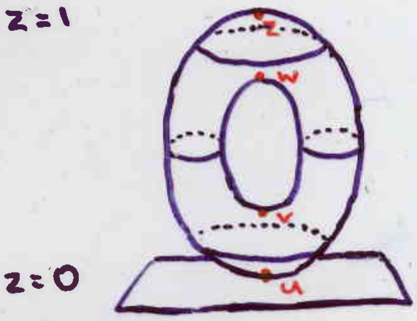
Classical Morse Theory : REVIEW

THE example : Let M be the 2-torus embedded in $\mathbb{R}^3$, laying on it's side and tangent to the two planes $z=0$ and $z=1$. Let $f: M \to \mathbb{R}$ be the function which takes a pt $p \in M$ to its z-coordinate in this embedding (that is, it's "height" function).

Let's study what this function can tell us about the topology of the manifold M.



Define the sets $M^a := \{ p \in M \mid f(p) \leq a \}$. Investigate M^a.

- $a < 0 = f(u)$ $\Rightarrow$ $M^a = \emptyset$
- $0 < a < 1/3 = f(v)$ $\Rightarrow$ M^a is a disk $\sim$ point
- $1/3 < a < 2/3 = f(w)$ $\Rightarrow$ M^a is a cylinder $\sim$ circle
- $2/3 < a < 1 = f(z)$ $\Rightarrow$ M^a is a capped torus $\sim$ figure-8
- $a > 1$ $\Rightarrow$ M^a is the complete torus M

Notice how the topology changes as you pass through a critical point (and conversely, how it doesn't when you don't!). To describe this change, look at the homotopy type of M^a.

It turns out f is a Morse function!

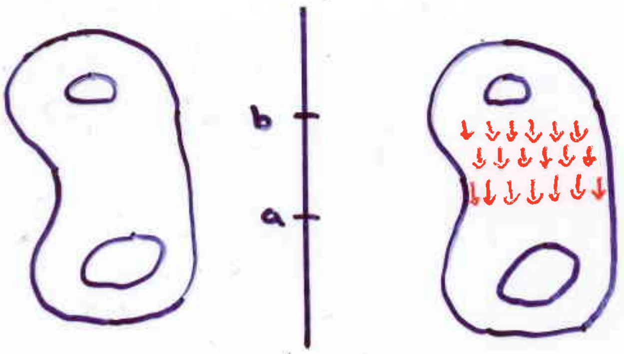
CHARACTERIZATION: $f: M \to \mathbb{R}$ is a <u>Morse function</u>
$\Leftrightarrow f \approx \sum_j \pm x_j^2$ near its critical pts.

<u>eg.</u> height fcn; M locally resembles the graph of
- $z = x^2 + y^2$ @ critical pt u
- $z = x^2 - y^2$ " " " v
- $z = y^2 - x^2$ " " " w
- $z = -x^2 - y^2$ " " " z.

The number of negative signs that appear in this local description of f is called the <u>index</u> λ of critical point p.

FAST SUMMARY:

Classical Morse theory links critical pts of a Morse fcn f with the topology of closed manifolds. More, specifically, the critical pts of f cause interuptions of the gradient flow ∇f, and the homology or even the homotopy type of a manifold M can be expressed in terms of such interuptions.



THE PLAYERS:

- a compact & connected $n = 2m$-dim symp. mfld (M, ω)
- $S^1 \circlearrowright M$ with moment map ϕ
- denote $M^{S^1} := \{$ fixed pts of S^1-action $\}$

Recall: by def of ϕ, x is a critical pt of ϕ
$\Leftrightarrow$ x is a fixed pt of the action

Assume the fixed pts are isolated.

GOAL: to understand $H^*_{S^1}(M)$

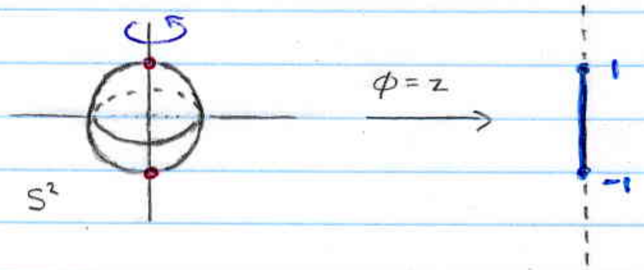
Things we know about our players (from lec.):

- ~~locally M is symplectomorphic to~~ $\cancel{\mathbb{R}^{2m} \cong \mathbb{C}^n}$
  ~~[Darboux's thm]~~

- $\cancel{S^1 \circlearrowright \mathbb{C}^n}$ ~~by~~ $\cancel{\lambda \cdot (z_1, \dots, z_n) = (\lambda z_1, \dots, \lambda z_n)}$
  ~~The moment map is~~ $\cancel{\phi(z_1, \dots, z_n) = \sum_{i=1}^{n} \frac{|z_i|^2}{2}}$

- Convexity thm: $\phi(M) \subset \mathfrak{g}^*$ is convex.
  In particular, $\phi(M) =$ convex hull of the images of (connected components) M^{S^1}.

  eg. (cover of our textbook!)
  $S^1 \circlearrowright S^2$



(will be used at various pts during presentation)

Notice ϕ is a Morse fcn !

Let $M^{\pm} := M^{\phi(p)\pm\varepsilon} = \phi^{-1}(-\infty, \phi(p)\pm\varepsilon)$ for ε sufficiently small

<u>Thm</u>: If p is a critical pt (fixed pt) for ϕ with index λ, and D^λ is the unit disk in $\mathbb{R}^\lambda$ and $S^{\lambda-1}$ the sphere of dim $\lambda-1$,

THEN $M^+ \sim M^- \cup_{S^{\lambda-1}} D^\lambda$ where the RHS is the result of gluing to M^- the D^λ disk along its boundary $S^{\lambda-1}$

[the gluing map is not specified here].

... getting back to our goal :

Equivariant Cohomology

Suppose mfld M is equipped with the action of a (compact) group.

A standard trick in algebraic topology is to replace $M \longrightarrow EG \times M$. This has a free G-action

so $\dfrac{EG \times M}{G} =: M_G$ (homotopy quotient) is a mfld.

Note:
- if G acts freely on M, then $M_G \simeq M/G$ in homotopy theory.
- in general, M_G can be regarded as a bundle over $BG := \dfrac{EG}{G}$ (classifying space) with fiber M.

Def The equivariant cohomology $H^*_G(M) := H^*(M_G)$
[Borel construction]

Example: If $G = S^1$ ($S^1 \circlearrowright S^{2n-1} \subset \mathbb{C}^n$ is free)
then $EG = S^\infty$ and $BG = \mathbb{CP}^\infty$. HOW? WHY?
We have $H^*_G(pt) = H^*(S^\infty/S^1)$

$= H^*(\mathbb{CP}^\infty)$

$= \mathbb{C}[X]$, the polynomial ring in one variable X.

The equivariant cohomology of a point is nontrivial!

Putting the Pieces Together

Recall: if $G \circlearrowright M$, bundles associated to M naturally acquire a compatible G-action.

If G acts on a bundle $\begin{smallmatrix} E \\ \downarrow \\ F \end{smallmatrix}$ and fixes F,

we get $\begin{smallmatrix} E_G \\ \downarrow \\ F_G = F \times BG \end{smallmatrix}$ to classify equivariant bundles.
Denote $\tilde{e}$ the Euler class of E_G.

Fact: Let $F \subseteq M^{S^1}$ be a connected component, λ the index. THEN $\exists$ a λ-dim bundle $\begin{smallmatrix} E \\ \downarrow \\ F \end{smallmatrix}$ s.t.

$$\begin{tikzcd}
\cdots \arrow[r, "1"] & H^*_{S^1}(M^+, M^-) \arrow[r, "2"] \arrow[d, phantom, "\cong" rotate=90] & H^*_{S^1}(M^+) \arrow[r, "3"] \arrow[d] & H^*_{S^1}(M^-) \arrow[r, "4"] & \cdots \\
& H^{*-\lambda}_{S^1}(F) \arrow[r, "\times \tilde{e}"] \arrow[d, phantom, "\cong" rotate=90] & H^*_{S^1}(F) & & \\
& H^*(F) \otimes H^*_{S^1}(pt) & & &
\end{tikzcd}$$

CLAIM: $\tilde{e}$ has no zero divisors.

This leads to a progression of natural consequences:

- $\times \tilde{e}$ is 1-1, and thus $1 = 4 = 0$, 2 is 1-1 and 3 onto.
- [Kirwan] $H^*_{S^1}(M) \to H^*_{S^1}(M^{S^1})$ is 1-1.
- $H^*_{S^1}(M) \cong H^*(M) \otimes H^*_{S^1}(pt)$ (as vector spaces).
  In fact, $H^*_{S^1}(M) \cong H^*(M) \otimes H^*(BG)$ (as vector spaces).
- $0 \to H^*(M^+, M^-) \to H^*(M^+) \to H^*(M^-) \to 0$ is exact.
  Also note $H^*(M^+, M^-) \cong H^{*-\lambda}(F)$ and $H^*(M) \cong \dfrac{H^*_{S^1}(M)}{H^*_{S^1}(pt)}$

<u>Consequence</u> :

Assume all the fixed pts are isolated. THEN $\forall p \in M^{S^1}$, $\exists$ $\alpha_p \in H^*_{S^1}(M)$ s.t.

(i) $\alpha_p|_p = \tilde{e}(E)$

and (ii) $\alpha_p|_{p'} = 0$ $\forall p'$ with $\phi(p') < \phi(p)$.

Furthermore, these α_p form a vector space basis for $H^*_{S^1}(M)$.

Example : $H^*_{S^1}(S^2) \cong H^*(S^2) \otimes \mathbb{C}[x]$

deg 0 : 1 generator

$H^0(S^2)$ generator

deg 2 : 1 generator

$H^2(S^2)$ generator