Morse Theory and Applications to Equivariant Topology

1 Morse Theory: the classical approach

Briefly, Morse theory is ubiquitous and "indomitable (Bott)". It embodies a far reaching idea: the geometry and topology of a smooth manifold are closely related to the nature of critical points of a smooth function on the manifold.

Let M be a compact n-dimensional smooth manifold (without boundary) and $f : M \to \mathbb{R}$ smooth.

- A point $p \in M$ is a *critical point* of f if there is a local coordinate system $(x_1, \dots, x_n)$ about p such that $\frac{\partial f}{\partial x_i} = 0 \ \forall i = 1, \dots, n$ i.e. $\nabla f(p) \equiv 0$ where p has local coordinates $x_1, \dots, x_n$ (assuming a Riemannian metric on M is fixed). The real number $f(p)$ is called a *critical value* of f.
- Such a critical point is *non-degenerate* iff the $n \times n$ matrix of second partials, called the *Hessian* of f at p, $H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j=1}^{n}$ is non-singular, i.e. $\det H_f(p) \neq 0$.
- The *index* λ of a non-degenerate critical point p is the number of negative eigenvalues of $H_f(p)$; it is the number of independent directions along which f will decrease from p. ($\exists$ many equivalent defs!)
- We say $f : M \to \mathbb{R}$ is a *Morse function* if every critical point of f is non-degenerate (these critical points are always isolated, as can be seen from the next basic lemma which gives a nice local description of Morse functions).

Lemma 1.1 (Morse Lemma)**.** *Let $p \in M$ be a non-degenerate critical point of $f : M \to \mathbb{R}$. Then there are local coordinates $x_1, \dots, x_n$ with $p = (0, 0, \dots, 0)$ such that $f(q) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ for every point q in a small neighbourhood U of p. λ is the index of f at p.*

Note. Given a Morse function f on M. The basic idea [more precise reformulation] of Morse theory is that the homotopy (or, in better situations, even the homeomorphism or diffeomorphism) type of the submanifold $M^a = f^{-1}(-\infty, a] = \{p \in M | f(p) \leq a\}$ changes ONLY at the critical points of f. If there is no critical value in the interval $[a, b]$ then the gradient flow, ∇f, of f provides a diffeomorphism between M^a and M^b. At a critical value a, we can suppose (after a small perturbation of f) that there is only one critical point p with $f(p) = a$. Now the **key result** is that we can get $M^{a+\epsilon}$ from $M^{a-\epsilon}$ by attaching/"gluing" a λ-disk D^λ (a cell of dimension of the index λ of f at the critical point $f(p) = a$). This disk D^λ can be constructed via the description given by the Morse Lemma, and the gluing map is between ∂D^λ and $\partial M^{a-\epsilon}$.

Theorem 1.2 (Fundamental theorem of Morse theory)**.** *Given a Morse function $f : M \to \mathbb{R}$. If p is a critical point for f with index λ and $f(p) = a$, then for all sufficiently small $\epsilon > 0$, the set $M^{a+\epsilon}$ has the homotopy type of $M^{a-\epsilon}$ with a disk $D^\lambda \subseteq \mathbb{R}^\lambda$ attached, i.e. $M^{a+\epsilon} \sim M^{a-\epsilon} \cup_{S^{\lambda-1}} D^\lambda$,where D^λ is the unit disk in $\mathbb{R}^\lambda$ and $S^{\lambda-1}$ is the sphere of $\dim \lambda - 1$.*

Notation: $M^{a-\epsilon} \cup_{S^{\lambda-1}} D^\lambda$ refers to the result of gluing to $M^{a-\epsilon}$ to the disk D^λ along its boundary $S^{\lambda-1}$ (gluing map is not specified here).

THE Example: *The upright torus.*

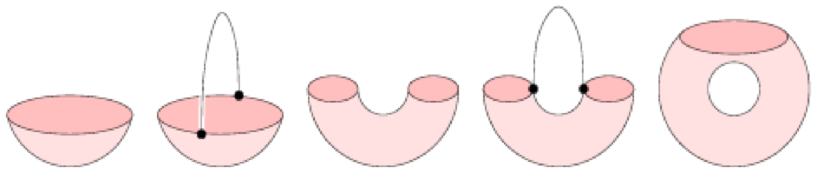

Figure 1: Going from a disk to a cylinder is homotopically the same as attaching a 1-disk, D^1. Similarly, going from the cylinder to the capped torus is homotopically the same as attaching another 1-disk, D^1.

2 Equivariant Cohomology - (mini) CRASH COURSE

It is typical in mathematics that a given object, cohomology groups for example, can be characterized or formulated in many different ways. This is true of equivariant cohomology. We will only consider Borel's topological definition.

Suppose M has a G-action, G a compact Lie group. If the G-action is free, then M/G (the space of orbits of the action) is a manifold. However, in general there are fixed points and M/G is **not** a manifold. Problem!? We will see Borel's construction replaces M by another space with the SAME homotopy type on which the G-action is always free. Thus replacing M/G by a better behaved quotient (known as the *homotopy quotient*).

Towards this end we introduce the following facts and definitions:

- There always exists a contractible space EG (not necessarily a manifold) on which G acts freely. Note that EG is unique up to G-equivariant homotopy. EG is the "platonic G-bundle".
- The *homotopy quotient* for a G-space M is $M_G := (M \times EG)/G$. M_G is well defined up to homotopy equivalence. As previously stated, $M \times EG$ has a free G-action so M_G is a manifold.
- The *classifying space* for G is $BG := EG/G$. Obviously, BG is unique up to homotopy equivalence.
- Recall that an important special case of G-spaces are principal G-bundles $P \to M$, i.e. G-spaces locally isomorphic to products $U \times G$. So a *classifying bundle** for G is a principal G-bundle $EG \to BG$, with the following universal property: For any principal G-bundle P over M, there is a map $f : M \to BG$, unique up to homotopy, such that P is isomorphic to the pull-back bundle f^*EG.

$$\begin{array}{ccc} P \cong f^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

  This map f is known as a *classifying map* of the principal bundle $P \to M$. Moreover, note that classifying bundles exist for all G, and are unique up to G-homotopy equivalence.

*To be precise, the base spaces of the principal bundles considered here must satisfy some technical condition. However, it's a basic fact that principal G-bundles with contractible total space are classifying bundles.

- The *equivariant cohomology* $H^*_G(M)$[†] for a G-space M is defined to be the ordinary cohomology ring of its homotopy quotient; $H^*_G(M) = H^*(M_G)$ [Borel construction].

Exercise: Consider the free $G = S^1$ action on $S^{2n-1} \subset \mathbb{C}^n$ where S^1 acts on each copy of $\mathbb{C}$ by the standard multiplication action. Compute EG, BG and $H^*_G(point)$.

Remark. It is not difficult to see that equivariant cohomology groups satisfy the following three properties;

1. **Normalization:** If G acts freely on M, then $H^*_G(M) \cong H^*(M/G)$.

2. **Homotopy Invariance:** If $h : M_1 \to M_2$ is a G-equivariant map inducing a homotopy equivalence, then $f^* : H^*_G(M_1) \to H^*_G(M_2)$ is an isomorphism.

3. **Mayer-Vietoris:** If $M = U \cup V$, where U and V are G-invariant open submanifolds of M, then there is a long exact sequence

$$\cdots \to H^{*-1}_G(U) \oplus H^{*-1}_G(V) \to H^{*-1}_G(U \cap V) \to H^*_G(M) \to H^*_G(U) \oplus H^*_G(V) \to H^*_G(U \cap V) \to \cdots$$

The beauty of equivariant cohomology is that it often computes something more interesting than standard cohomology, and we can often do computations simply.

3 Putting the Pieces Together

THE PLAYERS:

- We assume the $n = 2m$-dimensional symplectic manifold (M, ω) is compact and connected.
- We assume S^1 acts on M with moment map ϕ.
- If we denote the fixed point of the action by M^{S^1}, then (recall) by definition of ϕ, $x \in M$ is a critical point of $\phi \Leftrightarrow x$ is a fixed point of the action.

GOAL: Understanding $H^*_{S^1}(M)$, the equivariant cohomology of M.

We will assume from now on that the fixed points are isolated. Let $M^\pm := M^{\phi(p)\pm\epsilon} = \phi^{-1}(-\infty, \phi(p) \pm \epsilon)$ for ϵ sufficiently small.

Fact. Let $F \subseteq M^{S^1}$ be a connected component, and λ be the number of negative isotropy weights (i.e. the index). There is a λ-dimensional bundle $E \to F$ such that

$$\begin{array}{ccccccc}
\cdots \xrightarrow{①} H^*_{S^1}(M^+, M^-) & \xrightarrow{②} & H^*_{S^1}(M^+) & \xrightarrow{③} & H^*_{S^1}(M^-) & \xrightarrow{④} & \ldots \\
\wr\| & & \downarrow & & & & \\
H^{*-\lambda}_{S^1}(F) & \xrightarrow{\times \tilde{e}^{\ddagger}} & H^*_{S^1}(F) & & & & \\
\wr\| & & & & & & \\
H^*(F) \otimes H^*_{S^1}(\text{point}) & & & & & &
\end{array}$$

[†] We shall use cohomology with complex coefficients throughout.

Claim. $\widetilde{e}$ has no zero divisors, i.e. $\widetilde{e} \cdot z \neq 0$ if $z \neq 0$.

This leads to a progression of natural consequences:

- In the diagram above, $\times \widetilde{e}$ is one-to-one, and thus ①=④= 0, i.e. are 0-homomorphisms. Then by definition of exact ② is one-to-one and ③ onto.
- **[Kirwan]The restriction $H^*_{S^1}(M) \to H^*_{S^1}(M^{S^1})$ is one-to-one... (*)**
- $H^*_{S^1}(M) \cong H^*(M) \otimes H^*_{S^1}(\text{point})$ (as vector spaces, not as rings). In fact, $H^*_{S^1}(M) \cong H^*(M) \otimes H^*(BG)$ (again, not as rings).
- $O \to H(M^+, M^-) \to H^*(M^+) \to H^*(M^-) \to O$ is exact. Also note $H^*(M^+, M^-) \cong H^{*-\lambda}(F)$ (use general principles & Thm1.2 generalized to the -ve normal bundle of F) and $H^*(M) \cong H^*_{S^1}(M)/H^*_{S^1}(\text{point})$.

Consequence 1. *Assume all the fixed points are isolated. Then for every $p \in M^{S^1}$ there is a $\alpha_p \in H^*_{S^1}(M)$ such that $\alpha_{p|_p} = \widetilde{e}\ (E)$*[‡] *and $\alpha_{p|_{p'}} = 0$ for all p' with $\phi(p') < \phi(p)$. Furthermore, these α_p form a vector space basis for $H^*_{S^1}(M)$.*

This is giving us a relationship between our Morse theory and our equivariant cohomology with respect to our "players". More precisely, consequence 1 tells us that everything in the "equivariant cohomology" of the S^1 action is coming from the Euler class of the bundle E in the following way:

- each fixed point has a certain height measured by the Morse function ϕ.
- for each fixed point p, there is an equivariant cohomology class $\phi(p)$ that equals the Euler class at p but vanishes at fixed points below p.
- moreover, these generate. So consequence 1 is sort of decomposing the Euler class into summands, one for each fixed point, and doing so in order of their height on the manifold.

Is this the end of our story? Have we reached our goal? ... There is one thing we've swept under the rug here. Swept under the rug in the sense that we haven't yet paid it the attention it deserves. Namely **(*)** above.

(*) tells us that a class in equivariant cohomology is zero iff its restriction to all components of the fixed point set is zero. In other words, and equivariant cohomology class is uniquely characterized by its restriction to the components of the fixed point set.

Example: $H^*_{S^1}(S^2) \cong H^*(S^2) \otimes \mathbb{C}[X]$.

THEREFORE (the answer to our goal): we understand $H^*_{S^1}(M)$ exactly when we know $H^*(M)$ and understand the restriction of $H^*_{S^1}(M)$ to each of the connected components F of M^{S^1}.

[‡] If G acts on a bundle $E \to F$ and fixes F, we get the bundle of corresponding homotopy quotients $E_G \to F_G = F \times BG$ to classify equivariant bundles. Note: $\widetilde{e}$ is the Euler class of E_G, e is the Euler class of E.

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