y_q^h, Z_q^h))_Q(q)$, where $\xi_Q \in \mathfrak{X}(Q)$ is the infinitesimal generator of $\xi \in \mathfrak{g}$. Therefore, using the definition of the momentum map $\mathbf{J}$, the linearity of $hl^{\mathcal{M}}$ and $\mathbf{J}$ in the fibre variables, and the definition of Ξ, one has

$$\gamma([Y^h, Z^h])(q) = \frac{1}{f(\bar{q})} \left(d\bar{W}([Y, Z])(\bar{q}) - (d\bar{W})^* \Xi(Y, Z)(\bar{q}) \right).$$

By substituting the terms,

$$\begin{aligned} d\gamma(Y^h, Z^h) &= -\frac{1}{f^2} df \wedge d\bar{W}(Y, Z) + \frac{1}{f} (d\bar{W})^* \Xi(Y, Z) \\ &= -\frac{1}{f^2} (d\bar{W})^* \left(df \wedge d\bar{\Theta} - f\Xi \right) (Y, Z), \end{aligned}$$

which is identically zero because of the hypothesis of the theorem.

□

References

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