Department of Mathematics, University of Toronto

Term Test 1 – November 13, 2008

MAT 137Y, Calculus!
Time Alloted: 1 hour 50 minutes

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1. Evaluate the following limits. (Do not prove them using the formal definition of limit.)

(10%) **(i)** $\lim_{x\to 0} \frac{x \sin x}{1-\cos x}$.

(10%) **(ii)** $\lim_{x\to 3} \frac{\sqrt{5-x}-\sqrt{x^2-7}}{\sqrt{x+6}-3}$.

2.

(10%) **(i)** Find all solutions in the interval $[0,2\pi)$ that satisfy the equation $2\sin 3x - 1 = 0$.

(10%) **(ii)** Solve the inequality $|2x| + |x-3| < 5$ and express your answer as a union of intervals.

(12%) **3.** Recall that $\{F_n\}$, the Fibonacci sequence, is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let r be a positive constant which satisfies the property that $r^2 = r + 1$.

Prove for all $n \geq 2$ that

$$r^n = F_n r + F_{n-1}.$$

4.

(5%) **(a)** State the precise definition of the statement: $\lim_{x\to a} f(x) = L$.

(10%) **(b)** Prove, using the precise definition, $\lim_{x\to \frac{1}{2}} \frac{8x^2}{x-1} = -4$.

(6%) **(c)** Prove, using the precise definition, that $\lim_{x\to 5} \frac{1}{x-5}$ does not exist.

5.

(8%) **(i)** Suppose f, g, and h are functions such that $f(x) \leq g(x) \leq h(x)$ for all real numbers x, f is continuous at a, and h is continuous at a.
If $f(a) = h(a)$, prove that g is continuous at a.

(5%) **(ii)** Does there exist a function $F(x)$ defined for all real numbers such that $F(x)$ is not continuous <u>for any</u> a, but $(F \circ F)(x)$ is continuous <u>for all</u> a? Justify your answer.

6.

(8%) **(a)** Show that there exists a real number x which satisfies the equation $\sin x = x - 1$.

(6%) **(b)** Suppose $f(x)$ is a continuous function on $[a,b]$ and that $f(x)$ is always irrational. Prove that $f(x)$ is a constant function.