

CRASH COURSE ON FLOWS

Let M be a manifold.

A **vector field** X on M is a map that associates to each point $m \in M$ a tangent vector in $T_m M$, denoted $X|_m$ or $X(m)$, that is smooth in the following sense. In local coordinates x^1, \dots, x^n , a vector field has the form $X = \sum a^j(x) \frac{\partial}{\partial x^j}$; we require that the functions $x \mapsto a^j(x)$ be smooth.

A **flow** on M is a smooth one parameter group of diffeomorphisms $\psi_t: M \rightarrow M$. This means that $\psi_0 = \text{identity}$ and $\psi_{t+s} = \psi_t \circ \psi_s$ for all t and s in \mathbb{R} (so that $t \mapsto \psi_t$ is a group homomorphism from \mathbb{R} to $\text{Diff}(M)$, the group of diffeomorphisms of M), and that $(t, m) \mapsto \psi_t(m)$ is smooth as a map from $\mathbb{R} \times M$ to M .

Its **trajectories**, (or *flow lines*, or *integral curves*) are the curves $t \mapsto \psi_t(m)$. The manifold M decomposes into a disjoint union of trajectories. Moreover, if $\gamma_1(t)$ and $\gamma_2(t)$ are trajectories that both pass through a point p , then there exists an s such that $\gamma_2(t) = \gamma_1(t+s)$ for all $t \in \mathbb{R}$. Hence, the velocity vectors of γ_1 and γ_2 at p coincide.

Its **velocity field** is the vector field X that is tangent to the trajectories at all points. That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t_0 , which is a tangent vector to M at the point $p = \psi_{t_0}(m)$, is the vector $X(p)$. We express this as

$$\frac{d}{dt} \psi_t = X \circ \psi_t.$$

Conversely, any vector field X on M generates a *local flow*. This means the following. Let X be a vector field. Then there exists an open subset $A \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map $\psi: A \subset \mathbb{R} \times M$ such that the following holds. Write $A = \{(t, x) \mid a_x < t < b_x\}$ and $\psi_t(x) = \psi(t, x)$.

- (1) $\psi_0 = \text{identity}$.
- (2) $\frac{d}{dt} \psi_t = X \circ \psi_t$.
- (3) For each $x \in M$, if $\gamma: (a, b) \rightarrow M$ satisfies the differential equation $\dot{\gamma}(t) = X(\gamma(t))$ with initial condition $\gamma(0) = x$, then $(a, b) \subset (a_x, b_x)$ and $\gamma(t) = \psi_t(x)$ for all t .

Moreover, $\psi_{t+s}(x) = \psi_t(\psi_s(x))$ whenever these are defined. Finally, if X is compactly supported, then $A = \mathbb{R} \times M$, so that X generates a (globally defined) flow. Good references are chapter 8 of “Introduction to differential topology” by Bröcker and Jänich and chapter 5 of “A comprehensive introduction to differential geometry”, volume I, by Michael Spivak.

A *time dependent vector field* parametrized by the interval $[0, 1]$ is a family of vector fields X_t , for $t \in [0, 1]$, that is smooth in the following sense. In local coordinates it has the form $X_t = \sum a^j(t, x) \frac{\partial}{\partial x^j}$; we require a^j to be smooth functions of (t, x^1, \dots, x^n) .

An *isotopy* (or *time dependent flow*) of M is a family of diffeomorphisms $\psi_t: M \rightarrow M$, for $t \in [0, 1]$, such that $\psi_0 = \text{identity}$ and $(t, m) \mapsto \psi_t(m)$ is smooth as a map from $[0, 1] \times M$ to M .

An isotopy ψ_t determines a unique time dependent vector field X_t such that

$$(1) \quad \frac{d}{dt} \psi_t = X_t \circ \psi_t.$$

That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t , which is a tangent vector to M at the point $p = \psi_t(m)$, is the vector $X_t(p)$

A time dependent vector field X_t on M determines a vector field \tilde{X} on $[0, 1] \times M$ by $\tilde{X}(t, m) = \frac{\partial}{\partial t} \oplus X_t(m)$. In this way one can treat time dependent vector fields and flows through ordinary vector fields and flows.

In particular, a time dependent vector field X_t , $t \in [0, 1]$, generates a “local isotopy” $\psi_t(x) = \psi(t, x)$. If X_t is compactly supported then $\psi_t(x)$ is defined for all $(t, x) \in [0, 1] \times M$. If $X_t(m) = 0$ for all $t \in [0, 1]$ then there exists an open neighborhood U of m such that $\psi_t: U \rightarrow M$ is defined for all $t \in [0, 1]$.

The *Lie derivative* of a k -form α in the direction of a vector field X is

$$L_X \alpha = \left. \frac{d}{dt} \right|_{t=0} \psi_t^* \alpha$$

where ψ_t is the flow generated by X .

We have

$$L_v(\alpha \wedge \beta) = (L_v \alpha) \wedge \beta + \alpha \wedge (L_v \beta)$$

and

$$L_v(d\alpha) = d(L_v \alpha).$$

These follow from $\psi^*(\alpha \wedge \beta) = \psi^* \alpha \wedge \psi^* \beta$ and $\psi^* d\alpha = d\psi^* \alpha$.

Cartan formula:

$$L_v \alpha = \iota_v d\alpha + d\iota_v \alpha$$

where $\iota_v: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is

$$(\iota_v \alpha)(u_1, \dots, u_{k-1}) = \alpha(v, u_1, \dots, u_{k-1}).$$

(Outline of proof: it is true for functions. If it is true for α and β then it is true for $\alpha \wedge \beta$ and for $d\alpha$.)

Let α_t be a time dependent k -form and X_t a time dependent vector field that generates an isotopy ψ_t . Then

$$\left. \frac{d}{dt} \right|_{t=0} \psi_t^* \alpha_t = \psi_t^* \left(\left. \frac{d\alpha_t}{dt} \right|_{t=0} + L_{X_t} \alpha_t \right).$$

Outline of proof: if it is true for α and for β then it is true for $\alpha \wedge \beta$ and for $d\alpha$. Hence, it is enough to prove it for functions.

The left hand side, applied to a time dependent function f_t and evaluated at $m \in M$, is the limit as $t \rightarrow t_0$ of the difference quotient

$$\frac{f_t(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

This difference quotient is equal to

$$\left(\frac{f_t - f_{t_0}}{t - t_0} \right) (\psi_t(m)) + \frac{f_{t_0}(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

The limit as $t \rightarrow t_0$ of the first summand is

$$\left. \frac{df_t}{dt} \right|_{t=t_0} (\psi_{t_0}(m)) = \left(\psi_{t_0}^* \left. \frac{df_t}{dt} \right|_{t=t_0} \right) (m).$$

The limit as $t \rightarrow t_0$ of the second summand is the derivative of f_{t_0} along the tangent vector

$$\left. \frac{d}{dt} \right|_{t=t_0} \psi_t(m) = X_{t_0}(\psi_{t_0}(m));$$

this derivative is

$$(X_{t_0} f_{t_0})(\psi_{t_0}(m)) = (\psi_{t_0}^*(L_{X_{t_0}} f_{t_0}))(m).$$