As Easy As π

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Now I, even I, would celebrate
In rhymes unapt, the great
Immortal Syracusan, rivaled nevermore,
Who in his wondrous love,
Passed on before,
Left men his guidance
How to circles mensurate.
A.C. Orr

I leave it as an exercise for you to figure out why I included this poem about Archimedes. Perhaps I should leave a clue somewhere in this article, eh?

Archimedes, the greatest mathematician of antiquity, in his *On the Measurement of the Circle*, gave upper and lower bounds on π by inscribing and circumscribing polygons in a circle and then computing the perimeters of those polygons. The computations were quite complicated as lots of square roots were involved, so his method could not be extended indefinitely.

Many centuries later, but still quite some time ago, while mathematicians were tediously trying to calculate π's exact value, Johann Heinrich Lambert (1728-1777) must have ticked off a few people after discovery of the following theorem.

Theorem 1 *π is irrational.*

Poof [1]**.** Consider $\tan x$, where x is a nonzero rational number.

$$\{x : x \in \mathbb{Q}, x \neq 0\} \Rightarrow \tan x \notin \mathbb{Q} \quad \text{(Think about that!)}$$
$$\tan\frac{\pi}{4} = 1$$
$$\tan\frac{\pi}{4} \in \mathbb{Q} \Rightarrow \frac{\pi}{4} \notin Q$$
$$\Rightarrow \pi \notin \mathbb{Q}. \qquad \text{QED}$$

Whoa, you say! OK, so I had to compress the poof into four steps – the editor would have accused me of monopolizing this book with a complete proof. Therefore I must be cruel and (sadly enough)

[1] The word "poof" originated at Toronto and the word has several uses. A poof is (1) a proof that sneaks up on you and hits you like an uncountable number of bricks; then gets erased off the board before you absorb it, (2) a highly improbable construction, usually non-constructive, that produces the result by pulling a rabbit out of a hat, or (3) something which students supply, especially on exams, when asked to give a proof; such students do not usually continue in mathematics.

leave it as a real hard exercise for you. (Hint: $\tan x$ can be written as a continued fraction. 'Nuff said!)

Before Lambert had successfully proven π's irrationality, π had only been calculated to 112 decimal places. In 1844, Johann Martin Zacharias Dase spent two months calculating π *in his brain* and correctly computed it to 200 decimal places. In 1987, Hideaki Tomoyori *memorized* 40,000 digits of π, reciting it in only 17 hours (of course he got his name in the *Guiness Book of Records* for it). Today, 206,158,430,000 digits of π are known, thanks to Yasumasa Kanada and Daisuke Takahashi of the University of Tokyo. For those who don't remember what π is, it is the ratio of a circle's circumference to its diameter, approximately

3.14159265358979323846264338327950288419716939937510582097494459230781
6406286208998628034825342117067982148086513282306647093844609550582231
7253594081284811174502841027019385211055596446229489549303819644288109
7566593344612847564823378678316527120190914564856692346034861045432664
8213393607260249141273724587006606315588174881520920962829254091715364
3678925903600113305305488204665213841469519415116094330572703657595919
5309218611738193261179310511854807446237996274956735188575272489122793
8183011949129833673362440656643086021394946395224737190702179860943702
7705392171762931767523846748184676694051320005681271452635608277857713
427577896091 (No, I didn't randomly type the last 300 digits...)

Are the digits of π distributed randomly? Probably, but it has not been proven yet; which is one reason why mathematicians calculate π to such great lengths. The table below shows the distribution of digits for the first 200 billion digits of π beyond the decimal point, and the data suggests that perhaps the digits are distributed randomly.

Digit	Frequency	Digit	Frequency
0	20000030841	5	19999914711
1	19999914711	6	19999881515
2	20000136978	7	19999967594
3	20000069393	8	20000291044
4	19999921691	9	19999869180

Table 1: Frequency distributions for the first 200 billion digits of $\pi - 3$.
(Source: Yasumasa Kanada, University of Tokyo)

Interestingly enough, the sequence '0123456789' occurs in the decimal expansion of π a total of six times in the first 50 billion digits, while the sequence '9876543210' occurs five times, and the sequence '27182818284' occurs at the 45,111,908,393rd decimal place! Let me now show you some of the hideous methods used to calculate π...

From the beginning of the 18th century, π was calculated with John Machin's formula – which can be found in almost any elementary calculus textbook:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

Since Machin knew the power series for the arctangent function, the computation was done by evaluating the first bunch of terms of the power series. This formula became so powerful that most subsequent calculations of π were done by similar inverse-tangent identities. An extension of Machin's formula was used to calculate π to 1 million digits in 1973.

Since Machin's formula, other methods for calculating π were discovered. In 1914, Indian prodigy Srinvasa Ramanujan showed that

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,[1103 + 26390n]}{(n!)^4\, 396^{4n}}$$

(Try plugging $n = 0$ into your calculator and see what you get for π.)

In 1987, Peter and Jonathan Borwein, now professors at Simon Fraser University, formulated a different *Ramanujan-like convergent hypergeometric series*:

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n\,(6n)!\,[212175710912\sqrt{61} + 1657145277365 + n(13773980892672\sqrt{61} + 107578229802750)]}{(n!)^4\,(3n)!\,[5280(236674 + 30303\sqrt{61}]^{[3n+3/2]}}$$

(Phew!) Oh, by the way, what is amazing about this formula is that each term adds about 25 digits of accuracy so by the time you reach $n = 99$, the result for π will be accurate to about 2500 digits!

With the advent of computers, mathematicians have created iterative algorithms for π such as this one, also discovered by the Borweins', and appropriately called the *Borwein 4th-order convergent algorithm*:

$$y_0 = \sqrt{2} - 1, \qquad \alpha_0 = 6 - 4\sqrt{2}$$

$$y_{n+1} = \frac{1 - \left(1 - y_n^4\right)^{1/4}}{1 + \left(1 - y_n^4\right)^{1/4}},$$

$$\alpha_{n+1} = \left[(1 + y_{n+1})^4\, \alpha_n\right] - 2^{2n+3}\, y_{n+1}\left(1 + y_{n+1} + y_{n+1}^2\right),$$

$$\text{then we have } \frac{1}{\alpha_n} \to \pi \text{ as } n \to \infty.$$

With a fourth-order algorithm, each iteration approximately increases the number of correct digits by a factor of four. In fact, with $n = 15$, the result is guaranteed to agree with π for over 2 **billion** digits!

Kanada's computation of π on September 18, 1999 uses a relatively simple method called the *Gauss-Legendre algorithm*, which uses the Arithmetic-Geometric Mean:

$$a_0 = 1, \qquad b_0 = \frac{\sqrt{2}}{2}, \qquad t_0 = \frac{1}{4}, \qquad x_0 = 1$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n},$$

$$t_{n+1} = t_n - x_n (a_{n+1} - a_n)^2, \qquad x_{n+1} = 2x_n,$$

$$\text{then } \lim_{n\to\infty} \frac{(a_n + b_n)^2}{4t_n} = \pi.$$

Using one of the most powerful computers in the world, Kanada was able to generate over 206 billion digits of π in just over 31 hours with this second-order algorithm! Even though each iteration approximately doubles the number of correct digits, computing each iteration is much faster than the Borwein 4th-order algorithm; in fact Kanada verified his computation using Borwein's algorithm with the same computer and generated the same 206 billion digits in 46 hours.

David and Gregory Chudnovsky of Columbia University, who have also set records in calculating digits of π, use a Ramanujan-like series that they formulated in 1985:

$$\frac{1}{\pi} = \frac{163\cdot 8\cdot 27\cdot 7\cdot 11\cdot 19\cdot 127}{640320^{3/2}}\sum_{n=0}^{\infty}\left[\frac{13591409}{163\cdot 2\cdot 9\cdot 7\cdot 11\cdot 19\cdot 127}+n\right]\frac{(6n)!}{(3n)!(n!)^3}\frac{(-1)^n}{640320^{3n}}$$

Each term only adds 14 digits of accuracy, but the beauty of the formula is that it is the fastest converging series that only uses integer terms. In 1994, they were able to compute 4 billion digits of π with this algorithm. I will not bore you with details as to why the formula works, but it is a result from this cute observation:

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.9999999999992\ldots$$

A major snag in determining whether the digits of π are distributed randomly is that to determine, say, the billionth digit of π, one has to compute each of the preceding digits. However, some significant progress was made in 1996 when the Borweins' discovered this neat formula for π:

$$\pi = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right).$$

The advantage of this formula lies in the fact that it is very simple to calculate the n-th hexadecimal (base 16) digit of π directly without computing the values of any of its previous digits. The natural question to ask is whether a formula exists that allows simple computation of the n-th decimal digit of π. Of course, it is still an open question.

Well, folks, that's all for me today... Oops! I forgot to supply a clue to the puzzle in the beginning of our chit-chat. But heck, by now you should have no problem memorizing π to 30 digits, right? May I have a large container of coffee right now?

Further Reading

There is an abundance of literature dealing with π but we have space for just a few items: In the April 1992 issue of *The New Yorker* there is an article by Richard Preston about the Chudnovsky brothers, "Mountains of π," which won an award for scientific exposition from The American Association for the Advancement of Science. Dario Castellanos wrote about "The ubiquitous π" in *Mathematics Magazine*, 61(1988), 67-98 and 148-163. There is a paper in *The American Mathematical Monthly* by the Borweins entitled "Ramanujan, modular equations, and approximations to π, or how to compute one billion digits of π" (96(1989) 201-219). Finally, David Bailey, Jonathan Borwein, Peter Borwein, and Simon Plouffe have recently published a paper on June 25, 1996 called "The Quest for Pi," which is available on the World Wide Web at *http://www.cecm.sfu.ca/˜pborwein/.*