DIFFERENTIAL FORMS ON SYMPLECTIC QUOTIENTS

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Abstract. We summarize the de Rham model for symplectic quotients, which was introduced by Sjamaar [1]. This theory gives a nice definition for differential forms on the symplectic quotient of a Hamiltonian G-manifold even if the quotient itself is not a manifold. We prove a version of the Poincaré lemma and describe how this result implies a version of the de Rham theorem.

1. The symplectic quotient

Let (M, ω) be a connected symplectic manifold, and G a compact Lie group acting on M with moment map $\phi : M \to \mathfrak{g}^*$. Let

$$Z = \phi^{-1}(0)$$

and define the *symplectic quotient of* M *by* G to be $X := Z/G$. If G acts freely on Z, we have seen that Z and X are manifolds and $\Omega(X)$ can be identified with the basic forms on Z. But if G does not act freely on Z, we have no guarantee that either Z or X is a manifold, but we do know that they are both stratified spaces (Sjamaar and Lerman [2]).

Recall that for any closed subgroup H of G,

$$M_{(H)} = \{m \in M : \mathrm{Stab}_G(m) \text{ is conjugate to } H\}$$

is a smooth submanifold of M, and its connected components are the orbit-type strata. We have shown that there exists a partial order defined on the orbit type strata, given by $M_a \leq M_b$ if $M_a \subset \overline{M_b}$, and that there exists a unique maximal element $M_{\mathrm{top}} \subset M$, called *the principal stratum*, which is open and dense.

Define $Z_{(H)} = Z \cap M_{(H)}$. Then $Z_{(H)}$ is indeed a smooth G-stable submanifold of M , and the decomposition

$$Z = \coprod_{a \in A} Z_a$$

is a stratification of Z, where $\{Z_a : a \in A\}$ is the collection of connected components of manifolds of the form $Z_{(H)}$, so Z has a principal stratum, Z_{top}. In fact, the stratification is a Whitney stratification (Sjamaar and Lerman [2]), so it satisfies *Whitney's condition A*:

Given any two strata Z_a and Z_b, if a sequence of points $\{z_n\}_{n=1}^{\infty}$ in Z_a converges to $z \in Z_b$, and the sequence of tangent spaces $\{T_{z_n} Z_a\}_{n=1}^{\infty}$ converges to some tangent space T, then $T_z Z_b \subset T$.

This stratification on Z induces a stratification on X into symplectic manifolds, given by $X_a = Z_a/G$, and the principal stratum X_{top} is open and dense in X. Denote the standard inclusion and projection by $i_a : Z_a \hookrightarrow M$ and $\pi_a : Z_a \to X_a$ respectively. If U is an open subset of X, then we get a stratification on U from the stratification on X via $U_{(H)} := U \cap X_{(H)}$, so in particular, there is a principal stratum U_{top}, which is open and dense in U.

2. Differential forms on the symplectic quotient

We define a *differential form on the symplectic quotient* to be a differential form $\alpha \in \Omega(X_{\text{top}})$ such that there exists a form $\tilde{\alpha} \in \Omega(M)$ satisfying $\pi_{\text{top}}^* \alpha = i_{\text{top}}^* \tilde{\alpha}$. We say that $\tilde{\alpha}$ *induces* α, and denote the collection of such forms on X by $\Omega(X)$. Notice that if $X = X_{\text{top}}$, then X and Z are manifolds, and the lift of any form can be extended to M, so this definition agrees with the standard definition. By the standard averaging argument of replacing $\tilde{\alpha}$ by $\int_{g \in G} g^* \tilde{\alpha}\, dg$, we may assume without loss of generality that $\tilde{\alpha}$ is G-invariant.

If $\alpha \in \Omega(X)$, there exists $\tilde{\alpha}$ such that $\pi_{\text{top}}^* \alpha = i_{\text{top}}^* \tilde{\alpha}$. Hence,

$$\pi_{\text{top}}^*(d\alpha) = d(\pi_{\text{top}}^* \alpha) = d(i_{\text{top}}^* \tilde{\alpha}) = i_{\text{top}}^*(d\tilde{\alpha}),$$

so $d\alpha \in \Omega(X)$. Thus, $\Omega(X)$ is a subcomplex of $\Omega(X_{\text{top}})$. With a similar computation, we see that it is closed under the wedge product. We refer to this complex as *the de Rham complex of* X.

If U is an open subset of X, a *differential form on* U is a differential form on U_{top} such that, for every $x \in U$, there exists $\alpha' \in \Omega(X)$ and an open neighbourhood U' of x in U s.t. $\alpha|_{U'_{\text{top}}} = \alpha$. Denote the set of differential forms on U by $\Omega(U)$. It can be verified that $\Omega : U \to \Omega(U)$ is indeed a sheaf, and that its space of global sections is $\Omega(X)$.

We say that a differential form $\beta \in \Omega(M)$ is *ϕ-basic* if it is G-invariant and $i_{\text{top}}^* \beta \in \Omega(Z_{\text{top}})$ is G-horizontal. Denote the set of ϕ-basic forms by $\Omega_\phi(M)$. If $\tilde{\beta} \in \Omega_\phi(M)$, then $i_{\text{top}}^* \tilde{\beta}$ descends to a form $\beta \in \Omega(X_{\text{top}})$ because it is G-invariant. It is easy to verify that $i_{\text{top}}^*(d\tilde{\beta})$ is G-horizontal, and clearly $d\tilde{\beta}$ is G-invariant, so $d\tilde{\beta} \in \Omega_\phi(M)$, and thus $\Omega_\phi(M)$ is a subcomplex of $\Omega(M)$.

We get a natural surjection $\Omega_\phi(M) \to \Omega(X)$ given by sending $\tilde{\beta}$ to the form induced on the quotient from $i_{\text{top}}^* \tilde{\beta}$. It is clear that the kernel of this map is the ideal

$$I_\phi(M) = \{\tilde{\beta} \in \Omega(M)^G : i_{\text{top}}^* \tilde{\beta} = 0\}.$$

Thus, we conclude that this gives an isomorphism of complexes

$$\Omega(X) \cong \Omega_\phi(M)/I_\phi(M). \tag{2.1}$$

Lemma 2.2.

1. *Let $\beta \in \Omega_\phi(M)$. Then $i_a^* \beta$ is a horizontal form on Z_a for all a.*
2. *Let $\beta \in I_\phi(M)$. Then $i_a^* \beta = 0$ for all a.*
3. *There is a well-defined restriction map $\Omega(X) \to \Omega(X_a)$ for each stratum X_a.*

Proof. Let $\beta \in \Omega_\phi(M)$ and $z \in Z_a$. Since Z_{top} is dense, choose a sequence $\{z_n\}_{n=1}^\infty$ converging to z. Then by compactness of the Grassmanian, after passing to a subsequence, $\{T_{z_n} Z_{\text{top}}\}_{n=1}^\infty$ converges to a subspace T of $T_z M$. Then from Whitney's Condition A, we have $T_z Z_a \subset T$. By definition, $\iota(\xi_M)\beta|_{z_n} = 0$ on $T_{z_n} Z_{\text{top}}$ for all $\xi \in \mathfrak{g}$, so by continuity, $\iota(\xi_M)\beta|_z = 0$ on T for all ξ. In particular, $\iota(\xi_M)\beta|_z = 0$ on $T_z Z_a$. This proves the first claim. Similarly, if $\beta \in I_\phi(M)$, then $\beta|_{z_n} = 0$ on $T_{z_n} Z_{\text{top}}$, so by continuity, $\beta|_z = 0$ on T. Thus, $\beta_z = 0$ on $T_z Z_a$. Finally, if $\beta \in \Omega_\phi(M)$, then $i_a^* \beta$ descends to a form β_a on X_a. This map $\beta \mapsto \beta_a$ defines a homomorphism $\Omega_\phi(M) \to \Omega(X_a)$ for each a. Thus, by the isomorphism (2.1), we get the required restriction map $\Omega(X) \to \Omega(X_a)$. □

3. The Poincaré lemma

Our goal in this section is to prove the following lemma:

Lemma 3.1. *(Poincaré Lemma)*
Every $x \in X$ has a basis of open neighbourhoods U such that the sequence

$$0 \to \mathbb{R} \xrightarrow{i} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots$$

is exact, where $i : \mathbb{R} \to \Omega^0(U)$ is natural inclusion.

Let (M', ω', ϕ') be another Hamiltonian G-manifold, with $Z' := (\phi')^{-1}(0)$ and $X' = Z'/G$. As before, we have a stratification of Z' and X'.

We now give an appropriate definition of homotopy in this setting of stratified symplectic quotients. Call a smooth homotopy $F : M \times [0,1] \to M'$ *allowable* if:

(1) F is G-equivariant with respect to the given G-actions on M and M' and the trivial action on $[0,1]$;
(2) $F_t(Z) \subset Z'$ for all $t \in [0,1]$;
(3) $dF_{(z,t)}(T_z Z_{\text{top}}) \subset T_{F(z,t)} Z'_{a(z,t)}$ and $dF_{(z,t)}\left(\frac{\partial}{\partial t}\right) \in T_{F(z,t)} Z'_{a(z,t)}$ for almost all $t \in [0,1]$ and all $z \in Z_{\text{top}}$, where $Z'_{a(z,t)} \subset Z'$ is the stratum of $F(z,t)$.

Given an allowable homotopy F, define an operator $\kappa_F : \Omega(M') \to \Omega(M)$ via $\kappa_F(\gamma) = \int_0^1 \iota(\partial/\partial t) F^* \gamma \, dt$. Then κ_F is a chain homotopy, i.e. it lowers degree by 1 and

$$\kappa_F d + d\kappa_F = F_1^* - F_0^*.$$

Furthermore, for any $g \in G$, $\xi \in \mathfrak{g}$, $\gamma \in \Omega(M')$:

$$\kappa_F \circ g^*(\gamma) = g^* \circ \kappa_F(\gamma) \tag{3.2}$$

$$\begin{aligned} \kappa_F \circ \iota(\xi_{M'})(\gamma) &= \int_0^1 \iota\left(\frac{\partial}{\partial t}\right) F^* \iota(\xi_{M'}) \gamma \, dt \\ &= \int_0^1 \iota\left(\frac{\partial}{\partial t}\right) \iota(F_*(\xi_{M'})) F^* \gamma \, dt \\ &= \int_0^1 \iota\left(\frac{\partial}{\partial t}\right) \iota(\xi_M) F^* \gamma \, dt \\ &= -\int_0^1 \iota(\xi_M) \iota\left(\frac{\partial}{\partial t}\right) F^* \gamma \, dt \\ &= -\iota(\xi_M) \int_0^1 \iota\left(\frac{\partial}{\partial t}\right) F^* \gamma \, dt \\ &= -\iota(\xi_M) \circ \kappa_F(\gamma) \end{aligned} \tag{3.3}$$

Example 3.4. Let (V, ω) be a symplectic vector space on which G acts linearly and symplectically. Then a moment map is given by $\phi_V^\xi = \frac{1}{2}\omega(\xi v, v)$. Define $F : V \times [0,1] \to V$ via $F(v,t) = tv$. Clearly, this map is smooth, G-equivariant, and preserves Z. Also, for $t \neq 0$, $F_t(v)$ has the same stabilizer as v, so $F_t(Z_{\text{top}}) \subset Z_{\text{top}}$, so this is an allowable homotopy. If B is any G-invariant open ball centred at the origin, F defines an allowable homotopy $B \times [0,1] \to B$.

Lemma 3.5. *Let $F : M \times [0,1] \to M'$ be an allowable homotopy. Then the homotopy operator $\kappa_F : \Omega(M') \to \Omega(M)$ sends $\Omega_{\phi'}(M')$ to $\Omega_\phi(M)$ and $I_{\phi'}(M')$ to $I_\phi(M)$, so it induces a homotopy $\kappa_F : \Omega(X') \to \Omega(X)$.*

Proof. Let $\gamma \in \Omega^k_{\phi'}(M')$, and let $z \in Z_{\text{top}}$, $v \in \Lambda^{k-1}(T_z Z_{\text{top}})$. Then by (3.3),

$$\iota(\xi_M)(\kappa_F\gamma)_z(v) = \int_0^1 \psi(t)\, dt,$$

where $\psi(t) = -\gamma_{F(z,t)}(\xi_{M'}, F_*\frac{\partial}{\partial t}, (F_t)_* v)$. Let $Z'_{a(z,t)}$ be the stratum of Z' containing $F(z,t)$. Since F is allowable,

$$F_*\tfrac{\partial}{\partial t} \in T_{F(z,t)} Z'_{a(z,t)} \quad \text{and} \quad (F_t)_* v \in \Lambda^{k-1}(T_{F(z,t)} Z'_{a(z,t)})$$

for almost all t. But by Lemma 2.2, the restriction of γ to $Z'_{a(z,t)}$ is horizontal, so $\psi(t) = 0$ for almost all t. Thus, $\iota(\xi_M)(\kappa_F\gamma)_z(v) = 0$, i.e. $\kappa_F\gamma \in \Omega^{k-1}_\phi(M)$. A similar argument shows that $\kappa_F I_{\phi'}(M') \subset I_\phi(M)$. □

As a consequence, we conclude by Example 3.4 that $H(\Omega(B)) = H(\Omega(*))$ for any G-invariant ball in a symplectic vector space (with a linear, symplectic G-action).

Proof of Lemma 3.1. Let $x \in X$ and $z \in Z$ such that $\pi(z) = x$. Let $H = \text{Stab}(z)$, and $V = (T_z G \cdot z)^\omega/(T_z G \cdot z)$ be the symplectic slice at z. Let B be an H-invariant ball in V and let $O := (T^*G \times B)//H$, where H acts on T^*G by left multiplication. By the symplectic slice theorem, for a sufficiently small B, z has a G-invariant open neighbourhood that is isomorphic to O as a Hamiltonian G-manifold. Let $Y = B//H$ and $U = O//G$. Then $Y \cong U$ via a stratification-preserving isomorphism which restricts to a symplectomorphism on each stratum. By Proposition 4.2 in Sjamaar [1], $\Omega(U) \cong \Omega(Y)$. But Y is an invariant ball in a symplectic vector space, so $H(\Omega(Y))$ is trivial, and hence $H(\Omega(U))$ is trivial as well. Letting B shrink to a point yields a basis for the topology at x. □

4. A de Rham theorem

Let $\underline{\mathbb{R}}$ be the sheaf of locally constant real-valued functions on X. By Lemma 3.1, the sequence

$$0 \to \underline{\mathbb{R}} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots,$$

where $i : \underline{\mathbb{R}} \to \Omega^0$ is the natural inclusion, is an exact sequence. Furthermore, it can be shown that Ω is an acyclic sheaf (i.e. it has trivial sheaf cohomology), so that it is an acyclic resolution of the constant sheaf. Then arguments from sheaf theory (see Warner [3], Section 5) yield a de Rham theorem:

Theorem 4.1. *The de Rham cohomology ring $H(\Omega(X))$ is naturally isomorphic to $H(X;\mathbb{R})$, the singular (or Čech) cohomology ring of X.*

References

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