Localization and Cobordism

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1 Reduction and Equivariant Cohomology

Let (M, ω) be a compact symplectic manifold acted upon by a compact Lie group G with momentum map $\Phi : M \to \mathfrak{g}^*$. Then we have the *symplectic reduction* of M, denoted M_{red} and given by

$$M_{\text{red}} = \Phi^{-1}(0)/G.$$

Let us assume from now on that 0 is a regular value of Φ. Then M_{red} is at worst an orbifold, and inherits a unique symplectic form satisfying $\pi^*(\omega_{\text{red}}) = i^*(\omega)$, where $\pi : \Phi^{-1}(0) \to M_{\text{red}}$ is the projection and $i : \Phi^{-1}(0) \hookrightarrow M$ is the inclusion. This may be summarized by the following diagram:

$$\begin{array}{ccc} \Phi^{-1}(0) & \overset{i}{\hookrightarrow} & M \\ \pi \downarrow & & \\ M_{\text{red}} & & \end{array}$$

Next we recall the Cartan model of equivariant cohomology (see, for example [9] or [5]). The equivariant de Rham theorem says that the equivariant cohomology of M is the cohomology of the complex

$$(\Omega(M) \otimes S(\mathfrak{g}^*))^G$$

with differential d_G, defined by

$$(d_G\alpha)(\xi) = d\left(\alpha(\xi)\right) + i(\xi_M)\alpha(\xi).$$

Note that combining the momentum map Φ with the symplectic form ω gives an equivariant cohomology class c, given by

$$c(\xi) = \omega + \langle \Phi, \xi \rangle.$$

This is closed, since

$$\begin{aligned} \left(d_G c\right)(\xi) &= \left(d + i(\xi_M)\right)\left(\omega + \langle \Phi, \xi \rangle\right) \\ &= i(\xi_M)\omega + d\langle \Phi, \xi \rangle \\ &= 0. \end{aligned}$$

Now suppose that c is any equivariant cohomology class on M. Then there is an induced cohomology class c_{red} on M_{red}, given by the *Kirwan map*:

$$\kappa : H^*_G(M) \to H^*(M_{\text{red}}),$$

which is given by the composition $H^*_G(M) \to H^*_G\left(\Phi^{-1}(0)\right) \cong H^*(M_{\text{red}})$. This map satisfies $\pi^*(c_{\text{red}}) = i^*(c)$. It is well known that the Kirwan map is surjective (see [8]). That is, for every ordinary cohomology class c_{red} on M_{red}, there is an equivariant cohomology class c on M such that $c_{\text{red}} = \kappa(c)$.

Let $c_{\text{red}} \in H^*(M)$ be any class and consider the quantity

$$\int_{M_{\text{red}}} c_{\text{red}}.$$

Since $c_{\text{red}} = \kappa(c)$ for some $c \in H^*_G(M)$, we might expect to express the integral of c_{red} in terms of data on M that might be easier to compute. Similarly, we might consider the integral $\int_M c$ of an equivariant class c over M. It turns out that in the case of torus actions, these integrals may be computed in terms of data at the fixed point set of M. This is an example of a general phenomenon known as *localization*, originally discovered and expounded upon by Duistermaat-Heckman [3], Berline-Vergne [2], and Atiyah-Bott [1]. Formulas for nonabelian localization were later obtained by Witten [10] and Jeffrey-Kirwan [6], but are beyond the scope of this paper.

In what follows, we will take the approach of Ginzburg, Guillemin and Karshon (see [7] and [4]) to obtain (abelian) localization theorems from a suitable notion of noncompact cobordism.

2 Abstract Momentum Maps and Cobordism

Let T be a torus and M a smooth manifold with a smooth T action. The following definitions are due to [7].

Definition 2.1. *An* abstract momentum map *is a smooth map $\Phi : M \to \mathfrak{t}^*$ satisfying the following conditions:*

1. *Φ is T invariant.*
2. *For any subtorus $H \subset T$, the map Φ_H defined by the natural projection $\mathfrak{t}^* \to \mathfrak{h}^*$ is constant on each connected component of M^H.*

Definition 2.2. *Let M, M' be two smooth oriented manifolds acted upon by a torus T, with proper abstract momentum maps Φ, Φ', and let c, c' be equivariant cohomology classes on M and M', respectively. A* cobordism *between the triples (M, Φ, c) and (M', Φ', c') is a triple $(W, \tilde{\Phi}, \tilde{c})$ satisfying the following conditions:*

1. *W is an oriented manifold-with-boundary with a T-action.*
2. *$\partial W = M \sqcup M'$ inducing the given orientation on M and the opposite orientation on M'.*
3. *$\tilde{\Phi}$ is a proper abstract momentum map that restricts to $\Phi \sqcup \Phi'$ on ∂W.*
4. *$\tilde{c}$ is an equivariant cohomology class on W that restricts to $c \sqcup c'$ on ∂W.*

Note that in the above, we do not require that W be compact, but we do require that $\tilde{\Phi}$ be proper. The restriction that $\tilde{\Phi}$ be proper is absolutely essential, for otherwise every manifold would be cobordant to the empty set via the noncompact cobordism $W = M \times (0, 1]$ with $\tilde{\Phi}(m, x) = \Phi(m)$.

As we will see in theorem 2.4, the obstruction to constructing such a cobordism between M and the empty set is exactly the fixed point set of M.

Next we consider abstract momentum maps for linear actions. Let V be a vector space with T-action. Let $\pm\alpha_1, \dots, \pm\alpha_m$ be the weights for the action (note that there is an ambiguity in the weights because we are not assuming a complex or symplectic structure on V) and let V_i be the invariant subspace on which T acts by the weight $\pm\alpha_i$. Then V decomposes as $\oplus_i V_i$ and any $v \in V$ can be written as a sum $v = \sum v_i$ with each $v_i \in V_i$. Let $\eta \in \mathfrak{t}$ be any vector such that $\langle \alpha_i, \eta \rangle \neq 0 \ \forall i$. Let $a \in \mathfrak{t}^*$, and choose any T-invariant metric on V. For each weight $\pm\alpha$, let $\alpha^\eta \in \{\alpha, -\alpha\}$ be chosen so that $\langle \alpha^\eta, \eta \rangle > 0$. We will say that η is a *polarizing vector* for the abstract momentum map Φ. Then we define

$$\Phi^\eta(v) = a + \sum_i^m ||v_i||^2 \alpha_i^\eta.$$

Then Φ^η is an abstract momentum map taking the value a at the origin, and its η component is proper and bounded from below.

Lemma 2.3. *Let Φ' be any abstract momentum map on V whose value at the origin is a and whose η-component is proper and bounded from below. Then for any $c \in H_T^*(V)$, the triple (V, Φ^η, c) is cobordant to (V, Φ', c).*

Proof. Just take the abstract momentum map $\tilde{\Phi}(t, v) := (1-t)\Phi^\eta(v) + t\Phi'(v)$ on $V \times [0,1]$. □

Now let M be an oriented manifold with T action, Φ an abstract momentum map, and $c \in H_T^*(M)$. Suppose there exists a vector $\eta \in \mathfrak{t}$ such that the η-component of Φ is proper and bounded from below. Let M^η be the zero set of the vector field η_M. Each connected component F of M^η inherits a T action on its normal bundle NF from the T action on M, with abstract momentum map Φ_F^η given by

$$\Phi_F^\eta(p, v) = \Phi(p) + \Phi^\eta(v),$$

where $(p, v) \in NF$ and Φ^η is the abstract momentum map constructed above for linear actions. We now state the main theorem.

Theorem 2.4. *(Karshon [7]). The triple (M, Φ, c) is cobordant to the disjoint union*

$$\bigsqcup_F (NF, \Phi_F^\eta, c_F),$$

where the union runs over the connected components F of the fixed point set M^η, and c_F is the pullback of c via $NF \to F \hookrightarrow M$.

Proof. Let H be the closure in T of $\{\exp(t\eta) : t \in \mathbb{R}\}$. Then $M^\eta = M^H$, so M^η is a disjoint union of closed submanifolds of M. Let F be any connected component of M^η. Choose an invariant metric on M, and let $\epsilon > 0$ be such that the ϵ neighborhood of F in M is equivariantly diffeomorphic to NF. Let B_F be an ϵ neighborhood of $F \times \{0\}$ in $M \times (0,1]$, and by taking ϵ small enough we may assume that all of the B_F are disjoint. Let $W = (M \times (0,1]) \setminus \sqcup_F B_F$ (see figure 1). The removal of each B_F introduces a boundary component of NF, so that the boundary of W is the disjoint union of M together with $\sqcup_F NF$. All that remains is to construct a proper abstract momentum map $\tilde{\Phi}$ on W. We will take $\tilde{\Phi}$ to be of the form

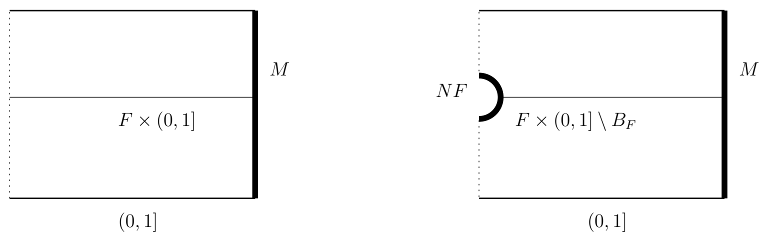

Figure 1: By removing an ϵ-neighborhood B_F of F, we introduce a boundary component equivariantly diffeomorphic to NF.

$$\tilde{\Phi}(m,x) = \Phi(m) + \rho(m,x),$$

where ρ is some function yet to be determined. Let

$$w : (0,1] \to \mathbb{R}_{\geq 0}$$

be any function which approaches infinity as $x \to 0$ and vanishes for $x \geq \epsilon$. We will use this function in the construction of ρ.

To construct ρ, we first work locally. Let $m \in M$ and U_m an invariant neighborhood of $T \cdot m$ chosen small enough that it retracts equivariantly to $T \cdot m$. We will define a function ρ_m on U_m, and then piece the ρ_m together using an invariant partition of unity. Let $\mathfrak{t}_m$ be the Lie algebra of the stabilizer of m. If $\eta \in \mathfrak{t}_m$, then set $\rho_m(m', x) = 0 \ \forall m' \in U$. Otherwise, let α_m be some element of the annihilator $\mathfrak{t}_m^0$ of $\mathfrak{t}_m$ such that $\langle \alpha_m, \eta \rangle > 0$ (the argument in [7] works best if we take $\langle \alpha_m, \eta \rangle = 1$, but this does not seem to be necessary). Then define $\rho_m(m', x) = \alpha_m w(x) \ \forall (m', x) \in U_m \times (0,1]$.

Since the U_m form an invariant open cover of M, we may find a sequence $\{y_i\}_{i \in \mathbb{N}}$ in M such that $\{U_{y_i}\}$ is a locally finite invariant subcover. Let $\{\lambda_i\}$ be an invariant partition of unity subordinate to this cover, and let the λ_i be chosen so that $\lambda_i(y_i) > 0$. Now set

$$\begin{aligned} \rho(m,x) &= \sum_i \lambda_i(m) \rho_{y_i}(m,x) \\ \tilde{\Phi}(m,x) &= \Phi(m) + \rho(m,x). \end{aligned}$$

It is easily seen that $\tilde{\Phi}$ is an abstract momentum map for the T-action on W, so it remains to check that it is proper.

Let $\langle \cdot, \cdot \rangle$ be an invariant inner product on $\mathfrak{t}$ and let us use it to identify $\mathfrak{t} \cong \mathfrak{t}^*$. Let $K \subset \mathfrak{t}$ be compact. Then it is closed and bounded, say by some constant C. Let $\{(m_n, x_n)\}$ be a sequence in $\tilde{\Phi}^{-1}(K)$. We would like to show the existence of a subsequence which converges in W. First, note that by construction $\langle \rho, \eta \rangle \geq 0$, so we have

$$C \geq \tilde{\Phi}^\eta(m_n, x_n) = \Phi^\eta(m_n) + \langle \rho(m_n, x_n), \eta \rangle \geq \Phi^\eta(m_n).$$

Hence $\Phi^\eta(m_n) \leq C$. But Φ^η is proper and bounded below, hence $\{m_n\}$ is contained in some compact subset of M. Thus by passing to a subsequence if necessary, we may assume $m_n \to m_\infty$ for some $m_\infty \in M$. Furthermore, we have $x_n \in (0,1] \subset [0,1]$, and by compactness of $[0,1]$ we may assume (again passing to a subsequence if necessary) that $x_n \to x_\infty$ for some $x_\infty \in [0,1]$.

Since W is a closed subset of $M \times (0,1]$, it suffices to show that $(m_\infty, x_\infty) \in M \times (0,1]$. If m_∞ is a fixed point, then clearly $x_\infty > 0$, so we are done. Otherwise, assume m_∞ is not a fixed point, and we must show that $x_\infty > 0$. By hypothesis, Φ^η is proper and bounded below. Without loss of generality, assume that it is bounded below by zero, so that $\Phi^\eta \geq 0$. Then we have

$$C \geq \tilde{\Phi}^\eta(m_n, x_n) = \Phi^\eta(m_n) + \langle \rho(m_n, x_n), \eta \rangle \geq \langle \rho(m_n, x_n), \eta \rangle.$$

Since $\{\lambda_i\}$ is a partition of unity, there is some $j \in \mathbb{Z}$ such that y_j is not a fixed point and $\lambda_j(m_\infty) > \delta > 0$. By continuity, there exists N such that $|\lambda_j(m_\infty) - \lambda_j(m_n)| < \delta$ for all $n > N$. Thus for $n > N$ we have

$$\begin{aligned} C &\geq \langle \rho(m_n, x_n), \eta \rangle = \sum_i \lambda_i(m_n) \rho_{y_i}(m_n, x_n), \eta \rangle \\ &= \sum_i \lambda_i(m_n) \langle \alpha_j, \eta \rangle w(x_n) \geq \lambda_j(m_n) \langle \alpha_j, \eta \rangle w(x_n) \\ &\geq (\lambda_j(m_\infty) - \delta) \langle \alpha_j, \eta \rangle w(x_n) \end{aligned}$$

Thus for $n > N$,

$$w(x_n) \leq \frac{C}{(\lambda_j(m_\infty) - \delta) \langle \alpha_j, \eta \rangle}.$$

Thus $w(x_n)$ is uniformly bounded, and since $w \to \infty$ as $x \to 0$, we have that x_n is uniformly bounded away from zero. Thus $x_\infty > 0$, and we have that $(m_\infty, x_\infty) \in W$.

Thus the pair $(W, \tilde{\Phi})$ give a cobordism of (M, Φ) and $\sqcup(NF, \tilde{\Phi}|_F)$. By the lemma above, $(NF, \tilde{\Phi}|_F)$ is cobordant to (NF, Φ_F), and composing these gives the desired cobordism between M and $\sqcup_F(NF, \Phi_F)$. □

3 Reduction and Localization

Now that we have the main theorem we use it to address the original question of deriving a formula for $\int_{M_{\text{red}}} c_{\text{red}}$.

Lemma 3.1. *If a and b are two values in $\mathfrak{t}^*$ such that c is a regular value of Φ for all $c \in [a,b]$, then $M_{\text{red}}(a)$ is cobordant to $M_{\text{red}}(b)$.*

Proof. Just consider $\Phi^{-1}([a,b])$. □

Lemma 3.2. *If (M, Φ, c) and (M', Φ', c') are cobordant oriented manifolds with T-actions, proper abstract moment maps, and $a \in \mathfrak{t}^*$ is regular for both Φ and Φ', then $(M_{\text{red}}, c_{\text{red}})$ and $(M'_{\text{red}}, c'_{\text{red}})$ are cobordant.*

Proof. Let $(W, \tilde{\Phi}, \tilde{c})$ be a cobordism between M and M'. Suppose $b \in \mathfrak{t}^*$ is regular for Φ, Φ', and $\tilde{\Phi}$. Then $\tilde{\Phi}^{-1}(b)/T$ gives a cobordism between $M_{\text{red}}(b)$ and M'_{red}. Furthermore, if we could choose b such that every value in $[a,b]$ is regular for both Φ and Φ', we could apply the previous lemma to obtain $M_{\text{red}}(a) \sim M_{\text{red}}(b) \sim M'_{\text{red}}(b) \sim M'_{\text{red}}(a)$. Thus it suffices to show that we can choose such a b. Since Φ and Φ' are proper, their sets of regular values are open. Hence by Sard's theorem we can choose such a b. □

This lemma implies that for a regular value $a \in \mathfrak{t}^*$, the integral $\int_{M_{\text{red}}} c_{\text{red}}$ is a cobordism invariant of (M, Φ, c). Together with theorem 2.4, we have the following localization formula, which is a generalization of Guillemin's topological form of the (abelian) Jeffrey-Kirwan formula.

Proposition 3.3. *Under the hypotheses of theorem 2.4, let $a \in \mathfrak{t}^*$ be regular for Φ and all the Φ_F^η. Then*

$$\int_{M_{\text{red}}} c_{\text{red}} = \sum_F \int_{(NF)_{\text{red}}} (c_F)_{\text{red}}.$$

Recall that for a symplectic manifold (M, ω), the *Liouville measure* is the measure induced by the volume form $\omega^n/n!$. If a torus T acts on M with momentum map Φ, then the *Duistermaat-Heckman* measure on the momentum polytope is the push-forward of the Liouville measure by the momentum map. The Duistermaat-Heckman measure is absolutely continuous with respect to the Lesbesgue measure on $\mathfrak{t}^*$, and it can be shown that the Radon-Nikodym derivative of the Duistermaat-Heckman measure is the *Duistermaat-Heckman function*, given by $a \mapsto \int_{M_{\text{red}}(a)} \omega_{\text{red}}^d/d! = \text{vol}(M_{\text{red}}(a))$ where $d = \dim M_{\text{red}}$ (this fact can be seen by application of Fubini's theorem). Using the preceding lemmas, one may show the following (see [7])

Proposition 3.4. *The Duistermaat-Heckman measure is an invariant of cobordism.*

Combining this with theorem 2.4, we have the following generalization of the Guillemin-Lerman-Sternberg formula for the Duistermaat-Heckman measure.

Proposition 3.5. (M, ω) *a pre-symplectic oriented manifold with Hamiltonian T-action, and let $\eta \in \mathfrak{t}$ be such that the η-component of Φ is proper and bounded below. Then*

$$DH(M) = \sum_F DH(NF),$$

where the sum runs over connected components F of the fixed-point set M^η, and each NF is equipped with a 2-form and abstract momentum map whose η-component is proper.

We now give a worked example. Let $T = (S^1)^2$ act on $\mathbb{CP}^2$ by

$$(t_1, t_2) \cdot [z_1, z_2, z_3] = [t_1 z_1, t_2 z_2, t_3 z_3].$$

Then the fixed points of this action are just $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$, corresponding to the three vertices in the momentum polytope. At each fixed point F, we have $NF \cong T_F\mathbb{CP}^2 \cong \mathbb{C}^2$. Now fix $\eta = (\eta_1, \eta_2) \in \mathfrak{t}$ generic with $\eta_1 > 0 > \eta_2$. At $[1, 0, 0]$, the isotropy representation has weights $(-1, 1)$ and $(-1, 0)$. Since $\eta_1 > \eta_2$, the momentum map $\Phi^\eta_{[1,0,0]}$ is given by

$$\Phi(x_1, x_2)^\eta_{[1,0,0]} = (1, 0) + |x_1|^2(1, -1) + |x_2|^2(1, 0).$$

Similarly, at $[0, 1, 0]$ the weights are $(1, -1)$ and $(0, -1)$, and the momentum map is

$$\Phi(x_1, x_2)^\eta_{[0,1,0]} = (0, 1) + |x_1|^2(1, -1) + |x_2|^2(0, -1).$$

Finally, at $[0, 0, 1]$ the weights are $(1, 0)$ and $(0, 1)$, so the momentum map is

$$\Phi(x_1, x_2)^\eta_{[0,0,1]} = (0, 0) + |x_1|^2(1, 0) + |x_2|^2(0, -1).$$

Taking orientations into account, we obtain the decomposition of the Duistermaat-Heckman measure of $\mathbb{CP}^2$ as depicted in figure 2.

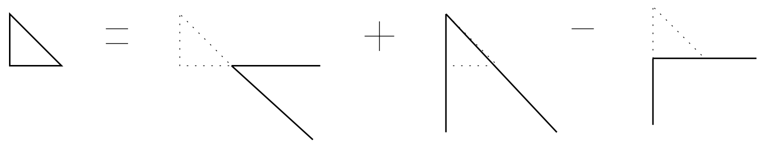

Figure 2: Decomposition of the Duistermaat-Heckman measure on $\mathbb{CP}^2$.

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