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CRASH COURSE ON FLOWS

Let M be a manifold.

A **vector field** X on M is a smooth section of its tangent bundle TM, that is, a smooth map $M \to TM$ whose composition with the projection map $TM \to M$ is the identity map. We denote the value of X at m sometimes by $X(m)$ and sometimes by $X|_m$. So $X|_m \in T_m M$ for each $m \in M$. In local coordinates $x^1, \ldots, x^n$, a vector field has the form $X = \sum a^j(x) \frac{\partial}{\partial x^j}$ where a^j are smooth functions.

A **flow** on M is a smooth one parameter group of diffeomorphisms $\psi_t \colon M \to M$, that is, $(t, m) \mapsto \psi_t(m)$ is smooth as a map from $\mathbb{R} \times M$ to M, and we have ψ_0 =identity and $\psi_{t+s} = \psi_t \circ \psi_s$ for all t and s. (That is, $t \mapsto \psi_t$ is a group homomorphism from $\mathbb{R}$ to $\mathrm{Diff}(M)$, the group of diffeomorphisms of M.)

Its **trajectories**, (or *flow lines*, or *integral curves*) are the curves $t \mapsto \psi_t(m)$. The manifold M decomposes into a disjoint union of trajectories.

Its **velocity field** is the vector field defined by $X(m) = \frac{d}{dt}\big|_{t=0} \psi_t(m)$. The property $\psi_{t+s} = \psi_t \circ \psi_s$ implies that X is tangent to the trajectories at *all* points. That is, the velocity vector of the curve $\gamma(t) = \psi_t(m)$ at time t_0, which is a tangent vector to M at the point $p = \gamma(t_0)$, is the vector $\dot{\gamma}(t_0) = X(p)$. In other words,

$$\frac{d}{dt}\psi_t = X \circ \psi_t$$

for all t.

Conversely, any vector field X on M generates a *local flow*. This means the following. Let X be a vector field. Then there exists an open subset $A \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map $\psi \colon A \subset \mathbb{R} \times M$ such that the following holds. Write $A = \{(t, x) \mid a_x < t < b_x\}$ and $\psi_t(x) = \psi(t, x)$.

(1) ψ_0 =identity.
(2) $\frac{d}{dt}\psi_t = X \circ \psi_t$.
(3) For each $x \in M$, if $\gamma \colon (a, b) \to M$ satisfies the differential equation $\dot{\gamma}(t) = X(\gamma(t))$ with initial condition $\gamma(0) = x$, then $(a, b) \subset (a_x, b_x)$ and $\gamma(t) = \psi_t(x)$ for all t.

Moreover, $\psi_{t+s}(x) = \psi_t(\psi_s(x))$ whenever these are defined. Finally, if X is compactly supported then $A = \mathbb{R} \times M$, so that X generates a (globally defined) flow. A good reference is chapter 8 of "Introduction to differential topology" by Bröcker and Jänich.

A *time dependent vector field* parametrized by the interval $[0,1]$ is a family of vector fields X_t, for $t \in [0,1]$, such that $(t,m) \mapsto X_t(m)$ is smooth as a map from $[0,1] \times M$ to TM. In local coordinates it has the form $X_t = \sum a^j(t,x)\frac{\partial}{\partial x^j}$ where a^j are smooth functions of $(t, x^1, \ldots, x^n)$.

An *isotopy* (or *time dependent flow*) of M is a family of diffeomorphisms $\psi_t \colon M \to M$, for $t \in [0,1]$, such that ψ_0 =identity and $(t,m) \mapsto \psi_t(m)$ is smooth as a map from $[0,1] \times M$ to M.

An isotopy ψ_t determines a unique time dependent vector field X_t such that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t. \tag{1}$$

That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t, which is a tangent vector to M at the point $p = \psi_t(m)$, is the vector $\dot\gamma(t) = X_t(p)$

Conversely, any time dependent vector field X_t, $t \in [0,1]$, generates a "local isotopy" $\psi(t,x)$. If X_t is compactly supported then $\psi(t,x) = \psi_t(x)$ is defined for all $(t,x) \in [0,1] \times M$. If $X_t(m) = 0$ for all t then there exists an open neighborhood U of m such that $\psi_t \colon U \to M$ is defined for all t.

Note that a time dependent vector field X_t on M determines a vector field $\tilde X$ on $[0,1] \times M$ by $\tilde X(t,m) = \frac{\partial}{\partial t} \oplus X_t(m)$. In this way one can treat time dependent vector fields and flow through ordinary vector fields and flows.

The *Lie derivative* of a k-form α in the direction of a vector field X is

$$L_X\alpha = \left.\frac{d}{dt}\right|_{t=0} \psi_t^*\alpha$$

where ψ_t is the flow generated by X.

We have

$$L_v(\alpha \wedge \beta) = (L_v\alpha) \wedge \beta + \alpha \wedge (L_v\beta)$$

and

$$L_v(d\alpha) = d(L_v\alpha).$$

These follow from $\psi^*(\alpha \wedge \beta) = \psi^*\alpha \wedge \psi^*\beta$ and $\psi^* d\alpha = d\psi^*\alpha$.

Cartan formula:

$$L_v\alpha = \iota_v d\alpha + d\iota_v\alpha$$

where $\iota_v \colon \Omega^k(M) \to \Omega^{k-1}(M)$ is

$$(\iota_v\alpha)(u_1, \ldots, u_{k-1}) = \alpha(v, u_1, \ldots, u_{k-1}).$$

(Outline of proof: it is true for functions. If it is true for α and β then it is true for $\alpha \wedge \beta$ and for $d\alpha$.)

Let α_t be a time dependent k-form and X_t a time dependent vector field that generates an isotopy ψ_t. Then

$$\frac{d}{dt}\psi_t^*\alpha_t = \psi_t^*(\frac{d\alpha_t}{dt} + L_{X_t}\alpha_t).$$

Outline of proof: if it is true for α and for β then it is true for $\alpha \wedge \beta$ and for $d\alpha$. Hence, it is enough to prove it for functions.

$$\frac{\psi_t^* f_t - \psi_{t_0}^* f_{t_0}}{t - t_0} = \psi_t^* \left(\frac{f_t - f_{t_0}}{t - t_0} \right) + \frac{\psi_t^* f_{t_0} - \psi_{t_0}^* f_{t_0}}{t - t_0}.$$

The limit as $t \to t_0$ of the first summand is $\psi_{t_0}^* \left.\frac{df_t}{dt}\right|_{t=t_0}$.

Write $\varphi = f_{t_0}$. The second summand, evaluated at $m \in M$, is

$$\frac{\varphi(\psi_t(m)) - \varphi(\psi_{t_0}(m))}{t - t_0}.$$

Its limit as $t \to t_0$ is equal to $v\varphi$ where v is the tangent vector to the curve $t \mapsto \psi_t(m)$ at time $t = t_0$. This tangent vector equals the value of the vector field X_{t_0} at the point $p = \psi_{t_0}(m)$. So this limit is $(X_{t_0}\varphi)(p) = (L_{X_{t_0}}\varphi)(\psi_{t_0}(m)) = (\psi_{t_0}^*(L_{X_{t_0}}\varphi))(m)$.