Non squeezing for ellipsoids/Leonid Shartser

Let $(V,\omega) = (\mathbb{R}^{2n},\omega)$ be the standard symplectic vector space with

$$\omega(x,y) =< Jx, y >,$$
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $Sp(n) = \{\phi : V \to V | \phi \text{ linear}, \phi^*\omega = \omega\}$ be the linear symplectic group. In this paper we treat the following question:
Let E_1 and E_2 be ellipsoids in V. When is it possible to "symplecticly" embed E_1 into E_2. i.e. when there exists a symplectic map $\phi : E_1 \hookrightarrow E_2$.
To give an invariant answer to this question we need some definitions. Let q be a quadratic positive definite form on V, that is $\exists S = S^T > 0$ such that $q(x) = \frac{1}{2} < Sx, x >$. Write $\tilde{q}(x,y) :=< Sx, y >$.An ellipsoid is a set of the form $E(q) = \{x|q(x) < 1\}$. Observe that we have a bijective correspondence between ellipsoids and quadratic positive definite forms. It can be expressed as $E(q_1) = E(q_2) \Leftrightarrow q_1 = q_2$. To see that we prove the following lemma.
Lemma 1: $E(q_1) \subset E(q_2) \Leftrightarrow q_2 \leq q_1$.
Proof:
$\Rightarrow$: Suppose $\exists x$ such that $q_1(x) < q_2(x)$. Find $k > 0$ such that $k^2q_1(x) = q_1(kx) < 1$ and therefore $q_2(kx) < 1$. Replacing kx with x denote $a = q_1(x), b = q_2(x)$. Pick $\frac{1}{\sqrt{b}} < r < \frac{1}{\sqrt{a}}$. Observe, $q_1(rx) = r^2q_1(x) = r^2a < 1$ but $q_2(rx) = r^2q_2(x) > 1$ that is $\Rightarrow\Leftarrow$.
$\Leftarrow$: Let $x \in E(q_1)$ then $1 > q_1(x) \geq q_2(x)$ so $x \in E(q_2)$.$\square$
Due to this correspondence we can associate to every ellipsoid $E(q)$, n positive numbers $\lambda(q) = (0 < \lambda_n \leq ... \leq \lambda_1)$. $\mp i\lambda_j$ are the eigenvalues of $X_q = -JS$.
Remark 1: $X_q(x)$ satisfies the Hamiltonian equation $d\omega(X_q(x), y) = -dq(x)(y)$ as can be easily verified.
Remark 2: $\lambda(q)$ is invariant under symplectic change of coordinates: Let $\tau \in Sp(n)$. We have then, $q \circ \tau(x) = \frac{1}{2} < S\tau x, \tau x >=< \tau^T S\tau, x >$. Note that τ is symplectic, therefore

$$< Jx, y >= \omega(x,y) = \tau^*\omega(x,y) = \omega(\tau x, \tau y) =< J\tau x, \tau y >=< \tau^T J\tau x, y > .$$

Hence $J = \tau^T J\tau$ multiplying by J both sides we get $I = J\tau^T J\tau$. So we have, $X_{q\circ\tau} = J\tau^T S\tau = (J\tau)^{-1}S\tau = \tau^{-1}JS\tau = \tau^{-1}X_q\tau$. Hence, $\lambda(q) = \lambda(q \circ \tau)$.
Now we are ready to answer our question in terms of this invariant $\lambda(q)$.

Theorem 1: Let $E(q_1)$ and $E(q_2)$ be ellipsoids, then $\exists\phi \in Sp(n), \phi(E_1) \subset E_2 \Leftrightarrow \lambda_j(q_1) \geq \lambda_j(q_2)$.
At this point we only give a short sketch of the proof whereas the details will be filled later on. First suppose that $\lambda_j(q_1) \geq \lambda_j(q_2)$, and find $\phi \in Sp(n)$ such that $\phi(E_1) \subset E_2$. Let's assume for a moment that we can find symplectic maps ψ_1 and ψ_2 such that:
$q_1 \circ \psi_1 = \frac{1}{2}\sum_{j=1}^n \lambda_j(q_1)(x_j^2 + x_{n+j}^2)$ and $q_2 \circ \psi_2 = \frac{1}{2}\sum_{j=1}^n \lambda_j(q_2)(x_j^2 + x_{n+j}^2)$
Then it is clear that $q_1\circ\psi_1 \geq q_2\circ\psi_2$ and thus by the previous lemma $E(q_1\circ\psi_1) \subset E(q_2 \circ \psi_2)$. Therefore, $\phi := \psi_2 \circ \psi_1^{-1}$ maps $E(q_1)$ into $E(q_2)$. Since if $x \in E(q_1)$ then $\psi_1^{-1}(x) \in E(q_1\circ\psi_1) \subset E(q_2\circ\psi_2)$ so $q_2(\psi_2(\psi_1^{-1}(x))) = q_2\circ\psi_2(\psi_1^{-1}(x)) < 1$. Therefore $x \in E(q_2)$. The fact that we can find ψ_1 and ψ_2 is the content of theorem 2.

Conversely, suppose we have $\phi \in Sp(n)$ such that $\phi(E_1) \subset E_2$ and we want to prove that $\lambda_j(q_1) \geq \lambda_j(q_2)$. Since $\phi(E(q_1)) = E(q_1 \circ \phi^{-1}) \subset E(q_2)$, we may assume w.l.o.g. that $E(q_1) \subset E(q_2)$ and thus by the lemma, $q_1 \geq q_2$. So the result we need to complete this proof is $q_1 \geq q_2 \Rightarrow \lambda_j(q_1) \geq \lambda_j(q_2)$. This result is lemma 2.
Theorem 2: If q be positive definite quadratic form as above, then $\exists \phi \in Sp(n)$ such that $q \circ \phi = \frac{1}{2}\sum_{j=1}^{n} \lambda_j(q)(x_j^2 + x_{n+j}^2)$.
Proof:
Define positive definite quadratic form K on $V \times V$, $K(x,y) = K_q(x,y) = q(x) + q(y)$. This quadratic form induces an inner product on $V \times V$ defined by the formula $<< (x,y),(x',y') >>= K(x+x', y+y') - K(x,y) - K(x',y')$. Denote $||x||^2 :=<< x,x >>= K(x)$ and $|x|^2 =< x,x >$. Observe that $\omega(x,y)$ is a quadratic form on $V \times V$ and hence, there is $4n \times 4n$ symmetric matrix Ω such that $\omega(x,y) =<< \Omega(x,y),(x,y) >>$.
Consider the following extremum problem with constrains:

$$(*) \max_{||(x,y)||=1} << \Omega(x,y),(x,y) >>$$

Let (a,b) be a such that $||(a,b)|| = 1$ and

$$\omega(a,b) = \max_{||(x,y)||=1} \omega(x,y) = \max_{||(x,y)||=1} << \Omega(x,y),(x,y) >> .$$

Note: The critical set of ω on $||.|| = 1$ lies in a circle of critical points:
Let $x(t) = cos(t)x + sin(t)y$, $y(t) = cos(t)y - sin(t)x$ then, it can be easily verified that, $K(x(t),y(t)) = K(x,y)$ and similarly we get $\omega(x(t),y(t)) = \omega(x,y)$.

Solve the extremum problem (*) above using Lagrange multipliers.Let $\lambda \in \mathbb{R}$ be the Lagrange multiplier of the critical point (a,b).

$$\nabla\omega(x,y) = (-Jy; Jx)$$

(vector of size 4n)

$$\nabla K(x,y) = (\tilde{q}(x,e_1),...,\tilde{q}(x,e_n),\tilde{q}(y,e_1),...,\tilde{q}(y,e_n))$$

So we have, $\nabla K(a,b) = \lambda\nabla\omega(a,b)$. That gives us:

$$(**)\tilde{q}(a,x) = \lambda\omega(x,b); \tilde{q}(a,x) = \lambda\omega(x,b)$$

And from that follows by computation that $\omega(a,b) = \frac{1}{\lambda} > 0$. The conclusion from here is that a,b are linearly independent in V (since ω is antisymmetric). $\omega(a,b) \neq 0$ so $V_1 = sp\{a,b\} \subset V$ is symplectic vector space. And it can be easily checked that V_1 is the eigen space of X_q corresponding to the eigen values $\mp i\lambda$.Once again simple computation shows that Lagrange multipliers equations (**) are equivalent to

$$X_q(a) = \lambda b, X_q(b) = -\lambda a.$$

So $X_q(a \pm ib) = \mp(i\lambda)(a \pm ib)$. Introduce $e_1 = \alpha a, f_1 = -\alpha b$ a symplectic basis with $\alpha^2 = \lambda$ so $\omega(e_1,f_1) = 1$ and $\tilde{q}(e_1,f_1) = 0, q(e_1) = q(f_1) = \frac{\lambda}{2}$. To see that use $X_q = -JS$ and $\omega(a,b) = \frac{1}{\lambda}$. Now, using the fact that $< Sx,y >= q(x+y) - q(x) - q(y)$ we get

$$q(\xi e_1 + \eta f_1) = \frac{\lambda}{2}(\xi^2 + \eta^2).$$

That is the desired result in V_1.
We may apply the same argument to $V_1^\perp = \{v \in V | \omega(v_1, v) = 0 \; \forall v_1 \in V\}$, Since $V = V_1 \oplus V_1^\perp$, and moreover, $\forall x \in V_1, y \in V^\perp \; \tilde{q}(x, y) = 0$ due to (**). So the result follows.
Observe that (a, b) is eigen vector of Ω (see for a proof of spectral theorem for selfadjoint operators) and moreover,
$<< (a, b), (v, w) >>= 0 \; \forall (v, w) \in V_1^\perp \times V_1^\perp$:
$<< (a, b), (v, w) >>= K(a + v, w + b) - K(a, b) - K(v, w) =$
$= q(a + v) + q(w + b) - q(a) - q(b) - q(v) - q(w) = 0$ since $a \in V_1, b \in V_1^\perp$.

Lemma 2: If q_1 and q_2 be quadratic positive definite forms then $q_1 \geq q_2 \Rightarrow \lambda_j(q_1) \geq \lambda_j(q_2)$.
Proof:
Recall from the previous proof that

$$\frac{1}{\lambda_n} = \max_{K_{q_1}=1} \omega = \omega(a, b)$$

$q_2(x) \leq q_1(x)$ for all x so find $k \geq 1$ such that $1 = K_{q_2}(ka, kb)$ so $\omega(ka, kb) = k^2\omega(a, b) \geq \omega(a, b)$. That implies $\frac{1}{\lambda_n(q_2)} \geq \frac{1}{\lambda_n(q_1)}$. To get the other inequalities recall that $\frac{1}{\lambda_j}$ are eigen values of the matrix Ω so we can use the minimax principle to get

$$\frac{1}{\lambda_j} = \min_{W \in F_j} \max_{W \cap K_q = 1} \omega$$

F_j denotes the family of subspaces of $V \times V$ of $dim = 2n + 2j$. Note: The circle action was taken into account. If (a, b) is critical point then $(-b, a)$ is also a critical point so each eigen value has at least multiplicity 2.