

Deformation Quantization of Poisson Structure on R^d

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Let M be a finite dimensional C^∞ manifold and \mathcal{A} be the algebra of smooth functions (over \mathbb{R}) on M . In classical mechanics, usually M is a symplectic manifold or more generally, Poisson Manifold. \mathcal{A} is the set of measurable quantities, called observables. Very naively, we have \mathcal{A} is a commutative algebra, i.e. $fg = gf$, for $f, g \in \mathcal{A}$. In quantum mechanics, we promote observables to operators which do not commute in general.

Here, we take a different approach. We deform \mathcal{A} , in the sense that we introduce a formal parameter \hbar and the *star product* $f *_\hbar g = fg + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \dots \in \mathcal{A}[[\hbar]]$ as a power series of \hbar , where $*$ is associative and $B_i : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ are bidifferential operators with the following properties: $B_i(f, g) = (-1)^i B_i(g, f)$ and $B_i(f, 1) = 0$, for $i \geq 1$. Furthermore, the product of $f, g \in \mathcal{A}[[\hbar]]$ is defined as

$$f(\hbar) *_\hbar g(\hbar) = \left(\sum_{n \geq 0} f_n \hbar^n \right) * \left(\sum_{n \geq 0} g_n \hbar^n \right) = \sum_{k, l \geq 0} f_k g_l \hbar^{k+l} + \sum_{m \geq 1, k, l \geq 0} B_m(f_k, g_l) \hbar^{m+k+l}$$

The moral is that B_i measures the i -th non-commutativity.

Remark: $\{f, g\} = \frac{f * g - g * f}{\hbar}$ defines a poisson bracket, since $\forall f, g, h \in \mathcal{A}$,

$$\begin{aligned} & \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \\ &= \frac{1}{\hbar^2} (f * (g * h) - (g * h) * f - f * (h * g) + (h * g) * f) \\ & \quad + \frac{1}{\hbar^2} (g * (h * f) - (h * f) * g - g * (f * h) + (f * h) * g) \\ & \quad + \frac{1}{\hbar^2} (h * (f * g) - (f * g) * h - h * (g * f) + (g * f) * h) \\ &= \frac{1}{\hbar^2} (f * (g * h) - (f * g) * h) + \dots = 0 \end{aligned}$$

Now, consider $D = D(\hbar) \in \text{Aut}(\mathbf{R}[[\hbar]])$ such that

$$D : f \mapsto f + \sum_{n \geq 1} \hbar^n D_n(f)$$

and

$$D : \sum_{n \geq 0} f_n \hbar^n \mapsto \sum_{n \geq 0} f_n \hbar^n + \sum_{m \geq 1, n \geq 0} D_m(f_n) \hbar^{n+m}$$

All such D form a gauge group on $*$, in sense that

$$\begin{aligned} D & : & * & \mapsto *' \\ f(\hbar) *_\hbar g(\hbar) & \mapsto & f(\hbar) *' g(\hbar) & = D(\hbar)(D(\hbar)^{-1}(f(\hbar)) * D(\hbar)^{-1}(g(\hbar))) \end{aligned}$$

Now, the equivalence classes of star-product modulo $O(\hbar^2)$ is classified by the Poisson bracket defined above. The converse is also true, though highly non-trivial, i.e., given a Poisson manifold, there is a $*$ such that $\frac{f*g-g*f}{\hbar}$ is the Poisson structure. More precisely, we have

Theorem 1 (*Kontsevich 97'*) *The set of gauge equivalence classes of star products on a smooth manifold M can be naturally identified with the set of equivalence classes of Poisson structures depending formally on \hbar :*

$\alpha = \alpha(\hbar) = \alpha_1\hbar + \alpha_2\hbar^2 + \dots \in \Gamma(M, \wedge^2 TM)[[\hbar]], [\alpha, \alpha] = 0 \in \Gamma(M, \wedge^3 TM)[[\hbar]]$
modulo the action of the group of formal paths in $\mathbf{Diff}(M)$.

Apart from Kontsevich, De Wilde & Lecomte (83') and B. Fedosov (85' in Russian, 94') proved existence of star-product on symplectic manifolds. Fedosov's construction also extends to regular Poisson manifolds. Later, others (Nest–Tsygan, Deligne, Bertelson–Cahen-Gutt, Xu etc.) using Fedosov's construction have classified all star-products for a given symplectic manifold. Etingof and Kazhdan (96') proved existence for Poisson Lie group. However, Kontsevich's 97' paper using Hochschild complex is the first attempt towards the case Poisson manifolds. In this summary, I will only attempt to construct an explicit formula for Poisson structure on \mathbb{R}^d , which is a special case.

As a warm-up, I will construct the Moyal-Weyl product for a constant Poisson structure on \mathbb{R}^d : $\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j, \alpha^{ij} = -\alpha^{ji} \in \mathbb{R}$, then Moyal product is

$$\begin{aligned} f * g &= fg + \hbar \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \alpha^{ij} \alpha^{kl} \partial_i \partial_k(f) \partial_j \partial_l(g) + \dots \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1, \dots, i_n, j_1, \dots, j_n} \prod_{k=1}^n \alpha^{i_k j_k} \left(\prod_{k=1}^n \partial_{i_k} \right) (f) \left(\prod_{k=1}^n \partial_{j_k} \right) (g) \end{aligned}$$

Now if α is not constant, the formula becomes somewhat more complicated. To properly construct the star-product, we use the concept of graph as a tool. We call $\Gamma = (V_\Gamma, E_\Gamma)$, where $E_\Gamma \subseteq V_\Gamma \times V_\Gamma$, an (oriented) graph. We will make use of a special class of graphs, G_n .

- $\Gamma \in G_n$, if 1) Γ has $n + 2$ vertices and $2n$ edges,
2) $V_\Gamma = \{1, \dots, n, L, R\}$. There are exactly two edges starting from each of $\{1, \dots, n\}$, labelled as e_j^1, e_j^2 , for $j = 1, \dots, n$.
3) There is no edge of the form (v, v) .

Clearly, $|G_n| = (n(n+1))^n$. Now for each Γ , we associate an operator $B_{\Gamma, \alpha}$

$$\begin{aligned} B_{\Gamma, \alpha}(f, g) &= \sum_{I: E_\Gamma \rightarrow \{1, \dots, n\}} \left[\prod_{k=1}^n \left(\prod_{e \in E_\Gamma, e=(*,k)} \partial_{I(e)} \right) \alpha^{I(e_k^1) I(e_k^2)} \right] \\ &\times \left(\prod_{e \in E_\Gamma, e=(*,L)} \partial_{I(e)} \right) (f) \times \left(\prod_{e \in E_\Gamma, e=(*,R)} \partial_{I(e)} \right) (g) \end{aligned}$$

Next, we assign a weight function w_Γ to each graph Γ .

Denote $\phi(p, q) = \text{Arg}((q - p)/(q - \bar{p}))$, for $p, q \in \mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. By analytical continuity, we can extend $\phi(p, q)$ to $\mathcal{H} \cup \mathbb{R}$. Let $L = 0, R = 1$. For $e = (p, q)$, let $\phi_e = \phi(p, q)$. Denote $\mathcal{H}_n = \{(p_1, \dots, p_n) \mid p_k \in \mathcal{H}, k \neq l \Rightarrow p_k \neq p_l\}$. w_Γ is defined as follows:

$$w_\Gamma = \frac{1}{n!(2\pi)^{2n}} \int_{\mathcal{H}_n} \bigwedge_{k=1}^n (d\phi_{e_k^1} \wedge d\phi_{e_k^2})$$

Here, we have an explicit formula for the star-product.

Theorem 2 *Let α be a Poisson vector field in a domain of \mathbb{R}^d .*

$$f * g = \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma, \alpha}(f, g)$$

defines a star-product which is gauge equivalent if we change coordinates.

Bibliography

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