COHOMOLOGICALLY FREE SYMPLECTIC GROUP ACTIONS

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1. Introduction

In his paper [9], Ono shows that for a symplectic circle action on a compact connected symplectic manifold (M, ω), the homology class of a corresponding orbit is Poincaré dual (up to a nonzero constant factor) to the cohomology class represented by $\xi \lrcorner \frac{\omega^n}{n!}$, where ξ is the vector field on M induced by the action. (Variants of this fact are proven in [4] and [8].) If (M, ω) satisfies the Lefschetz condition (defined below), then we have as a corollary that the action is hamiltonian if and only if the orbits contract in M.

Ginzburg extends these results to symplectic toric actions in his paper [5], in which he defines the notion of a cohomologically free action, which can be thought of as the "opposite" of a hamiltonian action. He then proves the following. Let G be any compact connected Lie group acting symplectically on a compact connected symplectic manifold (M, ω). Then there is a finite covering group $\tilde{G} \to G$ such that $\tilde{G} \cong \mathbb{T}^k \times G_0$, where G_0 is a compact connected Lie group, the induced action of the torus $\mathbb{T}^k$ on (M, ω) is cohomologically free, and the induced action of G_0 is hamiltonian.

This report is organised as follows. In Section 2, we list and prove some of the facts about circle actions mentioned above, as well as Ginzburg's generalisation of these facts to toric actions. In Section 3, we prove the above-mentioned "splitting" theorem of Ginzburg. We note that for this proof we use the flux conjecture, which was proven by Kaoru Ono in [10].

2. Properties of Cohomologically Free Actions

Theorem 2.1 (Ono [9])**.** *Let (M, Ω) be a compact connected manifold with volume form Ω. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ act on (M, Ω) via volume-preserving diffeomorphisms, and let ξ be the induced vector field on M. Then, for any orbit O_x of the action through a point $x \in M$, the (real) homology class of O_x is Poincaré dual to $[\frac{\xi \lrcorner \Omega}{V}]$, where $V := \int_M \Omega$.*

Proof. First, we note that all $\mathbb{S}^1$-orbits are homotopic, and hence in the same homology class. Let ϕ be any closed 1-form. Then there exists an $\mathbb{S}^1$-invariant closed 1-form $\tilde{\phi}$ in the same cohomology class (we can obtain this via averaging over $\mathbb{S}^1$). Note that since $\tilde{\phi}$ is $\mathbb{S}^1$-invariant and closed,

$$0 = \pounds_\xi \tilde{\phi} = d(\xi \lrcorner \tilde{\phi}).$$

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Hence, the function $\tilde{\phi}(\xi)$ is constant. Let $\pi_x : \mathbb{S}^1 \to O_x \hookrightarrow M$ be the map $\theta \mapsto \theta \cdot x \in M$. Then, $\tilde{\phi}(\xi) = \pi_x^*\tilde{\phi}\left(\frac{d}{d\theta}\right)$ is constant, and

$$\begin{aligned}\int_{O_x} \phi = & \int_{O_x} \tilde{\phi} \\ = & \int_{\mathbb{S}^1} \pi_x^*\tilde{\phi} \\ = & \pi_x^*\tilde{\phi}\left(\frac{d}{d\theta}\right) \\ = & \tilde{\phi}(\xi) \\ = & \frac{1}{V}\int_M \tilde{\phi}(\xi)\Omega \\ = & \frac{1}{V}\int_M \tilde{\phi} \wedge (\xi \lrcorner \Omega) \\ = & \int_M \phi \wedge \left(\frac{\xi \lrcorner \Omega}{V}\right).\end{aligned}$$

Since Ω is $\mathbb{S}^1$-invariant, $\frac{[\xi \lrcorner \Omega]}{V}$ is closed. Hence, $\frac{[\xi \lrcorner \Omega]}{V}$ is Poincaré dual to $[O_x]$ □

Corollary 2.2 (Ono [9]). *If (M, ω) is a $2n$-dimensional compact connected symplectic manifold, and $\mathbb{S}^1$ acts on (M, ω) symplectically, then the Poincaré dual of the homology class of an orbit of the action is equal to $\frac{1}{Vn!}[\xi \lrcorner \omega^n]$ (using the same notation as above).*

Definition 2.3. Let (M, ω) be a $2n$-dimensional compact symplectic manifold. Then (M, ω) satisfies the *Lefschetz condition* if

$$\smile [\omega]^k : H^{n-k}(M; \mathbb{R}) \to H^{n+k}(M; \mathbb{R})$$

is an isomorphism for each $k = 1, ..., n$.

Remark 2.4. Any Kähler manifold satisfies the Lefschetz condition.

Corollary 2.5 (Ono [9], Frankel [4]). *If (M, ω) satisfies the Lefschetz condition, then the $\mathbb{S}^1$-action is hamiltonian if and only if ξ has at least one zero.*

Proof. We already know that if the action is hamiltonian, then ξ has at least one zero. Conversely, if ξ has a zero $z \in M$, then $O_z = \{z\}$, in which case $[O_z] = 0$ in $H_1(M; \mathbb{R})$. By Corollary 2.2, $[\xi \lrcorner \omega^n] = 0$. Applying the Lefschetz condition, we obtain that $\xi \lrcorner \omega$ is exact, and hence the action is hamiltonian. □

Before we present Ginzburg's generalisation of these results, we need some ingredients. Let G be a compact connected Lie group acting symplectically on a compact symplectic manifold (M, ω), and let $\mathfrak{g}$ be the Lie algebra of G. For $\xi \in \mathfrak{g}$, we denote by ξ_M the induced vector field on M: for $x \in M$,

$$\xi_M|_x := \frac{d}{dt}\Big|_{t=0}(\exp(t\xi) \cdot x).$$

Proposition 2.6. *Let $\xi, \eta \in \mathfrak{g}$. Then, $[\xi, \eta]_M \lrcorner \omega$ is exact.*

Proof. Let X and Y be arbitrary smooth vector fields on M, and α a differential form. Recall the identities

(1) $\pounds_X(Y \lrcorner \alpha) = [X, Y] \lrcorner \alpha + Y \lrcorner \pounds_X \alpha$.

(2) $\pounds_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$.

Then,

$$\begin{aligned} [\xi, \eta]_M \lrcorner \omega &= \pounds_{\xi_M}(\eta_M \lrcorner \omega) && \text{by identity 1 and since } \xi_M \text{ is symplectic,} \\ &= d(\omega(\eta_M, \xi_M)) && \text{by identity 2 and since } \omega \text{ is closed.} \end{aligned}$$

□

Remark 2.7. Note that since G is compact and connected, semisimplicity is equivalent to $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. This in turn is equivalent to the de Rham cohomology group $H^1(G; \mathbb{R})$ being trivial. See [2], chapter V section 12.

With the above remark in mind, we obtain the following corollary immediately.

Corollary 2.8. *If G is semisimple, then the G-action is hamiltonian.*

Let the dimension of M be $2n$. Let ψ be the linear map

$$\psi : \mathfrak{g} \to H^1(M; \mathbb{R}) : \xi \mapsto [\xi_M \lrcorner \omega].$$

If we equip $H^1(M; \mathbb{R})$ with the trivial Lie algebra structure, then by Proposition 2.6, ψ is a Lie algebra homomorphism. The action is hamiltonian if and only if ψ is the zero map.

Definition 2.9. We say that the G-action is *cohomologically free* if ψ is a monomorphism.

For the rest of this section, we will focus on torus actions $\mathbb{T}^k \circlearrowright M$ where k is any integer greater than zero. Let $x \in M$ and let π_x be the natural map $\mathbb{T}^k \to O_x \hookrightarrow M$. This induces a well-defined map $\pi_{x*} : H_1(\mathbb{T}^k; \mathbb{R}) \to H_1(M; \mathbb{R})$ that is independent of x.

Proposition 2.10 (Ginzburg, [5])**.**

(1) *If π_{x*} is a monomorphism, then the $\mathbb{T}^k$-action is cohomologically free.*
(2) *If (M, ω) satisfies the Lefschetz condition and the action is cohomologically free, then π_{x*} is a monomorphism.*
(3) *In either case the action is locally free; that is, for any $x \in M$, $\mathbb{T}^k \to O_x$ is a covering map.*

Proof. By applying Corollary 2.2 to each element of a basis of $\mathfrak{t}^k$, there is a linear isomorphism $\mathrm{im}(\pi_{x*}) \to \mathrm{im}(\psi) \smile [\omega]^{n-1}$. Hence we have the following:

$$\dim \mathrm{im}(\pi_{x*}) = \dim(\mathrm{im}(\psi) \smile [\omega]^{n-1}) \leq \dim \mathrm{im}(\psi) \leq k.$$

If π_{x*} is a monomorphism, then $\dim \mathrm{im}(\pi_{x*}) = k$, and the inequalities become equalities. Thus $\dim \mathrm{im}(\psi) = k$ and so ψ is a monomorphism; that is, the action is cohomologically free. This proves (1).

If the action is cohomologically free, then the right inequality becomes an equality. If (M, ω) is Lefschetz, then the left inequality becomes an equality. Take both conditions together and we have that $\dim \mathrm{im}(\pi_{x*}) = k$. This proves (2).

Finally, assume $\mathbb{T}^k \to O_x$ is not a covering map. Then, O_x is isomorphic to some torus $\mathbb{T}^r$ for some $r < k$, and so $\pi_{x*} : H_1(\mathbb{T}^k; \mathbb{R}) \to H_1(M; \mathbb{R})$ is not injective. This proves (3). □

Remark 2.11. It is worth noting that McDuff provides in [8] a 6-dimensional example of a compact connected symplectic manifold that does not satisfy the Lefschetz condition, admits a symplectic circle action that is not hamiltonian, but has fixed points. This shows that the Lefschetz condition is required in the above proposition and in Corollary 2.5.

3. The "Splitting" Theorem of Ginzburg

The goal of this section is to prove the following theorem.

Theorem 3.1. *Let G be a compact connected Lie group that acts symplectically on a compact symplectic manifold (M,ω). Then there exists a finite covering $\mathbb{T}^r\times G_0\to G$ where G_0 is a compact connected Lie group, the induced action of $\mathbb{T}^r$ on (M,ω) is cohomologically free, and the induced action of G_0 on (M,ω) is hamiltonian.*

In order to prove this, we require some further ingredients.

Proposition 3.2. *There exists a finite covering group $\mathbb{T}^k\times K\to G$ such that K is a compact, connected and simply connected Lie group.*

For the proof, we refer the reader to [3].

Lemma 3.3. *Let $\mathbb{T}^s\hookrightarrow\mathbb{T}^r$ be a closed connected subgroup (*i.e.* a subtorus). Then, there exists a complementary subtorus $\mathbb{T}^{r-s}\hookrightarrow\mathbb{T}^r$ such that*

$$\mathbb{T}^r\cong\mathbb{T}^{r-s}\times\mathbb{T}^s.$$

Proof. Let $\mathfrak{t}^r$ and $\mathfrak{t}^s$ be the Lie algebras of $\mathbb{T}^r$ and $\mathbb{T}^s$, respectively. Let L^r and L^s be the kernels of the corresponding exponential maps, respectively, which are lattices in $\mathfrak{t}^r$. We identify these with $\pi_1(\mathbb{T}^r)$ and $\pi_1(\mathbb{T}^s)$.

Now, since $\mathbb{T}^s$ is a closed connected normal subgroup of $\mathbb{T}^r$, the quotient $Q:=\mathbb{T}^r/\mathbb{T}^s$ is a compact connected abelian Lie group. Hence Q is a torus of dimension $d=r-s$, and so $\pi_1(Q)\cong\mathbb{Z}^d$.

$\mathbb{T}^s\to\mathbb{T}^r\to Q$ is a fibration. Applying the long exact sequence of homotopy groups, we get

$$...\to\pi_2(Q)\to\pi_1(\mathbb{T}^s)\to\pi_1(\mathbb{T}^r)\to\pi_1(Q)\to 0.$$

Since Q is a torus, $\pi_2(Q)=0$, and we obtain the short exact sequence

$$0\to L^s\to L^r\to\pi_1(Q)\to 0.$$

Hence, L^r/L^s is isomorphic to $\mathbb{Z}^d$. Choosing representatives in L^r of its generators, we get a sublattice $L^d\subseteq L^r$ such that $L^r=L^s\times L^d$, and the Lie subalgebra $\mathfrak{t}^d:=L^d\otimes\mathbb{R}$ exponentiates to a subtorus $\mathbb{T}^d\hookrightarrow\mathbb{T}^r$ such that $\mathbb{T}^r\cong\mathbb{T}^s\times\mathbb{T}^d$. □

We now prove the "splitting" theorem for tori.

Proposition 3.4. *If G is the torus $\mathbb{T}^k$, then there is some $r\leq k$ such that $\mathbb{T}^k\cong\mathbb{T}^r\times\mathbb{T}^{k-r}$ where the induced action of $\mathbb{T}^r$ is cohomologically free and that of $\mathbb{T}^{k-r}$ is hamiltonian.*

Proof. Let $\rho:\mathbb{T}^k\to\mathrm{Symp}(M,\omega)$ be the representation $\rho(g)(x)=g\cdot x$. By the flux conjecture, proved by Ono in [10], we know that since M is closed, $\mathrm{Ham}(M,\omega)$ is a closed subgroup of $\mathrm{Symp}(M,\omega)$. Hence, $\rho^{-1}(\mathrm{Ham}(M,\omega))$ is a closed subgroup of $\mathbb{T}^k$; that is, a $(k-r)$-dimensional subtorus which we will denote $\mathbb{T}^{k-r}$. Its corresponding Lie subalgebra $\mathfrak{h}$ is the kernel of $\psi:\mathfrak{t}^k\to H^1(M;\mathbb{R})$, where ψ is as defined in the previous section. Indeed, since every element of $\mathfrak{h}$ induces a hamiltonian flow, these elements are in $\ker(\psi)$. Also, since the vector field on M induced by any element in $\mathfrak{t}^k \setminus \mathfrak{h}$ is not hamiltonian, we conclude $\mathfrak{h} = \ker(\psi)$.

Now, we apply Lemma 3.3, and obtain a complementary subtorus $\mathbb{T}^r$ such that $\mathbb{T}^k \cong \mathbb{T}^r \times \mathbb{T}^{k-r}$. The induced action of $\mathbb{T}^{k-r}$ is hamiltonian, and since the Lie algebra $\mathfrak{t}^r$ of $\mathbb{T}^r$ has zero intersection with $\mathfrak{h}$, the induced action of $\mathbb{T}^r$ is cohomologically free. □

Proof of Theorem 3.1. By Proposition 3.2, G has a finite covering $\mathbb{T}^k \times K$ where K is a compact, connected and simply connected Lie group. By Remark 2.7 and Corollary 2.8, the induced action of K on M is hamiltonian.

Next, we apply Proposition 3.4 to $\mathbb{T}^k$ and obtain a splitting $\mathbb{T}^k \cong \mathbb{T}^r \times \mathbb{T}^{k-r}$ where the induced action of $\mathbb{T}^r$ is cohomologically free, and the induced action of $\mathbb{T}^{k-r}$ is hamiltonian. Hence, define $G_0 := \mathbb{T}^{k-r} \times K$. The induced G_0-action is hamiltonian, and we are done. □

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