QUANTIZATION OF TORIC MANIFOLDS

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1. **Construction of Toric Manifolds**

1.1. **Symplectic construction.** Given a convex polytope $\triangle \in \mathbb{R}^n$, it gives rise to a symplectic manifold M^{2n}, together with an effective action of the torus $T^n \cong (S^1)^n$, whose image map is precisely $\triangle$. The polytope is required to satisty the Delzant conditions. Let $\triangle$ be a Delzant polytope in $\mathbb{R}^n$ with N facets (codimension 1 faces). For each facet of $\triangle$, let $v_j \in \mathbb{Z}^n$ be the primitive inward-pointing vector normal to the facet. Define a projection π from $\mathbb{R}^N$ to $\mathbb{R}^n$ by taking the jth basis vector in $\mathbb{R}^N$ to v_j :

$$\begin{aligned}\pi : \mathbb{R}^N &\to \mathbb{R}^n \\ e_j &\mapsto v_j\end{aligned} \tag{1}$$

Since the n-vectors normal to the facets meeting at any one vertex form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, the projection π maps $\mathbb{Z}^N$ onto $\mathbb{Z}^n$ and so induces a map (also called π) between tori

$$\pi : \mathbb{R}^N/\mathbb{Z}^N \to \mathbb{R}^n/\mathbb{Z}^n$$

Let K be the kernel of this map and k be the kernel of the map (1) which is the Lie algebra of K. We then get two exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i} & T^N & \xrightarrow{\pi} & T^n & \longrightarrow & 1 \\ 1 & \longrightarrow & k & \xrightarrow{i} & \mathbb{R}^N & \xrightarrow{\pi} & \mathbb{R}^n & \longrightarrow & 1 \end{array} \tag{2}$$

and the dual sequence

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\pi^*} \mathbb{R}^N \xrightarrow{i^*} k^* \longrightarrow 0 \tag{3}$$

(here $\mathbb{R}^n$ and $\mathbb{R}^N$ have been identifyed with $(\mathbb{R}^n)^*$ and $(\mathbb{R}^N)^*$ respectively). Using the vectors v_j, we can write the polytope as

$$\triangle = \{x \in \mathbb{R}^n : \langle x, v_j \rangle \geq \lambda_j, 1 \leq j \leq N\}$$

for some real numbers λ_j. This gives us a vector $\lambda \in \mathbb{R}^N$.

Claim 1. *Let $\nu = i^*(-\lambda) \in k^*$ and $\triangle' = i^{*-1}(\nu) \cap \mathbb{R}^N_+$, where $\mathbb{R}^N_+$ is the positive quadrant in $\mathbb{R}^N$. Then the map $\pi^* - \lambda$ restricts to an affine bijection from $\triangle$ to $\triangle'$. If $\lambda \in \mathbb{Z}^N$, the integer lattice points in $\triangle$ correspond to $\triangle' \cap \mathbb{Z}^N_+$ via the map $\pi^* - \lambda$.*

The torus T^N acts on $\mathbb{C}^N$ by componentwise multiplication; this action is Hamiltonian with moment map $\phi(z_1, \cdots, z_N) = \pi(\|z_1\|^2, \cdots, \|z_N\|^2)$ (here π means the *number* π). The inclusion $i : K \hookrightarrow T^N$ induces a Hamiltonian action of K on $\mathbb{C}^N$ with moment map

$$\mu = i^* \circ \phi$$

from $\mathbb{C}^N \to k^*$.

Let $M_\triangle = \mu^{-1}(\nu)/K$ and $\omega_\triangle$ be the symplectic reduced form. The action of T^N on $\mathbb{C}^N$ commutes with the action of K and thus descends to a Hamiltonian action on the quotient $M_\triangle$. This action is not effective; however, the quotient torus $T^n = T^N/K$ acts effectively. It is a theorem of Delzant that $M_\triangle$ with this action is a smooth toric manifold, with moment polytope $\triangle$.

1.2. **Complex construction.** Begin with a Delzant polytope $\triangle$ as in the previous section, and construct the map and the exact sequences (1) and (2) described there. If we complexify the sequence (2), we get an exact sequence

(4) $$1 \longrightarrow K_{\mathbb{C}} \xrightarrow{\ i\ } (\mathbb{C}^*)^N \xrightarrow{\ \pi\ } (\mathbb{C}^*)^n \longrightarrow 1$$

Let $F_1, F_2, \cdots, F_N$ be the facets of $\triangle$. Define a family $\mathcal{F}$ of subsets of $\{1, 2, \cdots, N\}$ as follows:

$\bullet \emptyset \in \mathcal{F}$

$\bullet I \in \mathcal{F}$ iff $\cap_{j\in J} F_j \neq \emptyset$

Given a point $(z_1, \cdots, z_N) \in \mathbb{C}^N$, let I_z be the set $\{j : z_j = 0\}$. Let $U_{\mathcal{F}} = \{z \in \mathbb{C}^N : I_z \in \mathcal{F}\}$. Notice that $U_{\mathcal{F}} = \mathbb{C}^N - Z_{\mathcal{F}}$, where

$$Z_{\mathcal{F}} = \bigcup_I \{(z_i, \cdots, z_N) : z_i = 0 \quad \forall i \in I\}$$

and where the union is taken over all the sets $I \subset \{1, \cdots, N\}$ for which $\cap_{i\in I} F_I = \emptyset$. Note that $Z_{\mathcal{F}}$ is the union of submanifolds of codimension at least 2.

Then it is a theorem that $M = U_{\mathcal{F}}/K_{\mathbb{C}}$, where $K_{\mathbb{C}}$ acts via the inclusion $i : k_{\mathbb{C}} \hookrightarrow (\mathbb{C}^*)^N$, is a smooth toric manifold.

Remark: These two constructions yield the same manifold, since $U_{\mathcal{F}} = K_{\mathbb{C}} \cdot \mu^{-1}(\nu)$, so that there is a natural diffeomorphism $U_{\mathcal{F}}/K_{\mathbb{C}} \cong \mu^{-1}(\mu)/K$.

2. **Quantization**

Let (M, ω) be a manifold with a closed two-form ω. A pre-quantization of (M, ω), or prequatization data for (M, ω), is a Hermitian line bundle (L, h) equipped with a hermitian connection ∇ whose curvature is ω. Such a line bundle exists if $[\omega] \in H^2(M, \mathbb{R})$ is integral. Equivalently, a pre-quantization of (M, ω) is a principal $U(1)$-bundle $\pi : P \to M$ and a connection form Θ on P with curvature ω. Recall that Θ is an $U(1)$-invariant one-form on P satisfying

$$\Theta\Big(\frac{\partial}{\partial\theta}\Big) = 1$$

where $\frac{\partial}{\partial\theta}$ is the vector field which generates the principal $U(1)$-action, and $\pi^*\omega = -d\Theta$

Recall that the one-to-one correspondence between Hermitian line bundles and principal $U(1)$-bundles associates to (L, h) its unit circle bundle

$$P = \{v \in L : h(v, v) = 1\},$$

and, conversely, associates to P the line bundle

$$L = P \times_{U(1)} \mathbb{C}.$$

The pre-quantization (L, h, ∇) uniquely determines the pre-quantization (P, Θ) and vice-versa, and the covariant derivative

$$\nabla : \Gamma(L) \to \Omega^1(M; L)$$

satisfies the equation

$$\frac{\nabla s}{s} = is^*\Theta$$

for any section $s \in P \subset L$.

Theorem 1. *The toric manifold $(M_\triangle, \omega_\triangle)$ constructed from a Delzant polytope $\triangle$ is prequantizable if the $\lambda \in \mathbb{R}^N$ appearing in the symplectic construction is in $\mathbb{Z}^n$. Moreover, if the toric manifold M is presented as $U_\mathcal{F}/K_\mathbb{C}$ then*

$$L = U_\mathcal{F} \times_{K_\mathbb{C}} \mathbb{C}$$

is a prequantization line bundle, where $K_\mathbb{C}$ acts on $\mathbb{C}$ with weight $\nu = i^*(-\lambda) \in k^*$

Proof. I will work with the same notation that was used in the section 1.1. Let $\omega = -\pi i \sum_{i=1}^N dz_i \wedge d\bar{z}_i$ be the standard symplectic form of $\mathbb{C}^N$ and let β be a K invariant one-form on $\mathbb{C}^N$ with $d\beta = -\omega$ and $\iota_{\xi_{\mathbb{C}^N}} \beta = -\mu^\xi$ for all $\xi \in k$. Then $(\mathbb{C}^N, \omega)$ can be pre-quantized by the trivial $U(1)$-bundle $P = \mathbb{C}^N \times U(1)$ with connection one-form $\Theta = d\theta + pr^*\beta$, where θ is the angle coordinate on $U(1)$ and $pr : P \to \mathbb{C}^N$ is the projection on the first component. Let the torus $K \subset T^n$ acts on $\mathbb{C}^N \times U(1)$ with weight ν on the second component, and via the inclusion $i : K \hookrightarrow T^N$ and the standard action of T^N on $\mathbb{C}^N$ on the first component.

Notice that $d\Theta = -pr^*\omega$, $\xi_P = \xi_{\mathbb{C}^N} + \nu^\xi \frac{\partial}{\partial\theta}$, and $\Theta(\xi_P) = pr^*(-\mu^\xi + \nu^\xi)$ for all $\xi \in k$. The quotient

$$((-\mu + \nu) \circ pr)^{-1}(0)/K = \mu^{-1}(\nu) \times_K U(1)$$

is a $U(1)$-bundle over $\mu^{-1}(\nu)/K$. Since $\Theta(\xi_P) = pr^*(-\mu^\xi + \nu^\xi) = 0$ on $\mu^{-1}(\nu)$, the restriction to

$$((-\mu + \nu) \circ pr)^{-1}(0) \in P$$

of the connection form Θ is horizontal with respect to the fibration

$$(pr \circ (-\mu + \nu))^{-1}(0) \to \mu^{-1}(\nu) \times_K U(1),$$

and hence is basic. Thus it descends to a connection Θ_0 on the principal bundle $\mu^{-1}(\nu) \times_K U(1) \to M$. The pair $(\mu^{-1}(\nu) \times_K U(1), \Theta_0)$ provide us with a pre-quantization for $(M_\triangle, \omega_\triangle)$ □

Definition 2.1. *If (M, ω) is a toric manifold, and (L, h, ∇) is a prequantization for (M, ω) where L is taken to be a holomorphic line bundle, the quantization space $\mathcal{Q}(M)$ is the space of holomorphic sections of L over M : $\mathcal{Q}(M) = \Gamma_\mathcal{O}(M, L)$.*

Theorem 2. *Let $M_\triangle$ be a toric manifold, with moment polytope $\triangle \subset \mathbb{R}^n$. Then the dimension of the space of holomorphic sections of $U_\mathcal{F} \times_{K_\mathbb{C}} \mathbb{C}$, the quantization space, is equal to the number of integer lattice points in $\triangle$,*

$$\dim \Gamma_\mathcal{O}(M, L) = \sharp(\triangle \cap \mathbb{Z}^n)$$

Proof (Sketch). A holomorphic section of $L = (U_\mathcal{F} \times \mathbb{C})/K_\mathbb{C}$ over $M = U_\mathcal{F}/K_\mathbb{C}$ corresponds to a $K_\mathbb{C}$-equivariant holomorphic function $s' : U_\mathcal{F} \to \mathbb{C}$. As a consequence of Hartog's theorem s' extends to a holomorphic function s on all of $\mathbb{C}^N$ (here is used that the complement of $U_\mathcal{F}$ is the union of submanifolds of codimension at least 2).

We are looking for a $K_\mathbb{C}$-equivariant, holomorphic function $s : \mathbb{C}^N \to \mathbb{C}$, where the action of $K_\mathbb{C}$ on $\mathbb{C}$ is with weight ν, and the action on $\mathbb{C}^N$ is via the inclusion $i : K_\mathbb{C} \hookrightarrow (\mathbb{C}^*)^N$ and the standard action of $(\mathbb{C}^*)^N$ on $\mathbb{C}^N$. Write such a function s with a Taylor series

$$s = \sum_{I \in \mathbb{Z}_+^N} a_I z^I.$$

Suppose that $s(z) = z^I$, and see when it is equivariant. First for $k \in K_\mathbb{C} \subset (\mathbb{C}^*)^N$ and $z \in \mathbb{C}^N$

$$s(k \cdot z) = s(i(k) \cdot z) = (i(k) \cdot z)^I = i(k)^I z^I = k^{i^*(I)} z^I.$$

On the other hand,

$$k \cdot s(z) = k^\nu \cdot z^I.$$

Thus $s(k \cdot z) = k \cdot s(z)$ when $i^*(l) = \nu$. Therefore a basis for $\Gamma_{\mathcal{O}}(M, L)$ is

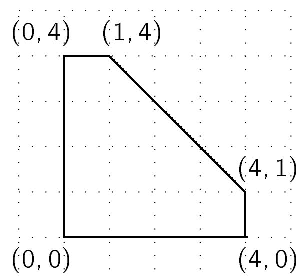
$$\{z^l : i^*(l) = \nu, l \in \mathbb{Z}_+^N\} = \{z^l : l \in \mathbb{Z}_+^N \cap i^{*-1}(\nu)\},$$

which corresponds with the set of integer lattice points in the moment polytope $\triangle$.

□

3. **Example**

Let's consider the polytope shown in the picture:



The five normal vectors are

$$v_1 = (0, 1) \quad v_2 = (-1, 0) \quad v_3 = (-1, -1) \quad v_4 = (0, -1) \quad v_5 = (1, 0)$$

and λ is $(0, -4, -5, -4, 0)$. The map $\pi : \mathbb{R}^5 \to \mathbb{R}^2$ is represented by the matrix

$$\begin{pmatrix} 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \end{pmatrix},$$

or writing the coordinates of $\mathbb{R}^5$ as $(x_1, x_2, x_3, x_4, x_5)$,

$$\pi(x_1, x_2, x_3, x_4, x_5) = (-x_2 - x_3 + x_5, x_1 - x_3 - x_4)$$

The kernel of this map is $k = \{x_1 = x_3 + x_4, x_5 = x_2 + x_3\} = span\{(0, 1, 0, 0, 1), (1, 0, 1, 0, 1), (1, 0, 0, 1, 0)\}$ which is identified with $\mathbb{R}^3$ by the map $i : \mathbb{R}^3 \to k \subset \mathbb{R}^5$, $i(x_1, x_2, x_3) = (x_2 + x_3, x_1, x_2, x_3, x_1 + x_2)$. The map π on tori is

$$\pi(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \left(\frac{\theta_5}{\theta_2 \cdot \theta_3}, \frac{\theta_1}{\theta_3 \cdot \theta_4}\right)$$

with kernel K which is identified with T^3 by the map $i : T^3 \to K \subset T^5$, $i(\theta_1, \theta_2, \theta_3) = (\theta_2\theta_3, \theta_1, \theta_2, \theta_3, \theta_1\theta_2)$. The map i^* is given by the transpose matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

or writing in coordinates $i^*(x_1, x_2, x_3, x_4, x_5) = (x_2 + x_5, x_1 + x_3 + x_5, x_1 + x_4)$, so $\nu = i^*(-(0, -4, -5, -4, 0)) = (4, 5, 4)$.

The hamiltonian action of $K = T^3$ on $\mathbb{C}^5$ is given by

$$(\theta_1, \theta_2, \theta_3) \cdot (z_1, z_2, z_3, z_4, z_5) = (\theta_2\theta_3 z_1, \theta_1 z_2, \theta_2 z_3, \theta_3 z_4, \theta_1\theta_2 z_5)$$

with moment map $\mu : \mathbb{C}^5 \to k^* \cong \mathbb{R}^3$, $\mu(z_1, z_2, z_3, z_4, z_5) = \pi(\|z_2\|^2 + \|z_5\|^2, \|z_1\|^2 + \|z_3\|^2 + \|z_5\|^2, \|z_1\|^2 + \|z_4\|^2)$ (here π is the number). So that $\mu^{-1}(\nu) = \{z \in \mathbb{C}^5 : \|z_2\|^2 + \|z_5\|^2 = \frac{4}{\pi}, \|z_1\|^2 + \|z_3\|^2 + \|z_5\|^2 = \frac{5}{\pi}, \|z_1\|^2 + \|z_4\|^2 = \frac{4}{\pi}\}$ and $M_\triangle = \mu^{-1}(\nu)/K$ which is diffeomorphic to $(\mathbb{CP}^1 \times \mathbb{CP}^1)\sharp\overline{\mathbb{CP}^2} \cong \mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}\sharp\overline{\mathbb{CP}^2}$.

For the complex construction, labelling the faces using the same numbering that was used for the normal vector the collection of subsets $\mathcal{F}$ corresponding to this polytope is

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}$$

and

$$U_{\mathcal{F}} = \mathbb{C}^5 \setminus \{\{z_1 = 0 = z_3\} \cup \{z_1 = 0 = z_4\} \cup \{z_2 = 0 = z_4\} \cup \{z_2 = 0 = z_5\} \cup \{z_3 = 0 = z_5\}\}$$

The quotient of $U_{\mathcal{F}}$ by the complexified action of $K_{\mathbb{C}}$ is $M_{\triangle}$. The prequantum line bundle will be $L = U_{\mathcal{F}} \times_{K_{\mathbb{C}}} \mathbb{C}$ where $K_{\mathbb{C}} \cong (\mathbb{C}^*)^3$ acts on $\mathbb{C}$ with weight $(4,5,4)$, i.e, for $(k_1, k_2, k_3) \in \mathbb{C}^3$, $(k_1, k_2, k_3) \cdot z = k_1^4 k_2^5 k_3^4 z$.

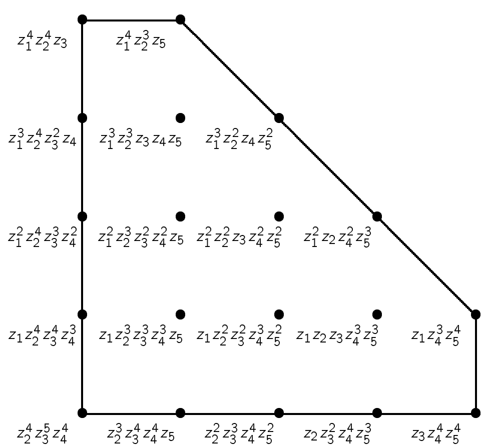
For the space of sections we are looking for $K_{\mathbb{C}}$-equivariant holomorphic functions $s : \mathbb{C}^5 \to \mathbb{C}$. So, take s to be a monomial $z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} z_5^{j_5}, j_i \in \mathbb{Z}_{\geq 0}$. For $(k_1, k_2, k_3) \in (\mathbb{C}^*)^3 \cong K_{\mathbb{C}}$,

$$\begin{aligned} s(k \cdot z) &= (k_1, k_2, k_3) \cdot (z_1, z_2, z_3, z_4, z_5) = (k_2 k_3 z_1)^{j_1} (k_1 z_2)^{j_2} (k_2 z_3)^{j_3} (k_3 z_4)^{j_4} (k_1 k_2 z_5)^{j_5} \\ &= k_1^{j_2 + j_5} k_2^{j_1 + j_3 + j_5} k_3^{j_1 + j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} z_5^{j_5}, \end{aligned}$$

on the other hand

$$k \cdot s(z) = k_1^4 k_2^5 k_3^4 z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} z_5^{j_5},$$

so that $j_2 + j_5 = 4, j_1 + j_3 + j_5 = 5, j_1 + j_4 = 4$; which is precisely the set of integer points in $i^*(4,5,4)^{-1} \cap \mathbb{Z}_+^3$. Recall that $\triangle \cap \mathbb{Z}^2$ is in correspondence with $i^*(4,5,4)^{-1} \cap \mathbb{Z}_+^3$ by the map $\pi^* - \lambda$. So every integer point of $\triangle$ represents a basis element of $\Gamma_{\mathcal{O}}(M, L)$:



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